Solutions 2

1. The chain is ergodic because it is irreducible (so all states are positive recurrent, as there is a finite number of states) and aperiodic. Because there is a clear tendency toward middle states (in particular, $p_{00} = 0$ and $p_{01} > p_{10}$), the stationary distribution is not uniform. Solving the equations gives:

$$\pi_0 = \frac{1}{9}\pi_1, \quad \pi_1 = \pi_0 + \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2, \quad \pi_2 = \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2 + \pi_3, \quad \pi_3 = \frac{1}{9}\pi_2, \quad \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1.$$

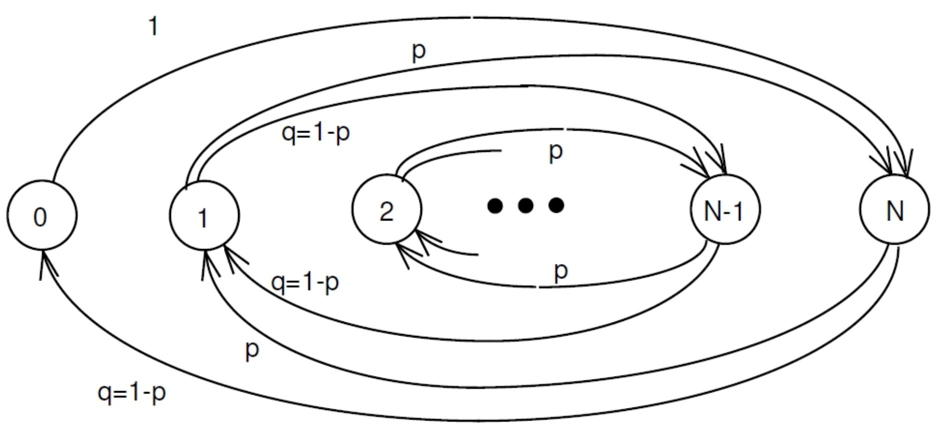
Because of the symmetry of these equations, we see that $\pi_0 = \pi_3$ and $\pi_1 = \pi_2$, so the solution is

$$(\pi_0^*, \pi_1^*, \pi_2^*, \pi_3^*) = \left(\frac{1}{20}, \frac{9}{20}, \frac{9}{20}, \frac{1}{20}\right)$$

b) The average number of red balls in the first urn is $\sum_{i=0}^{3} i\,\pi_i^* = 1.5$.

c) On 1000 steps in total, the first urn contains no red ball $\pi_0^* \times 1000 = 50$ times on average.

2. a) The state space $S = \{0, \ldots, N\}$, the transition graph is given by



and the corresponding transition matrix is given by

$$P = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & & & 0 & 1-p & p \\ \vdots & & & \iddots & & \\ & & \iddots & & & \vdots \\ 0 & 1-p & p & & & 0 \\ 1-p & p & 0 & \cdots & 0 & 0 \end{pmatrix}$$

The chain is ergodic, because it is irreducible (so all states are positive recurrent, as there is a finite number of states) and aperiodic (there is always one state in the middle of the chain that has a positive probability to come back to itself in one step, i.e., there is always a self-loop in the graph). Solving the equation for the stationary distribution gives:

$$\pi_0 = (1-p)\pi_N, \quad \pi_1 = (1-p)\pi_{N-1} + p\pi_N$$
$$\pi_2 = (1-p)\pi_{N-2} + p\pi_{N-1}$$
$$\vdots$$
$$\pi_{N-1} = (1-p)\pi_1 + p\pi_2, \quad \pi_N = \pi_0 + p\pi_1$$

and $\pi_0 + \pi_1 + \ldots + \pi_{N-1} + \pi_N = 1$. The solution reads

$$\pi_0^* = \frac{1-p}{N+1-p}, \quad \pi_i^* = \frac{1}{N+1-p}, \quad \text{for } 1 \le i \le N.$$

The probability of getting wet is therefore $p\,\pi_0^* = \dfrac{p\,(1-p)}{N+1-p}$.

b) In the numerical example, the probability of getting wet is 0.0414. The student would therefore get wet $0.041 \times 240 \approx 9.93 \approx 10$ times on average.

3. The chain is ergodic because it is irreducible (so all states are positive recurrent, as there is a finite number of states) and aperiodic.

Solving the equation for the stationary distribution gives:

$$\pi_0 = (1-p)\pi_0 + (1-p)\pi_1, \quad \pi_1 = p\pi_0 + (1-p)\pi_2$$
$$\pi_2 = p\pi_1 + (1-p)\pi_3$$
$$\vdots$$
$$\pi_{N-1} = p\pi_{N-2} + (1-p)\pi_N, \quad \pi_N = p\pi_{N-1} + p\pi_N$$

and $\pi_0 + \pi_1 + \pi_2 + \cdots + \pi_{N-1} + \pi_N = 1$. Rewriting these equations, we obtain

$$\pi_{i+1} = \frac{p}{1-p}\pi_i, \quad 0 \le i \le N-1, \quad \text{so} \quad \pi_0 \sum_{i=0}^{N} \left(\frac{p}{1-p}\right)^i = 1.$$

This leads to the solution:

$$\pi_0^* = \frac{1}{\sum_{i=0}^{N} \left(\frac{p}{1-p}\right)^i}, \quad \pi_i^* = \left(\frac{p}{1-p}\right)^i \pi_0^*, \quad 1 \le i \le N.$$

4. Let us proceed backwards and first solve the second part of the question.

b) We try solving the detailed balance equations:

$$\pi_k\, p = \pi_{k+1}\, q, \quad k \ge 0,$$

i.e. $\pi_{k+1} = (p/q)\,\pi_k$, so $\pi_k = (p/q)^k\,\pi_0$. Using the condition $\sum_{k\ge 0} \pi_k = 1$, we obtain that

$$\pi_0 \sum_{k\ge 0} (p/q)^k = 1.$$

This equation has a non-zero solution only if $p/q < 1$, i.e. $p < 1/2$. In this case,

$$\pi_k^* = (p/q)^k (1 - p/q), \quad k \geq 0.$$

So the chain admits a unique stationary distribution π^* when $p < 1/2$, which is moreover reversible. When $p \geq 1/2$, the chain does not admit a stationary distribution.

a) The intuition therefore suggests that the chain is positive recurrent when $p < 1/2$. When compared to the random walk on $\mathbb{Z}$, it can be guessed that the chain is null recurrent when $p = 1/2$ and transient when $p > 1/2$. We will come back to this question more precisely later in the course.