Background Notes on Real analysis

Let $\bar{\mathbb{R}} = [-\infty, \infty] = (-\infty, \infty) \cup \{-\infty, \infty\}$ denote the extended real line and let $E \subset \bar{\mathbb{R}}$ be a non-empty subset of the extended real line.

Definition 1 (Supremum and infimum of a set) *The supremum of E, which is written in several different ways*

$$\sup E = \sup_{x \in E} x = \sup\{x : x \in E\},$$

is the smallest element y in $\bar{\mathbb{R}}$ such that $y \geq x$ for all $x \in E$. The supremum is also known as the least upper bound of the set E.

The infimum of E, which can be written as any of

$$\inf E = \inf_{x \in E} x = \inf\{x : x \in E\},$$

is the greatest element $y \in \bar{\mathbb{R}}$ such that $y \leq x$ for all $x \in E$. The infimum is also known as the greatest lower bound of the set E.

Comment 1: For any $E \subset \bar{\mathbb{R}}$, it can be proven that both $\inf E$ and $\sup E$ always exist, although they can be $-\infty$ and ∞ respectively. For example if $E = \mathbb{R} = (-\infty, \infty)$, then $\inf \mathbb{R} = -\infty$ and $\sup \mathbb{R} = +\infty$.

Comment 2: Note that neither $\inf E$ nor $\sup E$ need be elements of E. For example, with $a < b$ consider the interval $[a, b)$, which contains a and every element from a to, but not including, b. Then $\inf\{x : x \in [a, b)\} = a \in [a, b)$ and $\sup\{x : x \in [a, b)\} = b \notin [a, b)$.

Comment 3: If $E \subset F \subset \bar{\mathbb{R}}$ then $\inf F \leq \inf E \leq \sup E \leq \sup F$.

Infimum and supremum pay the key role in determining if the limit of a sequence exists and, if it exists, identifying its value

Let $\{x_n \in \mathbb{R} : n \in \mathbb{N}\}$ be a sequence of real numbers.

Definition 2 (Limit superior and limit inferior of a sequence) *Define limit superior*

$$\limsup_{n \to \infty} x_n := \inf_{n \geq 1} \sup_{m \geq n} x_m \in [-\infty, \infty]$$

and limit inferior

$$\liminf_{n \to \infty} x_n := \sup_{n \geq 1} \inf_{m \geq n} x_m \in [-\infty, \infty].$$

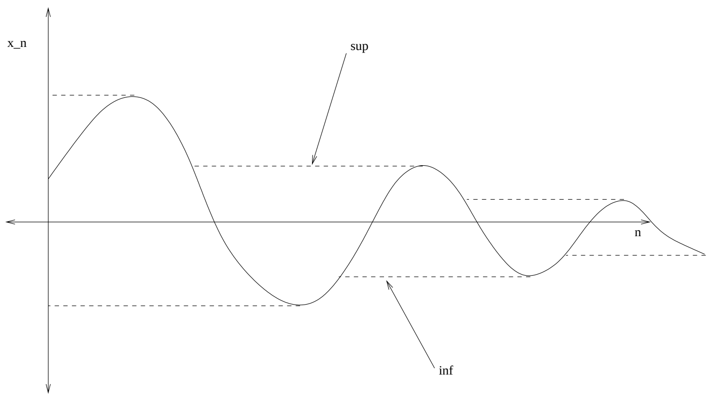

Figure 1: lim inf and lim sup for a sequence of real numbers

Comment 4: To understand $\liminf x_n$ and $\limsup x_n$, consider Figure 1. Let the solid line denote our sequence and consider $\limsup x_n = \inf_{n\geq 1} \sup_{m\geq n} x_n$. For each n, $y_n := \sup_{m\geq n} x_n$ is the least upper bound on the value the sequence takes after n, which is also denoted in the picture. Note that, from Comment 3, y_n is a non-increasing sequence. We then take an infimum over the sequence y_n, which is its greatest lower bound, to determine the $\limsup x_n$. The $\liminf x_n$ is determined in a similar fashion. For each n, $y_n := \inf_{m\geq n} x_n$ is the greatest lower bound on the value the sequence takes after n, which is denoted in the picture. We then take a supremum over the sequence y_n, which is its least upper bound, to determine the $\liminf x_n$. For the picture given, $\liminf x_n$ and $\limsup x_n$ coincide, but this is not necessarily the case. Consider what happens if our solid line was a sine wave oscillating between -1 and 1. Then $\sup_{m\geq n} x_m = 1$ for all n and thus $\limsup x_n = \inf_{n\geq 1} \sup_{m\geq n} x_n = 1$. Similarly $\liminf x_n = -1$.

Comment 5: Note that if $z > \limsup x_n$, then there exists N such that $z > x_n$ for all $n > N$. That is, $z > x_n$ for all sufficiently large n. Similarly if $z < \liminf x_n$, then there exists N such that $z < x_n$ for all $n > N$. That is, $z < x_n$ for infinitely many n.

Definition 3 (Limit of a sequence) *If* $-\infty < \liminf x_n = \limsup x_n < \infty$*, then the limit is said to exist and is defined by*

$$\lim x_n := \liminf x_n = \limsup x_n.$$

A sequence $\{x_n\}$ *is said to be convergent if* $\lim_n x_n$ *exists.*

Comment 6: A sequence $\{x_n\}$ is convergent with $x^* := \lim_n x_n$ if and only if given $\epsilon > 0$, there exists $N_\epsilon > 0$ such that $|x_n - x^*| < \epsilon$ for all $n > N_\epsilon$. Thus if a sequence is convergent, ultimately all elements of the sequence are arbitrarily close to its limit.

