NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

Switched systems, convex cones and common Lyapunov functions

by

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Ph.D Thesis

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy in Electronic Engineering is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

Signed: ________________________________ ID No. : 61143324

Date: 5^{th} May, 2004

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In finishing, the final two lines of W.B. Yeats' poem *The Municipal Gallery Revisited*, quoted below, go some way towards reflecting my feelings towards the numerous people whose help and support have made this thesis possible:

> *Think where man's glory most begins and ends,*
> *And say my glory was I had such friends.*

For Emer, Mom and Dad, and my family

Do Emer, mo thuismitheoirí, agus mo chlann uile

O you are not lying in the wet clay,
For it is a harvest evening now and we
Are piling up the ricks against the moonlight
And you smile up at us - eternally.

Patrick Kavanagh - In Memory of my Mother

Abstract

In this thesis, we are concerned with the problem of common Lyapunov function existence for families of linear time-invariant (LTI) systems, and with the precise nature of the relationship between common Lyapunov function existence and the exponential stability of switched linear systems. The work of the thesis is motivated by the practical importance of switched linear systems and the known fact that such systems can become unstable even when they are constructed by switching between individually stable constituent systems, giving rise to a need for verifiable conditions that guarantee the exponential stability of switched linear systems under arbitrary switching rules.

After introducing, and reviewing the literature on, the common quadratic Lyapunov function (CQLF) existence problem, we describe a novel approach to the question of CQLF existence for a pair of LTI systems that is based on analyzing convex cones of matrices. In particular, in Theorem 4.4.1, we derive a key result that provides a simple algebraic characterization of a marginal situation of two LTI systems that are on the 'boundary' of those pairs of systems that have a CQLF. This result provides insight into the issue of how conservative CQLF existence is as a criterion for the exponential stability of switched linear systems, and we show that it unifies two of the most powerful results giving necessary and sufficient conditions for CQLF existence to have previously appeared in the literature. Moreover, we explain how, for certain system classes, Theorem 4.4.1 provides a way of obtaining verifiable necessary and sufficient conditions for CQLF existence that can be interpreted in terms of the dynamics of switched linear systems. Based on the same underlying ideas, a corresponding result is derived for pairs of discrete-time LTI systems. We also extend a recent result giving necessary and sufficient conditions for CQLF existence for pairs of LTI systems whose system matrices are in companion form to the case of a general pair of exponentially stable LTI systems with system matrices differing by rank one,

and describe in Corollary 5.3.1 a class of switched linear systems for which CQLF existence is equivalent to uniform exponential stability under arbitrary switching.

Several problems relating to the stability of positive switched linear systems are also considered. In particular, for pairs of exponentially stable positive LTI systems, we present results on the CQLF existence problem, on the common diagonal Lyapunov function (CDLF) existence problem, and on the problem of common copositive Lyapunov function existence. We show that for switched linear systems constructed by switching between a pair of exponentially stable positive second order LTI systems, CQLF existence is equivalent to exponential stability under arbitrary switching. Moreover, we establish that for second and third order systems, any pair of exponentially stable positive LTI systems whose system matrices differ by rank one must have a CQLF, and that the associated switched linear systems must be uniformly exponentially stable under arbitrary switching. We also derive an algebraic condition that is necessary and sufficient for a generic pair of exponentially stable positive LTI systems, of any order, to have a CDLF, using similar arguments to those employed in the derivation of the result of Theorem 4.4.1 on the CQLF existence problem. Results giving necessary and sufficient conditions for common linear copositive Lyapunov function existence are also presented.

Finally, we discuss the possibility of extending our methods to non-linear switched systems and describe a number of open problems suggested by the work presented in the thesis.

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Notations and abbreviations

Common notations

Symbol	**Meaning**
$\mathbb{R}$ ($\mathbb{C}$)	the field of real (complex) numbers
$\mathbb{R}^n$ ($\mathbb{C}^n$)	the vector space of all n-tuples of real (complex) numbers
$\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$)	the space of all $n \times n$ matrices with real (complex) entries
$\mathrm{Re}(z)$ ($\mathrm{Im}(z)$)	the real (imaginary) part of the complex number z
a_{ij}	the (i, j) entry of a matrix A
A^T	the transpose of a matrix A
A^*	the Hermitian or conjugate transpose of a matrix A
A^{-1}	the inverse of an invertible matrix A
A^{-T}	abbreviation for $(A^{-1})^T$
$Sym(n, \mathbb{R})$	space of $n \times n$ symmetric matrices
$P > 0$ ($P < 0$)	the matrix P is positive (negative) definite
$P \geq 0$ ($P \leq 0$)	the matrix P is positive (negative) semi-definite
$\mathbf{P}_n$	the cone of positive definite matrices in $\mathbb{R}^{n\times n}$
$\det(A)$, $\lvert A\rvert$	the determinant of a matrix A in $\mathbb{R}^{n\times n}$

rank(A)	the rank of a matrix A in $\mathbb{R}^{n\times n}$
$\lvert x\rvert$	the modulus or absolute value of x in $\mathbb{R}$ or $\mathbb{C}$
$\sigma(A)$	the spectrum of a matrix A
$\rho(A)$	the spectral radius of A
$\mu(A)$	the maximal real part of the eigenvalues of a matrix A
$x \succeq 0$ $(x \succ 0)$	each entry of the vector x in $\mathbb{R}^n$ is non-negative (positive)
$\mathbb{R}^n_+$	the positive orthant of $\mathbb{R}^n$, given by $\{x \in \mathbb{R}^n : x \succ 0\}$
$A \succeq 0$ $(A \succ 0)$	each entry of the matrix A in $\mathbb{R}^{n\times n}$ is non-negative (positive)
Σ_A (Σ_A^d)	the continuous-time (discrete-time) linear time-invariant system $\dot{x} = Ax$ $(x(j+1) = Ax(j))$ where $A \in \mathbb{R}^{n\times n}$
$\mathcal{L}_A$	the (continuous-time) Lyapunov operator corresponding to the matrix A
$\mathcal{P}_A$	the cone $\{P = P^T > 0 : A^TP + PA < 0\}$
$\mathcal{P}_A^d$	the cone $\{P = P^T > 0 : A^TPA - P < 0\}$
$[A, B]$	the commutator of the matrices A and B, given by $AB - BA$
$\langle A, B\rangle$	the inner product trace(A^TB)
∇V	the gradient of the function $V : \mathbb{R}^n \to \mathbb{R}$

Abbreviations

LTI	linear time-invariant
RHP	right half plane
CQLF	common quadratic Lyapunov function
CDLF	common diagonal Lyapunov function

LCLF	linear copositive Lyapunov function
BIBO	bounded-input bounded-output
LDI	linear differential inclusion
LMI	linear matrix inequality
SISO	single input single output

Chapter 1

Introduction and Overview

In this introductory chapter, we motivate the study of switched and hybrid systems, and point out some of the issues associated with the stability of these systems. We also provide a brief overview of the work contained in the remainder of the thesis.

1.1 Introductory remarks on switched systems

The theory of switched or hybrid systems is by now a well-established research field, with the area attracting interest from across the engineering, mathematics and computer science communities. Loosely speaking, a switched system is one that combines continuous (or discrete) dynamics with a logic-based switching mechanism that determines abrupt mode switches in the system's operation at various points in time. The major improvements that have occurred in computational technology, as well as the need for control systems to satisfy more demanding performance criteria have led to a significant increase in the number of switched systems appearing in applications in the recent past [10, 11, 63, 140, 44]. Alongside this, there has been a corresponding increase in the level of interest in the theoretical aspects of these systems within the research community [69, 15, 133, 66]. From a control perspective, one of the major attractions of the switching paradigm is that it is often possible to satisfy complex performance objectives by switching between a number of relatively simple (often LTI) controllers. Before proceeding, it is worth noting some of the specific reasons for the recent surge of interest in systems that undergo switching.

(i) First of all, some engineering systems are inherently multi-modal, and for such systems, a model that incorporates switching is most appropriate. A classic and very familiar example of this type is given by the longitudinal dynamics of a four or five speed automobile [131], where the acceleration of the vehicle depends on the current speed, the throttle angle and the gear that is engaged. A change of gear may then be viewed as a mode-switch in the overall dynamics of the car.

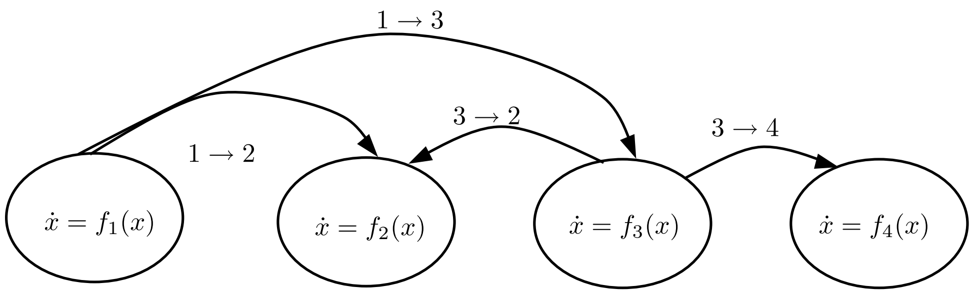

Figure 1.1: Sample switched system with four constituent modes or subsystems

(ii) Even for a plant that is not itself subject to switching, a switched controller offers greater flexibility than a single fixed controller and can deliver improved closed loop performance. In particular, it is sometimes possible to achieve multiple performance objectives by switching appropriately between a family of individual controllers, each one designed for a specific task. Further, it is often possible to achieve such complicated performance objectives by switching between a number of controllers of simple structure. For a number of examples of this type, including a controller for a computer disk-drive system providing improved response-time while also reducing overshoot, see [81]. Also, a switched controller, with three distinct modes of operation, for a variable-speed wind-turbine power generator is described in [64, 63].

(iii) The need for modern adaptive control systems to function effectively in uncertain and rapidly varying environments, and to cope with failures and external disturbances, has been a major motivation for investigating switched systems. In this context, the *Multiple Models, Switching and Tuning* paradigm [96, 95] has emerged as a powerful methodology for designing controllers for systems that are subject to sudden and large changes in their operating conditions.

Traditional adaptive control schemes use a single model and controller, whose parameters are tuned to improve the accuracy of the model and the performance of the controller. However, this type of adaptation can prove too slow in cases where the environment in which the controller is supposed to work is subject to rapid or abrupt changes. The *Multiple Models, Switching and Tuning* paradigm addresses several of the limitations of traditional adaptive control in dealing with uncertain and rapidly varying environments. Here, rather than use a single model to represent the plant, a number of different models are selected to represent the various situations in which the system has to operate. A controller is then designed for each of these models. At each instant, based on some performance criterion, the model that best describes the current environment is selected, and the corresponding control input is applied to the plant. Effectively, the system switches between the various controllers depending on what model best describes the system at each instant. A recent application of this paradigm has been to the design of reconfigurable flight control systems that can cope with a variety of sensor and actuator failure events [10, 11].

