Solutions 3

1. We first look for the stationary distribution $\pi^* = (\pi_0^*, \pi_1^*, \pi_2^*, \pi_3^*)$ satisfying the equations:

$$\pi^* = \pi^* P, \quad \sum_{i=0}^{3} \pi_i^* = 1.$$

If the solution exists, we check if the detailed balance equations of the form

$$\pi_i^* p_{ij} = \pi_j^* p_{ji}$$

are satisfied for all pairs of states, in order to conclude whether the chain is reversible or not. Notice that, in general,

1) they will not be satisfied if there exists a pair i, j such that $p_{ij} = 0$ and $p_{ji} > 0$;
(as it can actually be shown that if the chain is irreducible and π^* exists, then $\pi_i^* > 0$ for all $i \in S$)

2) they will be automatically satisfied for all i, j such that $p_{ij} = p_{ji} = 0$ (and a fortiori for $i = j$).

- Solving the stationary distribution equations for P_1, we obtain (remember that $0 < p, q < 1$);

$$\pi^* = \left(\frac{q^2}{q^2+q+p+p^2}, \frac{q}{q^2+q+p+p^2}, \frac{p}{q^2+q+p+p^2}, \frac{p^2}{q^2+q+p+p^2} \right)$$

Checking the detailed balance equations, we see that they hold for all pairs of states, so the chain is reversible.

- Solving the stationary distribution equations for P_2, we obtain

$$\pi^* = \left(\frac{1}{2+q}, \frac{p}{2+q}, \frac{q}{2+q}, \frac{q}{2+q} \right)$$

As $p_{13} = q > 0$ and $p_{31} = 0$, the detailed balance equations are not satisfied, so the chain is not reversible (another take on this is that in stationary state, the chain is “circulating” in the direction $\{0\} \to \{1, 2\} \to \{3\} \to \{0\}$).

- The matrix P_3 satisfies $\sum_{i \in S} p_{ij} = 1$, for all $j \in S$, so the corresponding stationary distribution π^* is the uniform distribution, as seen in the course. In this case, the chain is reversible if and only if the matrix P_3 is symmetric, which only occurs when $p = q = 1/2$ (when this condition is not met, we observe again that the chain is circulating in the direction $\{0\} \to \{1\} \to \{2\} \to \{3\} \to \{0\}$).

- The matrix P_4 satisfies also $\sum_{i \in S} p_{ij} = 1$, for all $j \in S$, and it is symmetric for all values of p and q, so the chain is reversible.

2. a) 0 and N are the two absorbing (and recurrent) states, while all the others are transient. We are interested in finding $h_{i,0}$ for $1 \le i \le N-1$. Writing down the equations, we obtain

$$h_{0,0} = 1, \quad h_{i,0} = p\, h_{i+1,0} + q\, h_{i-1,0}, \quad i = 1, \ldots, N-1, \quad h_{N,0} = 0.$$

Using the hint from the problem set, the general solution of this system of equations has the form $h_{i,0} = \alpha y_1^i + \beta\, y_2^i$ where y_1, y_2 are the roots of the quadratic equation

$$y = p\, y^2 + q, \quad \text{i.e.} \quad y_1 = 1 \text{ and } y_2 = q/p,$$

(these two roots are different, as we assumed that $p \neq q$). The solution has therefore the general form $h_{i,0} = \alpha + \beta\,(q/p)^i$. The boundary condition $h_{0,0} = 1$ imples now that $h_{1,0} = p\, h_{2,0} + q$, so after replacing $h_{1,0}$ and $h_{2,0}$ by their above expression, we obtain that $\alpha = 1 - \beta$, hence

$$h_{i,0} = 1 - \beta\, \left(1 - \,(q/p)^i\right).$$

Similarly, the boundary condition $h_{N,0} = 0$ implies that $h_{N-1,0} = q\, h_{N-2,0}$, so after replacing $h_{N-2,0}$ and $h_{N-1,0}$ by their above expression, we obtain that

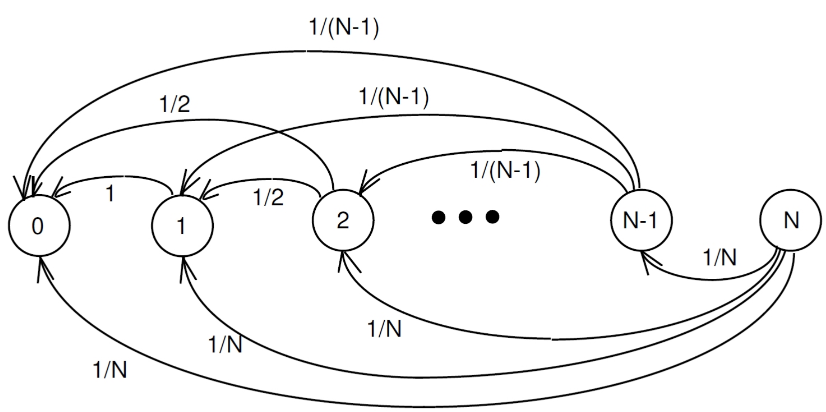
$$\beta = \frac{1}{1-(q/p)^N}, \quad \text{hence finally} \quad h_{i,0} = 1 - \frac{1-(q/p)^i}{1-(q/p)^N}.$$

b*) Following a similar procedure, we obtain that for $A = \{0, N\}$,

$$\mu_{i.A} = \frac{1}{p-q} \left(N\, \frac{1-(q/p)^i}{1-(q/p)^N} - i \right),$$

which corresponds to the average duration of the game starting from a fortune of i francs.

3. a) The state space $S = \{0, \ldots, N\}$ and the transition graph is given by



b) Again, 0 is the only absorbing (and therefore recurrent) state, while the others are transient. We are interested in computing here $\mu_{N,0}$ = the average time it takes for the ball to reach the hole form distance N. This requires to solve the system of equations:

$$\mu_{0,0} = 0, \quad \mu_{k,0} = 1 + \frac{1}{k} \sum_{j=0}^{k-1} \mu_{j,0}, \quad 1 \le k \le N.$$

Using twice this equation (once for k and once for $k-1$), we obtain

$$\begin{aligned}\mu_{k,0} &= 1 + \frac{1}{k}\sum_{j=0}^{k-1}\mu_{j,0} = 1 + \frac{1}{k}\,\mu_{k-1,0} + \frac{k-1}{k}\left(\frac{1}{k-1}\sum_{j=0}^{k-2}\mu_{j,0}\right)\\ &= 1 + \frac{1}{k}\,\mu_{k-1,0} + \frac{k-1}{k}\,(\mu_{k-1,0} - 1) = \mu_{k-1,0} + \frac{1}{k},\end{aligned}$$

so $\mu_{N,0} = 1 + \frac{1}{2} + \ldots + \frac{1}{N} \simeq \log(N)$.

c) In this case, we have

$$\mu_{0,0} = 0, \quad \mu_{2^k,0} = 1 + \frac{1}{2}\,\mu_{2^{k-1},0}, \quad k = 1, \ldots, M,$$

i.e. $\mu_{1,0} = 1$, $\mu_{2,0} = 1 + \frac{1}{2}$, $\mu_{4,0} = 1 + \frac{1}{2} + \frac{1}{4}$, so

$$\mu_{2^M,0} = \sum_{k=0}^{M} \frac{1}{2^k}$$

(this can be shown by induction). Therefore, as $N = 2^M$ gets large, $\mu_{N,0} = \sum_{k=0}^{M} \frac{1}{2^k} \simeq 2$ remains constant; the average time it takes to reach the hole indeed decreases with respect to the former case.