Definition 4 (Continuous function) *A function* $f : \mathbb{R} \mapsto \mathbb{R}$ *is continuous if, for all convergent sequences* $\{x_n\}$*,* $\lim_{n\to\infty} f(x_n)$ *exists and equals*

$$f\left(\lim_{n\to\infty} x_n\right).$$

Comment 7: ¿From the definition, a function is continuous if the limit of the function evaluated at the elements of a convergent sequence equals the value of the function evaluated at the limit of the sequence. This must hold true for every convergent sequence. For example, $f(x) = x^2$ is continuous, but the Heaviside step function

$$f(x) := \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

is not. To see this, consider the sequence $x_n = 1/n$ which is convergent with $\lim x_n = 0$. We have that $f(x_n) = 1$ for all n, so that $\lim f(x_n) = 1$, but $f(\lim x_n) = f(0) = 1/2$ so that $\lim f(x_n) \neq f(\lim x_n)$ for this convergent sequence and thus the Heaviside step function is not continuous.

Convex Functions - e.g. Roberts and Varberg's Convex Functions or Rockafeller's Convex Analysis

Recall that $\mathbb{R} := (-\infty, \infty)$.

Definition 5 (Convex function) *A function* $c : \mathbb{R} \to \mathbb{R}$ *is convex on an open interval* $(a, b) \subseteq \mathbb{R}$ *if*

$$c(tx + (1-t)y) \leq tc(x) + (1-t)c(y),$$

for all $t \in [0, 1]$*,* $x, y \in (a, b)$*. A function is strictly convex if*

$$c(tx + (1-t)y) < tc(x) + (1-t)c(y),$$

for all $t \in (0, 1)$*,* $x, y \in (a, b)$*.*

Comment 8: To understand convexity, consider Figure 2. The function $c(x)$ is convex on the interval (a, b) as for any x and y in (a, b), the chord (line segment) joining $c(x)$ and $c(y)$ (mathematically described as $tc(x) + (1-t)c(y)$, $t \in [0, 1]$) does not lie below the function. As drawn, this function is not convex outside (a, b), but is strictly convex within (a, b).

Comment 9: It is a standard result of cinvex analysis that c is continuous on all subsets of the set on which it is convex (a, b).

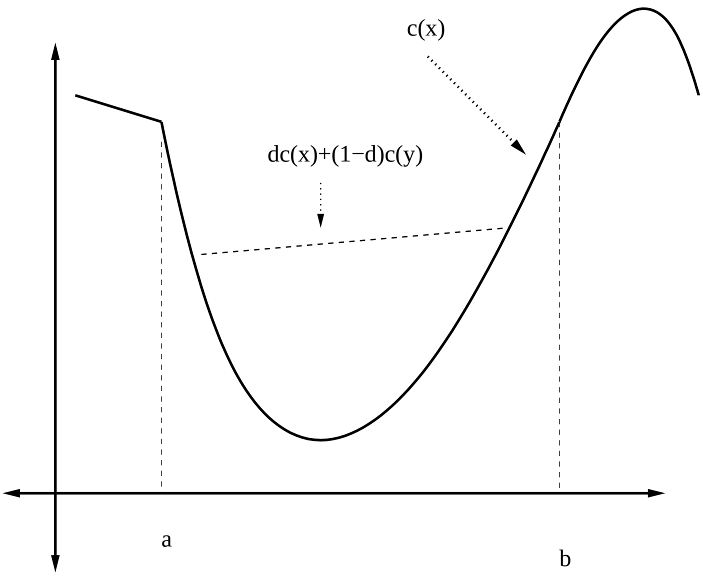

Figure 2: A function that is convex on (a, b)

<u>Comment 10:</u> Consider $c(x) := |x|$, it is convex, but not differentiable at 0. Convex functions are differentiable except on a countable set.

<u>Comment 11:</u> If a function c is twice differentiable on (a, b), a sufficient condition for c to be convex on (a, b) is that

$$\frac{d^2}{dx^2}c(x) \geq 0,$$

for all $x \in (a, b)$.

Definition 6 (Affine function) *A function $L : \mathbb{R} \to \mathbb{R}$, is affine if and only if there exists $a, b \in \mathbb{R}$ such that $L(x) = ax + b$ for all $x \in \mathbb{R}$.*

<u>Comment 12:</u> It is a result in convex analysis that convex functions can be represented by the supremum of a sequence of affine functions: if c is convex, then there exists a sequence $\{L_n, n \geq 1 \text{ and } L_n \text{ affine}\}$ such that $c = \sup_{n\geq 1} L_n$.

Definition 7 (Concave function) *A function c is concave on an open interval (a, b) if $-c$ is convex on (a, b). That is if*

$$c(tx + (1-t)y) \geq tc(x) + (1-t)c(y),$$

for all $t \in [0, 1]$, $x, y \in (a, b)$.

<u>Comment 11:</u> If a function c is twice differentiable on (a,b), a sufficient condition for c to be concave on (a,b) is that

$$\frac{d^2}{dx^2}c(x) \leq 0,$$

for all $x \in (a,b)$.