(iv) For a system that is subject to constraints, a switched controller is often the most natural way to meet performance objectives while still satisfying the constraints on the system [81].

Alongside their undeniable practical importance, switched systems have also emerged as a rich source of interesting and challenging mathematical problems. One possible approach to the analysis of systems that undergo switching would be to treat the continuous dynamics and the logic-based switching action separately. For instance, differential or difference equation models could be used to describe the dynamics with discrete event systems or finite state automata employed to encapsulate the switching action. Such an approach is unsatisfactory however, as it takes no account of the potentially complex interaction between the various dynamic modes and the switching mechanism. Ignoring this interaction could prove dangerous, as the interplay between the switching action and the dynamics can lead to unexpected or even catastrophic behaviour. This is of paramount importance given that switched systems are often applied in safety critical situations, such as the examples cited above from the aeronautical and automotive industries. As a simple example to illustrate the type of unusual behaviour that can arise as a result of switching, consider the following scenario of a "car in the desert" [126].

Example 1.1.1 *Consider the situation depicted in Figure 1.2 below.*

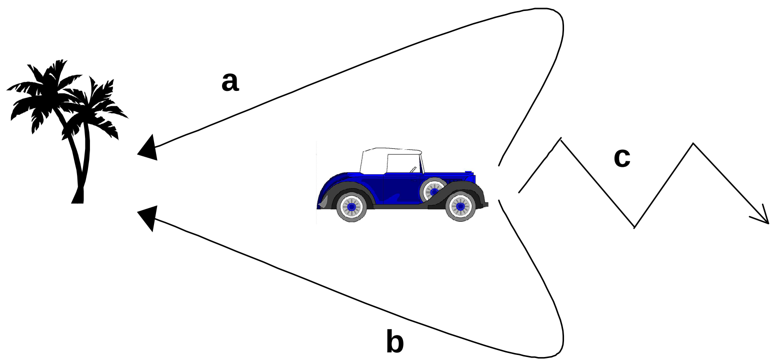

Figure 1.2: Car in the desert - switching and dehydration!

The driver is thirsty and would like to get back to the oasis in order to wet his palate. Either of the two paths **a** *or* **b** *in Figure 1.2 will lead the car safely to the oasis. However, by switching between these two paths it is possible to zig-zag away from the oasis along the path* **c***, leading to a parched throat and the possible demise of the driver!*

While the above example may appear somewhat frivolous it illustrates the key point that a switched system can exhibit behaviour that is not present in any of its constituent subsystems or modes. In a sense, we can think of the two paths **a** and **b** as representing safe or stable subsystems, while the path **c** shows that unsafe or catastrophic results can ensue from an inappropriate choice of switching rule. For this reason, it is important to develop analysis tools and results for these systems that take into account the effects of the interplay between the switching mechanism and the dynamics of the component modes of the system. In spite of the considerable body of work carried out in the area over the past ten years or so, several basic questions related to the properties of switched systems remain unanswered and at best partially understood. Perhaps the most important of these relates to their stability.

For engineering systems, the issue of stability is of critical importance. If the stability of a system can be guaranteed, then the designer is free to concentrate attention on performance-related issues. To quote from Narendra and Balakrishnan's paper of 1997 [95]:

> *It is well known that efficient design methods for various classes of control systems can be developed only when their stability properties are well understood.*

In this regard, a major complication that arises when considering switched systems is that it is possible for instability to arise even for systems constructed by switching between a number of individually stable subsystems. The simple example of the car in the desert above illustrates this point, and a number of analytic examples displaying the same sort of behaviour are given in [69, 28, 64]. In the next chapter, when we discuss some of these issues in more detail, we shall present an example of an unstable switched system that is constructed by switching between stable constituent systems. In the light of this fact and of the critical importance of stability for all control systems, it is not surprising that a major research effort has been dedicated to developing a fuller understanding of the stability properties of switched systems, and in particular to the determination of criteria that can be used to guarantee the stable operation of such systems. One way of approaching this problem is to look for verifiable conditions for the existence of a common Lyapunov function for the constituent subsystems of a switched system. It is known [69] that if such a function exists, then the overall system will be stable for arbitrary switching rules. Furthermore, a number of converse theorems have recently been established for various classes of switched systems [27, 86], essentially showing that, for a switched linear system, where the constituent subsystems are linear time-invariant (LTI) systems, a common Lyapunov function will always exist if the overall system is exponentially stable for arbitrary switching patterns.

It is worth pointing out that the converse theorems mentioned in the previous paragraph [27, 86] show that requiring the existence of a common Lyapunov function is not a conservative criterion for the exponential stability of switched linear systems. In fact, these results demonstrate that if such a system is uniformly exponentially stable under arbitrary switching, then all of its constituent subsystems will have a common Lyapunov function. We shall have more to say on the issue of common Lyapunov functions and conservatism later in the thesis. In fact, for a large part of the thesis we shall be concerned with determining verifiable conditions for the existence of a common quadratic Lyapunov function (CQLF) for the constituent systems of a switched linear system, as well as with understanding the precise connection between CQLF existence and the stability of switched linear systems. In particular, we shall investigate the 'myth' that CQLF existence is a conservative criterion for the stability of switched linear systems. In this context, while it has been established that requiring the existence of a CQLF *can* be conservative for the exponential stability of certain switched linear systems, it is important not to leave the argument at this point without further investigating the nature of this conservatism. We shall see that some of the results presented later in the thesis indicate that there are a number of classes of switched linear systems for which CQLF existence is not necessarily a conservative criterion for exponential stability under arbitrary switching. The identification of such system classes is a problem of considerable importance. In fact, knowing that the stability of a switched linear system was equivalent to the existence of a CQLF for its subsystems would be a significant advantage as quadratic Lyapunov functions are easier to search for, and are more fully understood, than the more complex Lyapunov functions that have emerged in recent years.

Another related 'myth' about CQLFs is that they generally provide little dynamical insight into the workings of a system. We shall be presenting results that indicate that quite the opposite can be the case and that, in certain cases, there is a strong connection between the existence of a CQLF and the possible dynamic behaviours that a switched linear system may exhibit.

Before drawing these introductory paragraphs to a close and proceeding to give an overview of the work described in the remainder of the thesis, it seems appropriate to formulate something of a ‘mission statement’. Broadly speaking, our goal is to add to the current understanding of the stability issues associated with switching systems; providing, where possible, verifiable and dynamically meaningful conditions for the stability of such systems, based for the most part on the existence or non-existence of common quadratic Lyapunov functions. Hopefully, this preamble has made clear both the importance of, and the need for, such conditions.

1.2 Overview and Contributions

When presenting new work, it is of course important to set it in context, and for this reason the next two chapters are given over to describing the background to the results presented in the remainder of the thesis. In Chapter 2, we introduce mathematical definitions for switched linear systems as well as for several notions associated with such systems that are needed in our later discussions. Those types of stability that are most relevant for the work of later chapters are defined, and we present a numerical example to emphasize that switched linear systems are capable of exhibiting behaviours not present in any of their constituent systems. In particular, we show that a switched linear system can become unstable for certain switching patterns, even when its constituent systems are individually stable. A brief review of some of the recent work done on the general stability theory of switched linear systems and on non-quadratic Lyapunov functions is also given, with the most relevant results stated explicitly.

For a large part of the thesis, we shall be concerned with the question of common quadratic Lyapunov function (CQLF) existence for families of LTI systems and with the precise relationship between CQLF existence and the stability question for switched linear systems. The systematic study of this issue begins in Chapter 3 with a comprehensive review of the various approaches taken to the CQLF existence question and the results obtained so far. Apart from its practical importance arising from the stability issues associated with switching systems, the question of CQLF existence is also a challenging problem in linear algebra. For this reason, it has attracted the attention of both the mathematics and the engineering communities, and the results that we shall consider are drawn from the literature on linear algebra as well as systems' theory. A number of questions on the relationship between CQLF existence and the stability of switched linear systems are also raised in this chapter.

In spite of the volume of work done on the CQLF existence problem, it is fair to say that results giving simple, interpretable necessary and sufficient conditions for a number of LTI systems to have a CQLF are somewhat scarce. With few exceptions, most of the results currently available in the literature either give sufficient conditions for CQLF existence, or else rely on numerical methods that do not have a meaningful dynamical interpretation, and that provide little or no insight into the CQLF existence problem itself. In fact, this is true even when we consider the problem of determining conditions for a pair of systems to have a CQLF. In Chapter 4, we describe a new approach to the CQLF existence problem for pairs of LTI systems based on analyzing convex sets of matrices. The essence of this approach is to consider a marginal situation where a pair of systems are on the 'boundary' (in a sense to be made precise in the text) of possessing a CQLF. We show, under mild additional assumptions, that in this situation the two system matrices satisfy simple algebraic conditions that relate directly to the stability of switched linear systems. The power of this approach is underlined by showing that the same techniques can be applied to the CQLF existence problem in discrete-time, yielding similar conditions to those found in the continuous-time case.

In Chapter 5, we describe how the results of Chapter 4 can be used to obtain necessary and sufficient conditions for CQLF existence for certain system classes, and show that, in a sense, these results provide a unifying framework within which two of the most powerful results on CQLF existence in the literature can be treated. First of all, an intuitively appealing and insightful proof of the known conditions for a pair of second order LTI systems (in both continuous-time and discrete-time) to have a CQLF is given, and we point out how the conditions obtained relate the question of CQLF existence to the stability of the associated class of switched linear systems in this case. A key point is that the proofs given here indicate *why* the results for second order systems take the simple form that they do. We next show that a result recently derived in [128] on CQLF existence for pairs of LTI systems whose system matrices are in companion form can be extended to the case of two LTI systems whose system matrices differ by a general rank one matrix. It is pointed out that this result can be thought of as a time-domain extension (to the case of general rank one difference between system matrices) of the classical SISO Circle Criterion. Furthermore, much of the work of the chapter is given over to demonstrating the key role played by the results of Chapter 4 in determining the simple conditions for CQLF existence in this case. As a by-product of this analysis, a class of switched linear systems is identified for which CQLF existence is not a conservative way of establishing exponential stability under arbitrary switching. It is important to appreciate that the conditions for CQLF existence derived in this chapter are not only simple to state, but also have clear implications for the dynamics and stability of switching systems.

In a sense, one could say that the classes of switching systems considered in Chapter 5 are classical as they have been the subject of interest in the control community for some time now. With the advent of new application areas for control theory in such fields as communications and systems' biology, *positive systems* and systems that switch between a number of different positive systems (positive switched systems) seem likely to assume a position of considerable importance in years to come. At this point, it should be stated that by a positive system, we mean one whose state vector is constrained to be non-negative for all time. The practical importance of such systems has long been recognized, and a rich theory of positive LTI systems now exists with close connections to the theory of non-negative matrices [32, 72].

In Chapter 6, we turn our attention to the stability issues associated with positive switched linear systems. Initially, we present some results on the CQLF existence problem for families of positive LTI systems and describe another class of switched linear systems for which CQLF existence is not a conservative criterion for exponential stability under arbitrary switching. However, when considering the stability question for positive switched linear systems, the specific properties of positive LTI systems naturally suggest certain types of Lyapunov function other than CQLFs. For instance, if a positive LTI system is exponentially stable, then it not only has a quadratic Lyapunov function, but has a diagonal quadratic Lyapunov function, or diagonal Lyapunov function [32]. In view of this, it is natural to consider the question of when two or more stable positive LTI systems have a common diagonal Lyapunov function (CDLF), and to investigate the possibility of basing stability criteria for positive switched linear systems on CDLF existence. After presenting some simple sufficient conditions for CDLF existence, we derive an algebraic condition that is necessary and sufficient for a generic pair of n-dimensional exponentially stable positive LTI systems to have a CDLF. An important point to note is that this result is derived using the same underlying ideas that inspired the analysis of the CQLF existence problem presented in Chapters 4 and 5.

Given that the trajectories of a positive system are constrained to remain within the positive orthant, there is no need to impose the global conditions of a traditional Lyapunov function when analysing the stability of these systems. In fact, the usual requirements of a positive definite function with negative definite derivative along system trajectories need only be satisfied in the positive orthant in this case. A function satisfying these relaxed conditions is known as a copositive Lyapunov function, and it is natural to consider such functions when dealing with the stability of positive systems. In view of this, in Chapter 6, we shall also address the question of the existence of common copositive Lyapunov functions for families of positive LTI systems and present a number of preliminary results in this direction for both linear and quadratic copositive Lyapunov functions. We shall see that it is once again possible to obtain significant results in this context following the methods applied to the CQLF existence problem in Chapters 4 and 5. In fact, we shall derive necessary and sufficient conditions for a general pair of positive LTI systems to have a common linear copositive Lyapunov function following this approach.

Throughout the discussions in Chapters 4 through 6, the issue of determining when certain convex sets of matrices intersect is something of a recurrent motif. For instance, when considering the general CQLF existence problem, the set of positive definite solutions of the Lyapunov inequality for a given Hurwitz matrix plays a key role. In particular, the geometrical properties of the boundary of this set are crucial to the proofs of several of the results described in Chapters 4 and 5. Hence, it seems that a thorough understanding of the boundary structure of sets of the form

$$\mathcal{P}_A = \{P = P^T > 0 : A^T P + PA < 0\},$$

(where A is Hurwitz) will lead to further advances in our understanding of the general CQLF existence problem and its relation to the stability of switched systems. With this in mind, in Chapter 7 we present a collection of preliminary technical results on the boundary structure of these sets, highlighting the role played by their geometry in the proofs of the main results of earlier chapters. We also give some time to a discussion of how the work presented here for switched linear systems may be extended to the non-linear case, and describe a number of open problems relating to the work of thesis that should form the basis of future research. Finally in Chapter 8 we review the contents of the thesis and present our conclusions.

The work described in the thesis has led to a number of publications in international conference proceedings and peer-reviewed journals. In particular, work has been presented at the IFAC world congress, the American Control Conference, the European Control Conference and the IEEE Conference on Decision and Control [63, 122, 77, 76, 124], and [38] has been accepted for presentation at the 16th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2004). Also four journal papers [129, 64, 78, 123] have either appeared or been accepted for publication while [79] has recently been accepted for publication in Linear Algebra and its Applications.

Chapter 2

Stability of switched linear systems

In this chapter, we define switched linear systems as well as those types of stability that shall concern us throughout the thesis. In addition, we discuss a number of problems connected with the stability of switched linear systems and provide a brief review of the known results on the stability of switched systems that are available in the literature.

2.1 Introductory remarks

A key issue in the study of switched systems is that a system constructed by switching between a number of individually stable subsystems can become unstable for certain switching patterns. Essentially, the interaction of the switching mechanism with the individual dynamic modes can lead to instability. It is this fact that provides the motivation for much of the work presented in this thesis. In fact, our primary concern throughout shall be with finding conditions on families of subsystems that rule out the possibility of this type of behaviour; that is with determining conditions that guarantee the stability of the overall system, irrespective of how and when switching occurs. In this chapter, we introduce the definitions necessary for a formal treatment of the issues related to this question. In particular, mathematical definitions are given for switched linear systems as well as for the relevant notions of stability. Some of the major questions that arise in the study of stability theory for switched linear systems are described and a brief survey of recent results in the area is provided.

2.2 Switched linear systems - definitions

In this section we define the class of *switched linear systems*, as well as a number of basic notions related to these systems that we shall need in our later discussions. While some of the definitions given below are quite formal, it should be kept in mind that, essentially, a switched linear system is a system of the form

$$\dot{x} = A(t)x,$$

where $A(t)$ can switch between some given finite collection of matrices $A_1, \ldots, A_k$ in $\mathbb{R}^{n \times n}$. Thus, a switched linear system is obtained by switching between a number of linear time-invariant (LTI) systems $\Sigma_{A_i} : \dot{x} = A_i x$, $1 \leq i \leq k$, referred to as its constituent systems or modes. We shall be considering systems that are free to switch in a more or less arbitrary manner between their constituent systems, and we generally assume that the precise switching pattern, determined by the mapping $t \to A(t)$ is not known. However, we do require that once the system switches into a given mode, it remains in that mode for a specified finite time, which can be chosen to be arbitrarily small; thus excluding the possibility of infinitely fast switching and ensuring that only a finite number of switches can occur in a finite time. Note that a switched linear system can thus be thought of as a family of linear time-varying systems obtained as the matrix-valued function $t \to A(t)$ varies over all allowable switching patterns. We shall now make these ideas more formal.

Suppose that we are given a set of matrices $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ in $\mathbb{R}^{n\times n}$. We now introduce the set of matrix-valued mappings, $\mathbf{PC}(\mathcal{A})$ to formalize the notion of an allowable switching pattern. $\mathbf{PC}(\mathcal{A})$ is defined to be the set of matrix-valued functions $t \to A(t)$ from $\mathbb{R}$ into the set $\mathcal{A}$ with the following properties.

(i) $A(.)$ is piecewise constant and continuous from the right. This means that for each $i \in \{1, \ldots, k\}$, the set of t in $\mathbb{R}$ such that $A(t) = A_i$ is a disjoint union of intervals of the form $[s_0, s_1)$ where $s_1 > s_0$. Essentially, $A(t)$ is switching between the matrices in $\mathcal{A}$.

(ii) There is some constant $\tau_{min} > 0$, such that given any two points t, s in $\mathbb{R}$ such that $A(t) \neq A(s)$, $|t - s| > \tau_{min}$. This guarantees that each of the intervals in (ii) are of length at least τ_{min}. The constant τ_{min} is independent of t, s and may be arbitrarily small.

The switched linear system $\Sigma_{\mathcal{A}}$, associated to $\mathcal{A}$, is now defined to be the family of time-varying systems, defined on $\mathbb{R}^n$, given by

$$\dot{x}(t) = A(t)x(t) \quad t \geq t_0, x(t_0) = x_0 \tag{2.1}$$

where $t_0 > 0$ and x_0 are initial times and states respectively and $t \to A(t)$ belongs to $\mathbf{PC}(\mathcal{A})$. Note that we are defining $\Sigma_{\mathcal{A}}$ to be the collection of time-varying systems of the above form obtained as $t \to A(t)$ varies over all elements of $\mathbf{PC}(\mathcal{A})$.

Comments:

The above definition is just a formalization of the ideas that were discussed at the beginning of the section. By considering mappings $t \to A(t)$ in $\mathbf{PC}(\mathcal{A})$, we are ensuring that:

(i) $t \to A(t)$ is piecewise constant;

(ii) there can only be finitely many switches in any finite time-interval.

Constituent systems:

For each $i \in \{1, \dots, k\}$, the LTI system

$$\Sigma_{A_i} : \dot{x}(t) = A_i x(t)$$

is referred to as the i^{th} constituent system of the switching system (2.1). The overall system (2.1) is constructed by switching between the constituent systems $\Sigma_{A_1}, \dots, \Sigma_{A_k}$.

The switching signal and sequence:

Given a function $A(.)$ in $\mathbf{PC}(\mathcal{A})$ and some initial time t_0, we can define a piecewise constant function $\sigma : [t_0, +\infty) \to \{1, 2, \dots, k\}$ by

$$A(t) = A_{\sigma(t)} \quad \text{for all } t \geq t_0.$$

In keeping with the standard practice in the literature [28, 69] we shall call σ the *switching signal.*

Furthermore, we can form the sequence of ordered pairs

$$(t_0, i_0), (t_1, i_1), \dots, (t_N, i_N), \dots \tag{2.2}$$

where $t_N \in [t_0, +\infty)$, $i_N \in \{1, \dots, k\}$ for all $N \geq 0$ and

$$\begin{aligned} A(t) &= A_{i_N} \quad \text{for } t_N \leq t < t_{N+1}; \\ A(t_{N+1}) &\neq A(t_N) \text{ for all } N \geq 0. \end{aligned}$$

In the spirit of [15] we shall refer to this sequence as the switching sequence. The times $t_0, t_1, \dots, t_m, \dots$ are known as the *switching instances.*

The switching signal can be thought of as an input that determines the switching action of the system (2.1), specifying the instances when switching takes place and which system is to be activated at each such time. Similarly, the switching sequence lists the times when switching occurs together with which system is active between any two such instances.

As a final piece of notation $x(t, t_0, x_0, \sigma)$ denotes the state of the system (2.1) at time t starting from an initial state x_0 at time t_0 with the switching signal given by σ. When the meaning is clear from context, we shall simply use $x(t)$ to denote a trajectory of the system.

Of course, before proceeding to discuss the properties of solutions to (2.1), it is important to establish that such solutions actually exist. Keeping in mind that between any two switching instances, (2.1) evolves in the same manner as a standard LTI system, it is fairly easy to see that (2.1) has continuous piecewise C^1 solutions $x(.)$ for all initial states x_0 and all initial times t_0[1]. In fact, there is one such solution corresponding to each function $A(.)$ in $\mathbf{PC}(\mathcal{A})$. To see this, let the switching instances be $t_0, t_1, t_2, \ldots, t_m, \ldots$, and remember that for $t_\gamma \leq t < t_{\gamma+1}$ the system (2.1) evolves as the LTI system $\Sigma_{A(t_\gamma)}$. Then it follows by combining the transition matrices of the LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ [118] appropriately, that for any $t > t_0$, the unique solution corresponding to a given $A(.)$ is given by

$$x(t) = [e^{A(t_m)(t-t_m)} e^{A(t_{m-1})(t_m - t_{m-1})} \ldots e^{A(t_1)(t_2-t_1)} e^{A(t_0)(t_1-t_0)}]x_0,$$

where t_m is the last switching instant before t.

To illustrate how systems such as (2.1) evolve, suppose that the dynamics are given by $\dot{x}(t) = A_i x(t)$ over some finite time interval $[t_\gamma, t_{\gamma+1})$, where A_i is one of the matrices in $\mathcal{A}$. At $t_{\gamma+1}$ the system switches and the dynamics in the following interval $[t_{\gamma+1}, t_{\gamma+2})$ are given by $\dot{x}(t) = A_j x(t)$ for some $j \neq i$. We assume that the state vector $x(t)$ does not jump discontinuously at $t_{\gamma+1}$. Hence the initial state at

[1]A piecewise C^1 function is a continuous function that is piecewise differentiable with a piecewise continuous first derivative.

time $t_{\gamma+1}$ for $\dot{x}(t) = A_j x(t)$ is the terminal state of $\dot{x}(t) = A_i x(t)$ at $t_{\gamma+1}$.

Figure 2.1 below depicts a sample trajectory of a second order switched linear system with two constituent systems. Notice that while the trajectory is continuous, its derivative has discontinuities at the switching instances.

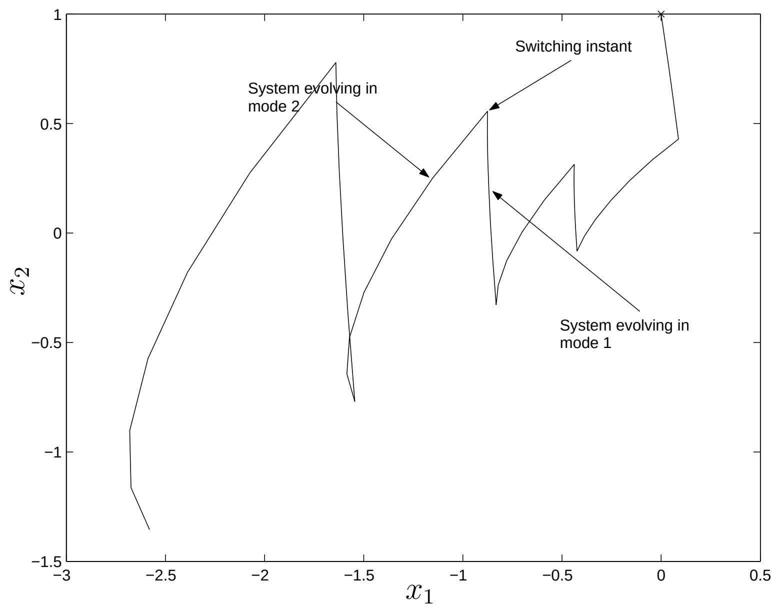

Figure 2.1: Sample trajectory of a second order switched linear system

Linear differential inclusions:

In the next section we shall define the various types of stability with which we are going to be concerned throughout the thesis. Before doing this, it is worth noting briefly that it is possible to treat some of the questions relating to the stability of switched linear systems within the framework of linear differential inclusions (LDIs) [137]. Specifically, given the set of matrices $\mathcal{A}$, we could consider the LDI

$$\dot{x}(t) \in \{A_1 x, A_2 x, \ldots, A_k x\}. \tag{2.3}$$

This is a more general setting than that which we have chosen, allowing for the possibility of infinitely fast switching and non-measurable selections from the set

$\mathcal{A}$. Some of the results that we quote later in this chapter are taken from the mathematics literature on LDIs. However, for our purposes the framework provided by the definition given in (2.1) is more than adequate as this covers the vast majority of switched linear systems encountered in practice.

2.3 Stability of switched linear systems

Stability is a fundamental requirement for all control systems and switched linear systems are no exception to this general rule. In view of this, it is hardly surprising that the recent growth of interest in these systems has led to a considerable effort being expended on the investigation of their stability properties. As we shall see, there are a number of questions pertaining to the stability of switched linear systems that are as yet unanswered. Before embarking on a discussion of these questions, we now introduce the formal definitions of the types of stability that are of interest to us.

2.3.1 Definitions of stability

A point $p \in \mathbb{R}^n$ is an equilibrium point of the system (2.1) if $A_i p = 0$ for $1 \leq i \leq k$. In particular, the origin is always an equilibrium point for (2.1). In the following definitions, for a point $x \in \mathbb{R}^n$, $\|x\|$ denotes the usual Euclidean norm of x, given by $\|x\|^2 = x^T x$.

Definition 2.3.1 *The origin is said to be a uniformly stable equilibrium point of (2.1) if given any $\epsilon > 0$, there is some $\delta > 0$ such that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0, \sigma)\| < \epsilon$ for all $t \geq t_0$ and all switching signals σ. The choice of δ does not depend on t_0 or σ.*

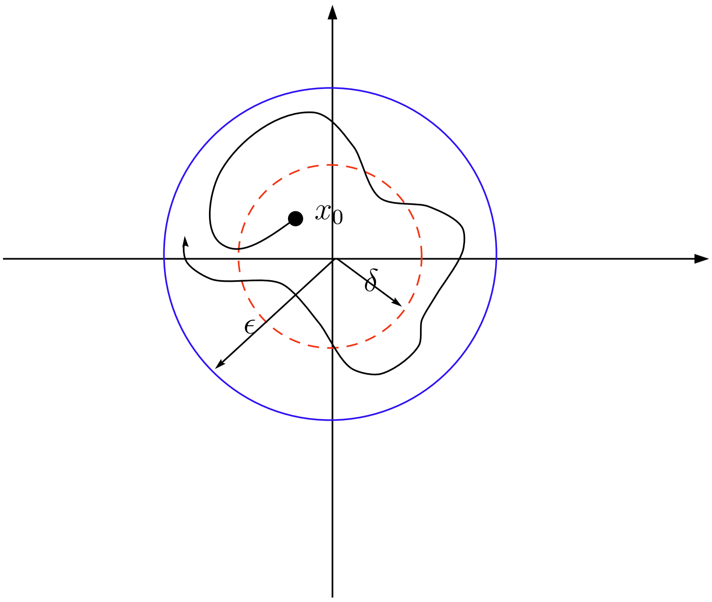

Figure 2.2: Stability

Comments:

> Roughly speaking, this means that we can ensure that a trajectory of the system (2.1) remains close to the origin by choosing the initial state sufficiently close to the origin. Note that in Definition 2.3.1 the uniformity is with respect to both the initial time t_0 and the switching signal σ.

In addition to stability, an equilibrium point is often required to be *attractive* in the sense of the following definition. The idea here is that any trajectory that starts close to an equilibrium point eventually converges to that equilibrium.

Definition 2.3.2 *The origin is said to be globally uniformly attractive if given any* $R > 0$ *and* $\epsilon > 0$*, there exists some* $T > 0$ *such that if* $\|x_0\| < R$*, then* $\|x(t, t_0, x_0, \sigma)\| <$

ϵ for all $t > t_0 + T$ and all switching signals σ. Once again the choice of T does not depend on t_0 or σ.

The definition of asymptotic stability is now obtained by combining the notions of uniform stability and attractivity.

Definition 2.3.3 *The origin is said to be a uniformly asymptotically stable equilibrium if it is uniformly stable and globally uniformly attractive.*

We next define the related notion of uniform exponential stability where an estimate on the rate at which the state converges to the origin is also given.

Definition 2.3.4 *The origin is said to be uniformly exponentially stable if there exist constants $\gamma, \lambda > 0$ such that $\|x(t, t_0, x_0, \sigma)\| < \gamma e^{-\lambda(t-t_0)}\|x_0\|$ for all $t \geq t_0$ and all switching signals σ.*

It is important to understand that all of the above definitions of stability are uniform not only with respect to the initial time t_0 but also with respect to the switching signal σ. So for instance, if the origin is uniformly exponentially stable in the sense of Definition 2.3.4, then it is exponentially stable for any admissible switching signal σ. For most of the thesis, we shall be interested in obtaining conditions for the exponential or asymptotic stability of (2.1) under arbitrary switching signals. It is also possible to investigate whether or not (2.1) is stable for some restricted class of switching signals, and some of the research on switched systems has focussed on identifying switching signals that result in stability. We shall say more of this later in the chapter.

Exponential and Asymptotic Stability:

Before proceeding we should note that, for switched linear systems, the notions of global uniform attractivity, uniform asymptotic stability and uniform exponential stability are all equivalent [4]. In fact this result is proven for homogeneous (not necessarily linear) switched systems. The equivalence of uniform exponential stability and uniform asymptotic stability has also been established in [27], where the authors use the term linear polysystem for the switched linear system (2.1). In view of this equivalence, we shall speak of the uniform exponential stability and uniform asymptotic stability of switched linear systems interchangeably.

Bounded-input bounded-output (BIBO) stability:

Thus far, the definitions that we have considered concern the unforced system (2.1), without taking into account any input or output signals. In practice of course, it is necessary to allow for control inputs, and to deal with the stability issues associated with input-output systems. It is in this context that the important notion of bounded-input bounded-output (BIBO) stability , which we now introduce, arises.

Given the sets of matrices $\{A_1, \ldots, A_k\} \subset \mathbb{R}^{n\times n}, \{B_1, \ldots, B_p\} \subset \mathbb{R}^{n\times m}, \{C_1, \ldots, C_q\} \subset \mathbb{R}^{p\times n}$, consider the switched input-output equations

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \qquad (2.4) \\ y &= C(t)x. \end{aligned}$$

Here $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output and $A(t), B(t), C(t)$ switch between the matrices in the sets $\{A_1, \ldots, A_k\}$, $\{B_1, \ldots, B_p\}$, $\{C_1, \ldots, C_q\}$ respectively, in such a way that the mapping $t \to (A(t), B(t), C(t))$ satisfies properties analogous to those imposed on the mapping $t \to A(t)$ in the definition of the unforced switched linear system (2.1). Switching signals and switching sequences can be defined for input-output systems in much the same way as we have defined them for the system (2.1). The idea behind the definition of BIBO stability is that a bounded input signal u should give rise to a bounded output signal y under the assumption of zero initial state.

Definition 2.3.5 *The input-output system (2.4) is uniformly BIBO stable if there exists a positive constant η such that for any initial time t_0, any bounded input signal u and any switching signal, the zero-state response y satisfies*

$$\sup_{t \geq t_0} \|y(t)\| \leq \eta \sup_{t \geq t_0} \|u(t)\|$$

In a sense, if a system is BIBO stable, this means that an input signal cannot be amplified by a factor greater than some finite constant η, in passing through the system. We shall not deal explicitly with BIBO stability in this thesis, but will rather concentrate on the internal stability properties of the system (2.1). However, it is important to keep in mind that there are strong connections between the asymptotic or exponential stability of (2.1) and the BIBO stability of (2.4) [118, 85]. In particular, if the system (2.1) is uniformly exponentially stable, then the corresponding input-output system (2.4) will be uniformly BIBO stable provided the matrices $B(t), C(t)$ are uniformly bounded in time. This will of course be the case when $B(t), C(t)$ switch between a finite family of matrices.

2.3.2 Stability problems for switched linear systems

For an LTI system

$$\Sigma_A : \dot{x}(t) = Ax(t) \quad A \in \mathbb{R}^{n \times n},$$

establishing whether Σ_A is uniformly exponentially stable or not is relatively straightforward. In fact, it is well-known that Σ_A is uniformly exponentially stable if and only if all of the eigenvalues of the system matrix A lie in the open left half of the complex plane [118, 50]. Such matrices are known as *Hurwitz* matrices. Unfortunately, for switched linear systems of the form (2.1) the situation is considerably more complicated. In particular, the exponential stability of each of the constituent LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ is not sufficient to ensure that the overall system is uniformly exponentially stable for arbitrary switching signals. This fact has been demonstrated by several examples presented in [69, 28, 64] and elsewhere. In fact, it is not overly difficult to find examples of systems constructed by switching between exponentially stable LTI systems, for which there exist switching signals that result in divergent trajectories.

Unstable switching system with stable constituent systems:

Example 2.3.1 *Consider the two second order LTI systems Σ_{A_1}, Σ_{A_2} where*

$$A_1 = \begin{pmatrix} 4.4908 & -35.1122 \\ 41.8926 & -6.2049 \end{pmatrix}, A_2 = \begin{pmatrix} -5.3169 & 64.9156 \\ -22.0802 & 4.0163 \end{pmatrix}.$$

It is easily checked that both A_1 and A_2 are Hurwitz so that the LTI systems Σ_{A_1} and Σ_{A_2} are uniformly exponentially stable. However, if we run a simulation of the system

$$\dot{x} = A(t)x \quad A(t) \in \{A_1, A_2\},$$

starting at $t = 0$ with $A(0) = A_1$ and $x(0) = (0.7, 0.3)$ and switching every $\pi/80$ seconds we obtain the results shown in Figure 2.3 below.

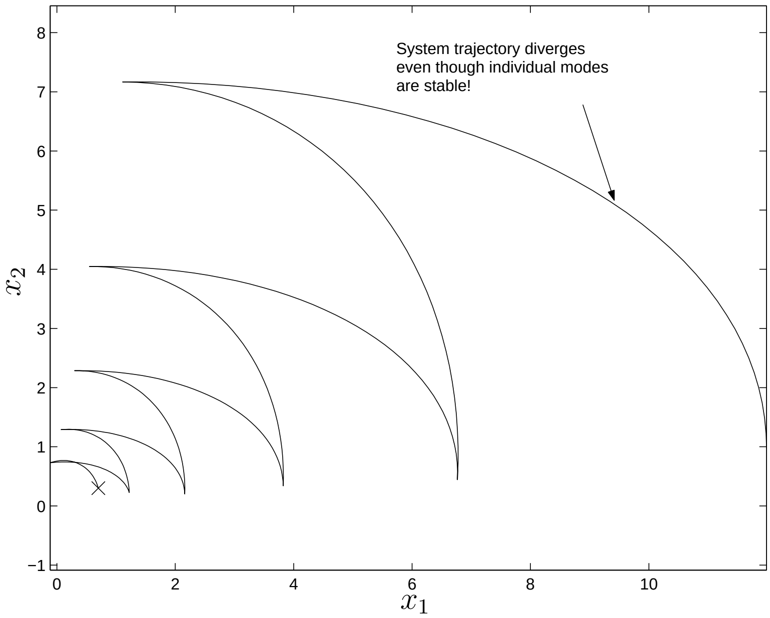

Figure 2.3: Switching between stable systems can lead to instability

Clearly, for this particular switching signal, the switched system is unstable, even though the constituent systems Σ_{A_1}, Σ_{A_2} are exponentially stable.

The next result, taken from [130] describes a simple condition on a family of exponentially stable subsystems that guarantees the existence of a destabilizing switching signal, such as that in the above example, for the associated switched linear system.

Theorem 2.3.1 *Let $A_1, \ldots, A_k$ be Hurwitz matrices in $\mathbb{R}^{n\times n}$ so that the associated LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ are exponentially stable. Suppose that there exist non-negative real numbers $\alpha_1, \ldots, \alpha_k$, not all zero, such that $\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_k A_k$ has an eigenvalue λ with non-negative real part. Then the associated switched linear system (2.1) is not uniformly exponentially stable for arbitrary switching signals. Furthermore, if $\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_k A_k$ has an eigenvalue with positive real part, then there exists a periodic switching signal that results in a divergent trajectory.*

<u>Comments:</u>

Note that it is possible to use Theorem 2.3.3 below to show that if there exist constants $\alpha_1, \ldots, \alpha_k$ such that $\alpha_1 A_1 + \cdots + \alpha_k A_k$ has an eigenvalue in the closed right half plane, then the system (2.1) is not uniformly exponentially stable for arbitrary switching signals. However, the result of Theorem 2.3.1 is stronger than this, as it establishes that if $\alpha_1 A_1 + \cdots + \alpha_k A_k$ has an eigenvalue in the open right half plane, then it is possible to obtain an *unbounded* solution of (2.1) through an appropriate choice of switching signal. In fact, the destabilizing switching signal can be taken to be periodic. In this context, the work of Pyatnitskii and Rapoport on the relationship between absolute and periodic stability should be noted [111].

It should be noted that the approach taken in [130] relies on identifying a specific way in which instability can arise in the switched linear system (2.1). Specifically, it is shown that if the hypotheses of Theorem 2.3.1 are satisfied then the system (2.1) can become unstable through so-called 'chattering', in a similar manner to Example 2.3.1. This is reminiscent of the describing function approach taken in [82] (for example) where conditions for the existence of an oscillatory instability of a non-linear feedback system are considered.

The above observations give rise to a number of natural questions regarding the stability of systems of the form (2.1) [69]. We now list three of the major such questions to have received attention in the literature in the recent past.

(i) **Arbitrary switching:**

Here we are interested in determining when the system (2.1) is uniformly exponentially stable in the sense of Definition 2.3.4? This requires the system to be exponentially stable for all possible switching signals σ. In particular this must be true for any constant signal $\sigma(t) = i$, and hence a necessary condition for stability under arbitrary switching is that each of the constituent LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ is exponentially stable.

(ii) **Dwell Time:**

When the system (2.1) fails to be exponentially stable for all switching signals, there may be a subclass of signals for which the system is exponentially stable. In particular, it is known [69, 92] that provided the interval between successive switches is sufficiently long (switching is slow enough), the system will be stable. The essence of the dwell-time problem is to determine the minimum time that needs to be left between successive switches in order to preserve stability.

(iii) **Stabilizing switching sequences:**

For switched systems constructed by switching between a number of individually unstable systems, it is sometimes possible to construct specific switching signals that result in exponentially stable trajectories [33, 144]. Such signals are commonly referred to as stabilizing switching signals or sequences and this problem is concerned with identifying these signals when they exist.

While all of the above problems are both interesting and important, for the most part we shall concentrate on problem (i); that of guaranteeing the exponential stability of switched linear systems for arbitrary switching signals. Our approach to this problem shall be to look for conditions for the existence of *common Lyapunov functions* for families of LTI systems. Some of the basic ideas and key results underpinning the use of common Lyapunov functions in the stability analysis of switched systems are discussed in the next section.

2.3.3 Lyapunov theory for switched linear systems

The concept of a Lyapunov function has played a key role in the stability theory of linear and non-linear systems for some time [73, 118, 141, 56]. In view of this, it is not surprising that a considerable amount of recent work has focussed on applying similar ideas to switched systems. In particular, many authors have attempted to derive conditions for the stability of switched systems based on the existence of common Lyapunov functions for their constituent systems. In this context, two fundamental facts have now been established for switched linear systems. First of all, if the constituent systems of the switched linear system (2.1) have a common Lyapunov function, then the overall system is exponentially stable for arbitrary switching signals. Conversely, a number of authors have shown that if the system (2.1) is uniformly exponentially stable for arbitrary switching signals, then its constituent systems have a common Lyapunov function. We shall now discuss some of the major results of this type in more detail.

To begin with we state a result due to Molchanov and Pyatnitskiy. This theorem was originally proven in [86] for linear differential inclusions but we give the result as it applies to switched linear systems of the form (2.1). Before stating the theorem, we need to introduce some notation as well as recall a few basic concepts from convex analysis [116].

Given a point $x \in \mathbb{R}^n$, and the set $\mathcal{A} = \{A_1, \ldots, A_k\} \subset \mathbb{R}^{n \times n}$, $\mathcal{A}x$ shall denote the set $\{A_1 x, \ldots, A_k x\} \subset \mathbb{R}^n$. For a positive integer k, we say that a function $V : \mathbb{R}^n \to \mathbb{R}$ is (positively) homogeneous of degree k if $V(\lambda x) = \lambda^k V(x)$ for all $x \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$ ($\lambda > 0$). Finally, we recall the definition of the one-sided directional derivative for a convex function $V : \mathbb{R}^n \to \mathbb{R}$ at a point $x \in \mathbb{R}^n$ in the direction of $y \in \mathbb{R}^n$ [116].

$$\frac{\partial V(x)}{\partial y} = \inf_{t>0} \frac{V(x+ty) - V(x)}{t}$$

Theorem 2.3.2 *The origin is a uniformly exponentially stable equilibrium of the* *system (2.1), in the sense of Definition 2.3.4 (that is, for arbitrary switching signals), if and only if there exists a strictly convex, positive definite function $V(x)$, homogeneous of degree 2, of the form*

$$\begin{aligned} V(x) &= x^T\mathcal{L}(x)x \quad \text{where } \mathcal{L}(x) \in \mathbb{R}^{n\times n} \text{ and} \\ \mathcal{L}(x)^T &= \mathcal{L}(x) = \mathcal{L}(cx) \quad \text{for all non-zero } c \in \mathbb{R}, x \in \mathbb{R}^n, \end{aligned}$$

whose derivative along trajectories of the system (2.1) satisfies

$$\max_{y\in\mathcal{A}x} \frac{\partial V(x)}{\partial y} \leq -\gamma\|x\|^2$$

for some $\gamma > 0$.

Comments:

The function $V(.)$ appearing in Theorem 2.3.2 defines a common Lyapunov function for each of the constituent LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ associated with the switched linear system (2.1). Thus if a switched linear system is uniformly exponentially stable for arbitrary switching signals, such a common Lyapunov function is guaranteed to exist. Note however that the function $V(.)$ need not be a smooth (or even C^1) function[2]. Based on the above result, Molchanov and Pyatnitskiy have also shown that the constituent systems of a uniformly exponentially stable switched linear system always have common Lyapunov functions of piecewise quadratic and piecewise linear type. These particular types of Lyapunov functions shall be discussed in more detail in later sections.

While the Lyapunov function appearing in Theorem 2.3.2 is not necessarily smooth, a similar result, due to Dayawansa and Martin [27], shows that if the system (2.1) is uniformly exponentially stable for arbitrary switching signals, then there is a smooth common Lyapunov function for its constituent systems. More specifically, the fol-

[2] A C^1 function is a differentiable function with continuous first partial derivatives.

lowing result is proven in [27] where the authors use the term linear polysystem for what we are calling a switched linear system.

Theorem 2.3.3 *For the switched linear system (2.1) the following are equivalent.*

(i) The origin is a uniformly exponentially stable equilibrium for arbitrary switching signals.

(ii) The origin is a uniformly asymptotically stable equilibrium for arbitrary switching signals.

(iii) There is a C^1 positive definite function $V : \mathbb{R}^n \to \mathbb{R}$, positively homogeneous of degree 2 such that $\nabla V(x)A_i x$ is a negative definite function of x for all $A_i \in \mathcal{A}$.

(iv) There is a C^∞ positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla V(x)A_i x$ is a negative definite function of x for all $A_i \in \mathcal{A}$.[3]

The two results above establish that the existence of a common Lyapunov function for the constituent systems of a switched linear system is sufficient for its uniform exponential stability and, perhaps more significantly, that the converse is also true. That is, that the constituent systems of a switched linear system that is uniformly exponentially stable under arbitrary switching always have a common Lyapunov function. A number of other authors have obtained similar results, including Brayton and Tong in [16] where a converse theorem for discrete-time systems was established. The work presented in [70] and [114] is also worth mentioning in this regard. Furthermore, a version of Theorem 2.3.3 for switched non-linear systems was proven in [75].

The abstract results of this section assure us that if a switched linear system is uniformly exponentially stable for arbitrary switching signals, then its constituent

[3] A function is C^∞ if it has continuous partial derivatives of all orders.

systems have a common Lyapunov function. In order to make use of these results in practice however, we need ways of searching for common Lyapunov functions and of determining whether or not they exist. In the next section, we shall briefly review some of the techniques that have been developed to search for Lyapunov functions of various types for switched linear systems.

2.4 Switched linear systems and Lyapunov functions

In order to apply the results described in the previous section to analyse the stability of switched linear systems, we need reliable methods of searching for, or determining the existence of, Lyapunov functions. Of course, it is not feasible to conduct a search over the entire class of possible Lyapunov functions, and for this reason, the common practice is to restrict ourselves to searching for Lyapunov functions of specific forms. The most common types of functions used in practice are quadratic, piecewise-quadratic and piecewise-linear functions. In this section, we shall describe some of the techniques for applying Lyapunov functions to the stability analysis of switched systems that have been developed in recent years.

2.4.1 Common quadratic Lyapunov functions

Perhaps the most familiar type of Lyapunov function is that given by a quadratic form $V(x) = x^T Px$. The stability analysis of LTI systems is completely covered by such functions as exponential stability is equivalent to the existence of a quadratic Lyapunov function in this case [118, 141]. For the switched linear system (2.1), it follows from Theorem 2.3.3 that if there exists a common quadratic Lyapunov function (CQLF), $V(x) = x^T Px$, for the constituent systems $\Sigma_{A_1}, \dots, \Sigma_{A_k}$, then the overall system is uniformly exponentially stable for arbitrary switching signals. Part of the appeal of using such functions comes from the fact that the condition for $V(x) = x^T P x$ to be a CQLF for $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ can be written in a very simple linear algebraic form. Specifically [69, 133], $V(x) = x^T P x$ is a CQLF for $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ if and only if

$$\begin{aligned} P = P^T &> 0 \\ A_i^T P + P A_i &< 0 \quad \text{for } 1 \leq i \leq k. \end{aligned} \tag{2.5}$$

To date, much work has been carried out on the problem of CQLF existence for families of LTI systems and a number of theoretical results are now available [101, 133, 129, 88]. Furthermore, it is important to note that the conditions given by (2.5) define a system of linear matrix inequalities (LMIs) in the decision variable P, and hence modern numerical techniques from the theory of convex optimization can be applied to determine whether or not a CQLF exists [12, 13]. However, it should be noted that there are some issues with the numerical approach to CQLF existence, and that the general theoretical problem of determining analytic and verifiable conditions that are equivalent to CQLF existence is still open. The problem of CQLF existence shall be a major topic throughout this thesis and we shall have much more to say on this question and the approaches to it, numerical and otherwise, in later chapters. In particular, detailed background on the problem and a review of the relevant literature are presented in the next chapter.

While quadratic Lyapunov functions possess many advantages, it has been noted by various authors [27, 17, 125] that CQLF existence may be a conservative way of establishing exponential stability for general switched linear systems or linear time-varying systems. In particular, an example has been described in [27] that illustrates that it is possible for a switched linear system to be uniformly exponentially stable under arbitrary switching even if its constituent systems have no CQLF. This has naturally led to the consideration of other, potentially less restrictive, classes of Lyapunov functions in the literature. One early example of the use of a non-quadratic Lyapunov function for demonstrating the stability of a time-varying system is given by the so-called Lur'e Postnikov Lyapunov functions. These functions combine quadratic and integral terms and arose in connection with the classical Popov stability criterion [141, 56, 109]. We shall now discuss a number of other more general types of Lyapunov function that have appeared in the literature on the stability of switched systems over the past number of years.

2.4.2 Piecewise linear systems and the S-procedure

For certain classes of switched linear systems, the global nature of the conditions in (2.5) can be unduly restrictive. An example of such a class is that of piecewise-linear systems. For such systems, the switching action is state-dependent and typically the state space is partitioned into a number of regions (often polyhedral) $\Omega_1, \ldots, \Omega_k$, with the dynamics within Ω_i given by

$$\dot{x} = A_i x, \qquad \text{for } 1 \leq i \leq k. \tag{2.6}$$

Thus, for each $i \in \{1, \ldots, k\}$, A_i only determines the system dynamics within Ω_i. For this reason, it is unnecessary to demand that the derivative of $V(x) = x^T P x$ along trajectories of $\dot{x} = A_i x$, given by $x^T(A_i^T P + PA_i)x$, is negative on the entire space. In fact, it is enough to require that $x^T(A_i^T P + PA_i)x$ is negative for x in Ω_i. This is the underlying idea of the so-called S-procedure [12, 47].

In the S-procedure, we construct matrices $S_1, \ldots, S_k$ such that $x^T S_i x \geq 0$ for x in Ω_i and then search for a positive definite P satisfying the conditions

$$A_i^T P + PA_i + S_i < 0 \quad \text{for } 1 \leq i \leq k. \tag{2.7}$$

It follows from the fact that $x^T S_i x \geq 0$ for x in Ω_i that if (2.7) is satisfied then $x^T(A_i^T P + PA_i)x$ is negative when $x \in \Omega_i$ for $1 \leq i \leq k$. However, it is possible for the expression $x^T(A_i^T P + PA_i)x$ to take either negative or positive outside of Ω_i, and for this reason the conditions of (2.7) may be easier to satisfy than those of (2.5). As with the conditions for CQLF existence, (2.7) defines a system of LMIs in the variable P.

2.4.3 Piecewise quadratic Lyapunov functions

In the hope of obtaining less restrictive stability criteria for piecewise linear systems of the form (2.6), a number of authors have considered piecewise quadratic Lyapunov functions. Here, rather than searching for a single quadratic Lyapunov function $V(x) = x^T Px$ for the system (2.6), the idea is to search for a family of such functions satisfying certain local conditions, and then to piece these together appropriately to form a Lyapunov function for the overall system.

For example, in [47] under the assumption that the individual regions Ω_i are polyhedra, Johansson and Rantzer describe a procedure for searching for a piecewise quadratic Lyapunov function of the form [4]

$$V(x) = x^T P_i x \quad \text{for } x \in \Omega_i, \tag{2.8}$$

where $P_i \in \mathbb{R}^{n\times n}$ for $i = 1, 2, \ldots, k$. The conditions for stability that they derive relax those of (2.5) in a number of ways.

(i) The use of different quadratic forms for the different operating regions Ω_i can lead to greater flexibility in the definition of the Lyapunov function V.

(ii) The matrices P_i are not required to be globally positive definite. In fact, by applying the S-procedure, the inequality $x^T P_i x > 0$ is only required to hold for $x \in \Omega_i$.

[4] The function takes a slightly different form in regions Ω_i that do not contain the origin. For details consult [47]

(iii) Similarly, the condition $x^T(A_i^T P_i + P_i A_i)x < 0$ is only required to hold for $x \in \Omega_i$.

The results presented in [47] were derived for the class of piecewise affine systems and can also be applied to more general switched or hybrid systems.

A few specific points relating to the results described in [47] are worth noting.

(i) The matrices P_i are parameterized so as to ensure that the overall function $V(x)$ is continuous.

(ii) The conditions for (2.8) to define a piecewise quadratic Lyapunov function for the system (2.6) are expressed in the form of a system of linear matrix inequalities (LMIs) which means that modern convex optimization algorithms can be applied to search for piecewise quadratic Lyapunov functions.

(iii) It is possible to search for a piecewise quadratic Lyapunov function defined with respect to a partition of the state space other than that dictated by the dynamics. This means that if an initial search is unsuccessful, it may be possible to find a piecewise quadratic Lyapunov function defined with respect to an alternative, possibly finer, partition of the space. The problem of selecting an initial partition, and of devising automatic methods of successively refining the partition, in order to systematically search for piecewise quadratic Lyapunov functions is in general far from straightforward.

The same authors have extended the results and ideas of [47] in [48, 113], and similar results leading to LMI-based conditions have been reported in [104, 105] by Pettersson and Lennartson. Piecewise quadratic Lyapunov functions have appeared before in stability analysis in the work of Power and Tsoi and also of Weissenberger [110, 143]. An important point about this type of Lyapunov function is that the constituent systems of a switched linear system of the form (2.1) that is exponentially stable under arbitrary switching always have a common Lyapunov function of piecewise quadratic type. This was shown by Molchanov and Pyatnitskiy in [86] and some related work has been presented in [149].

2.4.4 Piecewise linear Lyapunov functions

Another class of Lyapunov functions that has received attention in relation to stability questions is that of piecewise linear Lyapunov functions [106, 107, 147, 9, 151]. While the study of such functions is not new [117, 143] it has undergone something of a resurgence of late. A number of factors have contributed to this phenomenon. For instance, given that the trajectories of switched systems can be non-smooth at the switching instances, intuitively it seems appropriate to use Lyapunov functions that can themselves be non-smooth to analyse these systems. Moreover, for systems constructed by switching between LTI systems with real eigenvalues, the nature of the trajectories of such systems suggests that piecewise linear (rather than quadratic) Lyapunov functions are particularly suited to their stability analysis. Hence the emergence of switched systems has naturally led to renewed interest in the study of this type of Lyapunov function. Furthermore, the advances made in computational technology and optimization algorithms in recent years have rendered it far easier to carry out numerical searches for piecewise linear (and quadratic) Lyapunov functions than was previously the case.

Typically, piecewise linear Lyapunov functions are defined using either the l_1 or the l_∞ vector norms on the state space $\mathbb{R}^n$. As usual [42], the l_1 norm of a vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ is defined as

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

with the l_∞ norm given by

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

As an example of the use of Lyapunov functions based on the l_1 norm, Wulff et al [147] define a unic function on $\mathbb{R}^n$ to be a function of the form

$$V(x) = \|Mx\|_1 \tag{2.9}$$

for some non-singular matrix $M \in \mathbb{R}^{n\times n}$. They then derive a simple necessary and sufficient condition for a second order LTI system to have a unic Lyapunov function. Specifically, given a matrix A in $\mathbb{R}^{2\times 2}$, they show that the associated LTI system

$$\Sigma_A : \dot{x} = Ax$$

has a unic Lyapunov function if and only if all of the eigenvalues λ of the system matrix A lie in the so-called 45°-region given by

$$\{z \in \mathbb{C} : |\mathrm{Im}(z)| < |\mathrm{Re}(z)|, \mathrm{Re}(z) < 0\}.$$

Furthermore, a sufficient condition for a pair of exponentially stable second order LTI systems to have a common unic Lyapunov function is also derived in the same paper. Specifically, the following is shown.

Theorem 2.4.1 *Let two Hurwitz matrices A_1, A_2 in $\mathbb{R}^{2\times 2}$ be given such that the eigenvalues of A_1 and A_2 are all real. Suppose that for some $\alpha_0 \in [0,1]$, $\alpha_0 A_1 + (1-\alpha_0)A_2$ has a complex eigenvalue and that for all $\alpha \in [0,1]$, the eigenvalues of $\alpha A_1 + (1-\alpha)A_2$ lie in the interior of the 45°-region. Then the associated LTI systems Σ_{A_1}, Σ_{A_2} have a common unic Lyapunov function.*

Comments:

> It should be noted that the above result provides a simple eigenvalue-based condition for a pair of LTI systems to have a *common* piecewise linear Lyapunov function, and moreover that the condition that it describes is co-ordinate independent.

As well as the l_1-based unic functions, several authors have considered Lyapunov functions, defined on $\mathbb{R}^n$, of the form

$$V(x) = \|Wx\|_\infty \tag{2.10}$$

where $W \in \mathbb{R}^{m\times n}$ is a matrix of rank n and $m \geq n$. In fact it has been shown in [86], that the switched linear system (2.1) is uniformly exponentially stable under arbitrary switching if and only if its constituent systems have a common Lyapunov function of this form. In the same paper, the following necessary and sufficient conditions for $\|Wx\|_\infty$ to define a Lyapunov function for the single LTI system Σ_A were derived. A constructive proof of this same result, based on linear programming, was given by Polanski in [107].

Theorem 2.4.2 *Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz. Then a necessary and sufficient condition for $V(x)$, given by (2.10) to be a Lyapunov function for the LTI system Σ_A is that there exists a matrix $Q \in \mathbb{R}^{m\times m}$ such that*

$$WA = QW$$

and

$$q_{ii} + \sum_{j=1, j\neq i}^{m} |q_{ij}| < 0$$

for $1 \leq i \leq m$.

The level curves of Lyapunov functions of the form (2.10) are centrally symmetric polyhedra, with the number of faces on the polyhedron being determined by the integer value m. Those Lyapunov functions of the form (2.10) for a system Σ_A, whose level curves have the minimum possible number of faces, $\nu(A)$, were considered in a recent paper by Bobyleva [9]. In that paper, a number of results were presented relating this minimum number $\nu(A)$ for the LTI system Σ_A to the location of the spectrum of the system matrix A. Some of these results are related to the so-called '$45°$' criterion of [147].

In connection with piecewise linear Lyapunov functions, the work of Yfoulis and Shorten [151] on numerical techniques for searching for such functions is worth noting. Currently, the difficulty of systematically searching for such functions increases dramatically with the dimension of the state space and general search methods only appear to work effectively up to dimension three.

Lyapunov functions defined using the l_1 and l_∞ norms are special cases of Lyapunov functions defined using general vector norms. For a general such norm $\|.\|$ on $\mathbb{R}^m$, and $W \in \mathbb{R}^{m \times n}$ of rank n $(m \geq n)$, the problem of determining when $\|Wx\|$ defines a Lyapunov function for the LTI system Σ_A has been considered by Kiendl et al. in [57]. The conditions presented there are only sufficient in general, although Polanski has subsequently identified situations where they are also necessary in [108].

2.4.5 Multiple Lyapunov functions

The concept of *multiple Lyapunov functions* is another related approach to the stability analysis of switched systems, with the key idea being to use a number of different Lyapunov functions, rather than a single Lyapunov function, to investigate stability. Here, the focus is not on guaranteeing stability for arbitrary switching. Rather, through the use of multiple Lyapunov functions, it is possible to identify ways of restricting the switching action so as to ensure that the system remains stable. To date a number of papers dealing with multiple Lyapunov functions have appeared and several closely related results are now available [15, 103, 104]. Theorem 2.4.3 below is a fairly typical example of the results in this area, and should hopefully illustrate the key ideas behind the method.

While several of the theorems on multiple Lyapunov functions in the literature appear involved and technical in nature, the underlying ideas are quite simple. In fact, the technique itself is related to the intuitive observation that a switched system will be stable provided the switching between its constituent systems occurs at a sufficiently slow rate [69]. It should be noted that very similar ideas have arisen before in the context of time-varying systems in the work of deSoer and others on the stability of slowly-varying systems [29, 135].

In this context it should be pointed out that it is not a generally true precept that 'slower switching means greater stability'. For instance, consider again the two second order systems of Example 2.3.1. We have seen that switching between the two systems every $\pi/80$ seconds leads to instability. Now if we start from the same initial conditions but slow down the rate of switching to one switch every $\pi/40$ seconds, we obtain the stable trajectory shown in Figure 2.4 below. However, if we switch at the slower rate of one switch every $\pi/30$ seconds, we obtain the trajectory shown in Figure 2.5.

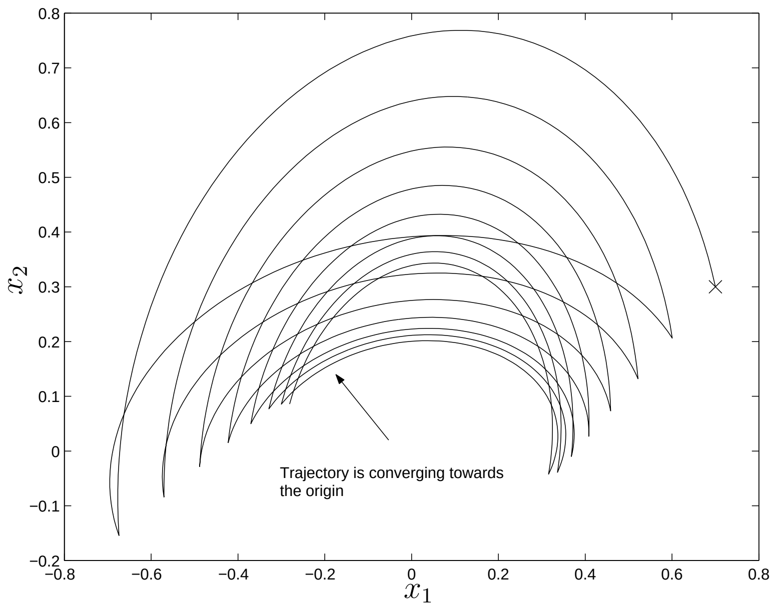

Figure 2.4: Switching every $\pi/40$ seconds

Now, with the same initial conditions once again and the yet slower rate of switching of one switch every $\pi/30$ seconds, we obtain the trajectory shown in Figure 2.5.

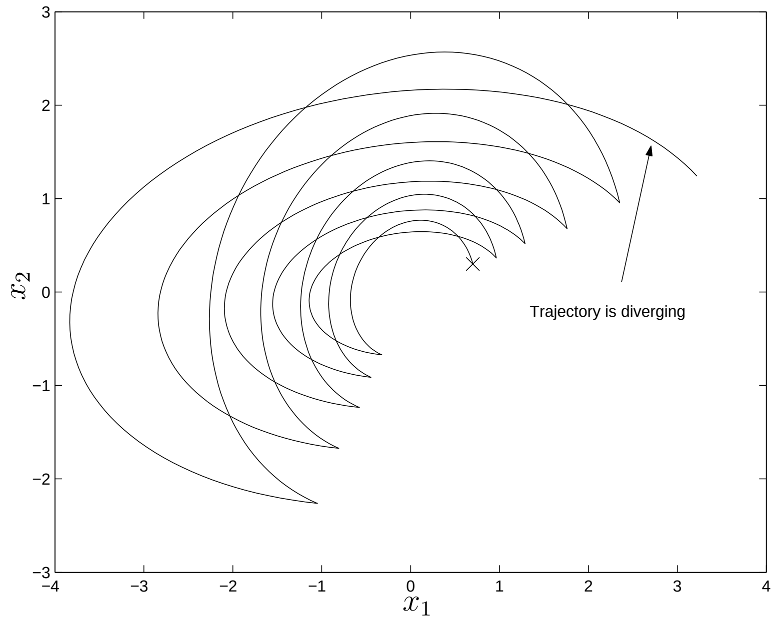

Figure 2.5: Switching every $\pi/30$ seconds

This simple example illustrates that it is possible for a switched system to be stable at one rate of switching, and unstable for a *slower* rate.

In order to state the next theorem succinctly, we need to introduce a piece of notation associated with switching signals for the system (2.1).

Consider the system (2.1). Given $i \in \{1, \ldots, k\}$ and a switching signal, we denote by $t_j^{(i)}$ the j^{th} instant when the constituent system Σ_{A_i} is switched on or activated. While we state the following theorem for switched linear systems, it has been proven by Branicky in [15] for switched non-linear systems.

Theorem 2.4.3 *Consider the system (2.1). Suppose that we have a family of C^1 positive definite functions V_i, $1 \leq i \leq k$, such that, V_i decreases along the trajectories* *of the system whenever Σ_{A_i} is active, and moreover that $V_i(x(t_{j+1}^{(i)})) \leq V_i(x(t_j^{(i)}))$ for all j. Then the system is uniformly stable.*

The idea of the theorem is illustrated in Figure 2.6 below.

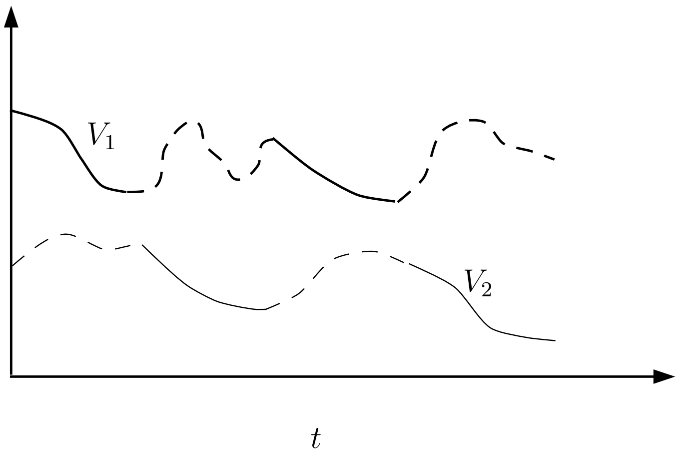

Figure 2.6: Multiple Lyapunov functions V_1, V_2: solid line indicates that the corresponding system is active

Comments:

It follows from Theorem 2.4.3 that a switched system will be stable if we can find a set of Lyapunov functions $V_1, \ldots, V_k$, one for each of its constituent systems, such that each time the i^{th} constituent system is switched on, the value of V_i is no greater than it was the previous time the i^{th} system was switched on. Thus, to ensure stability, we need to wait until the value of V_i has decreased sufficiently before we switch back to the i^{th} subsystem. In a sense, the essence of this method is the observation, related to earlier work on slowly-varying systems [29], that if we switch sufficiently slowly, then the system will be stable. Multiple

> Lyapunov functions provide a way of deciding when and how to switch once the individual functions V_i have been chosen.
>
> However, a drawback of the method is that the Lyapunov functions V_i must be selected in advance, and in order to do this, the individual systems must be stable. Thus the method cannot effectively be used to obtain stabilizing switching laws for systems switching between unstable systems. Also, while the results of [15] were proven for non-linear systems, finding appropriate Lyapunov functions for non-linear constituent systems is far from straightforward. Furthermore, in the linear case, the wrong choice of functions V_i may well lead to unnecessarily conservative restrictions on allowable switching signals.

As mentioned previously, a number of variations on the basic theme of Theorem 2.4.3 exist. For instance, it is possible to obtain a similar result where we compare the initial value of V_i each time Σ_{A_i} is activated with the terminal value of the last interval when it was active [14]. A version of Theorem 2.4.3 which can be used to demonstrate asymptotic stability for switched linear systems has been proven by Peleties and deCarlo [103, 28] and Pettersson and Lennartson have derived similar results in [104].

2.5 Concluding remarks

In this chapter, we have introduced the class of switched linear systems as well as the types of stability that are of interest to us. We have also presented a numerical example to illustrate some of the well-known issues that can arise in studying the stability of this class of systems. Throughout the thesis, we shall mainly be interested in obtaining verifiable conditions for a switched linear system to be exponentially stable under arbitrary switching. However, in this chapter we have also discussed a number of other problems related to the stability of switched linear systems, and presented a brief review of the general literature on the stability theory of switched linear systems. In the next chapter, we move on to discuss one of the major topics of the thesis; namely the common quadratic Lyapunov function (CQLF) existence problem.

Chapter 3

The common quadratic Lyapunov function (CQLF) existence problem

In this chapter, we introduce the problem of CQLF existence for families of LTI systems and discuss the relationship between CQLF existence and the stability of switched linear systems. We also summarize the results on CQLF existence that have appeared in the systems theory and linear algebra literature. Background on the CQLF existence problem for discrete-time systems is also presented.

3.1 Introductory remarks

It is now known that CQLF existence can be a conservative criterion for the exponential stability of certain switched linear systems [27]. For this reason, much recent work has focused on deriving conditions for stability based on more general types of

Lyapunov functions, such as those outlined in the previous chapter. In this light, it may seem natural to question whether there is a point to continuing to work on the CQLF existence problem and its relation to the stability of switched linear systems. However, there are a number of very valid reasons for still being concerned with this problem. In fact, the very observation that CQLF existence can be a conservative criterion for the exponential stability of switched linear systems raises a number of important questions.

First of all, exactly how conservative is CQLF existence for the exponential stability of general switched linear systems? This important question has been raised in relation to the classical Circle Criterion by Megretski in [8]. No answer can be given to this without a more thorough theoretical understanding of the problem of CQLF existence than we currently have.

Furthermore, the conservatism of CQLF existence has been established by providing specific examples of switched linear systems that are exponentially stable under arbitrary switching, while their constituent systems do not have a CQLF. It is surely natural to ask if there is anything special about these examples, or more importantly, whether it is possible to identify classes of systems, of practical relevance, for which CQLF existence is not conservative? This is a question of great interest and importance, as quadratic Lyapunov functions are well-understood, and modern optimization techniques can be used to efficiently test for the existence of CQLFs for families of LTI systems. Thus, if we know that we are dealing with a system for which CQLF existence is not a conservative stability criterion, then the question of whether the system is stable or not can be settled simply and efficiently, without having to resort to the more sophisticated types of numerical search techniques required by piecewise Lyapunov functions. As before, in order to make progress on this question, we must first obtain more knowledge and a deeper understanding of the theoretical nature of CQLF existence for families of stable LTI systems.

Finally, a thorough understanding of the CQLF existence problem may well lead to the development of insights and techniques that can then be used to further our understanding of the more complicated types of Lyapunov function described in the previous chapter.

Our main purpose in this chapter is to present an overview of what is currently known about the problem of CQLF existence for families of exponentially stable LTI systems. We shall describe a number of different approaches that authors have taken to this problem and discuss the major results on CQLF existence that have appeared in the literature. While most of the chapter is taken up with the CQLF existence problem for continuous-time systems, in the final section we shall also briefly review the work that has been done on the related problem of CQLF existence for discrete-time systems.

3.2 Background on the CQLF existence problem

Recall from (2.5) that the conditions for $V(x) = x^T P x$ to be a CQLF for the LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ are that $P = P^T > 0$ and $A_i^T P + P A_i < 0$ for $1 \leq i \leq k$. For the sake of clarity, we now state explicitly what we understand by the CQLF existence problem for continuous-time LTI systems.

The CQLF existence problem:

Given the Hurwitz matrices $A_1, \ldots, A_k$ in $\mathbb{R}^{n \times n}$ (with the associated exponentially stable LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$) , determine if there exists a single positive definite $P = P^T > 0$ in $\mathbb{R}^{n \times n}$ such that

$$A_i^T P + P A_i < 0 \quad \text{for } 1 \leq i \leq k.$$

If such a P exists, then $V(x) = x^T P x$ is said to be a CQLF for the LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$.

Before discussing the CQLF existence problem, we first need to record a number of facts about the Lyapunov inequality for a single matrix. Perhaps the most fundamental result in this area is the following well-known theorem [118, 43, 62] relating the location of the spectrum of a matrix A to the existence of a positive definite solution to the corresponding Lyapunov inequality.

Theorem 3.2.1 *Let $A \in \mathbb{R}^{n\times n}$ be given. There exists a positive definite $P = P^T > 0$ in $\mathbb{R}^{n\times n}$ such that*

$$A^T P + PA = Q \tag{3.1}$$

with $Q < 0$ if and only if A is Hurwitz. Furthermore, if A is Hurwitz, then there is a unique Hermitian solution P to (3.1) for every choice of Hermitian Q, and P is positive definite if and only if Q is negative definite.

Comments:

(i) The Lyapunov equation (3.1) is a special case of the more general matrix equation, known as the *Sylvester equation*, given by

$$AX + XB = C, \tag{3.2}$$

where A, B, C in $\mathbb{R}^{n\times n}$ are given and X is an unknown. It is known that (3.2) has a unique solution X for every C in $\mathbb{R}^{n\times n}$ if and only if $\sigma(A) \cap \sigma(-B)$[1] is empty [62, 43]. It follows from this that there is a unique Hermitian solution P of (3.1) for every Hermitian Q if and only if $\sigma(A) \cap \sigma(-A)$ is empty. Note that this is certainly the case when A is Hurwitz.

[1] $\sigma(A)$ denotes the spectrum of the matrix A. So if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, $\sigma_A = \{\lambda_1, \ldots, \lambda_n\}$.

(ii) Given a Hurwitz matrix A, and a Hermitian Q, the unique solution P of (3.1) is given by the expression [118]

$$P = -\int_0^\infty e^{A^T t} Q e^{At} dt \tag{3.3}$$

<u>**The set $\mathcal{P}_A$ of solutions of the Lyapunov inequality:**</u>

It follows from Theorem 3.2.1 that if A in $\mathbb{R}^{n\times n}$ is Hurwitz, then there exists a positive definite P such that $A^T P + PA < 0$. In fact, the set of all positive definite solutions of the Lyapunov inequality corresponding to A forms a non-empty open convex cone in the space of symmetric matrices $Sym(n,\mathbb{R})$. Formally, if we define $\mathcal{P}_A$ as

$$\mathcal{P}_A = \{P = P^T > 0 : A^T P + PA < 0\}, \tag{3.4}$$

then we have the following simple facts about the set $\mathcal{P}_A$.

(i) $\mathcal{P}_A$ is a convex cone. This means that given matrices $P_1, P_2 \in \mathcal{P}_A$ and real numbers $\lambda > 0, \mu > 0$, the matrix $\lambda P_1 + \mu P_2$ is in $\mathcal{P}_A$ also.

(ii) If we denote the closure of $\mathcal{P}_A$ in the space $Sym(n,\mathbb{R})$ by $\overline{\mathcal{P}}_A$, then

$$\overline{\mathcal{P}}_A = \{P = P^T \geq 0 : A^T P + PA \leq 0\}.$$

The following result, derived by Loewy in [71], completely characterises pairs of Hurwitz matrices A, B in $\mathbb{R}^{n\times n}$ for which $\mathcal{P}_A = \mathcal{P}_B$.

Theorem 3.2.2 *Let A, B be Hurwitz matrices in $\mathbb{R}^{n\times n}$. Then $\mathcal{P}_B = \mathcal{P}_A$ if and only if $B = \mu A$ for some real $\mu > 0$ or $B = \lambda A^{-1}$ for some real $\lambda > 0$.*

An immediate consequence of Theorem 3.2.2 is that, for a Hurwitz matrix A, the two sets $\mathcal{P}_A$ and $\mathcal{P}_{A^{-1}}$ are identical. This is a highly significant fact as it effectively means that a result on CQLF existence for a family of LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ can also be applied to any family of systems obtained by replacing some of the matrices A_i with their inverses A_i^{-1}. The convex cones $\mathcal{P}_A$ shall play a major role in much of the work presented later in the thesis.

The Lyapunov operator $\mathcal{L}_A$:

A slightly more abstract way of thinking about the Lyapunov equation (3.1) is to consider the linear operator $\mathcal{L}_A$ defined on the space $Sym(n, \mathbb{R})$ by

$$\mathcal{L}_A(H) = A^T H + HA \quad \text{for all } H \in Sym(n, \mathbb{R}). \tag{3.5}$$

It is important to appreciate that $\mathcal{L}_A$ maps the space of symmetric matrices into itself, and as such is a linear operator defined on a vector space of dimension $\frac{n(n+1)}{2}$. We assume that $Sym(n, \mathbb{R})$ (and $\mathbb{R}^{n\times n}$) is equipped with the inner product

$$\langle A, B \rangle = \text{trace}(A^T B) \tag{3.6}$$

Note the following important points about the operators $\mathcal{L}_A$.

(i) It is possible to obtain a matrix representation of $\mathcal{L}_A$ as follows. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. Then the symmetric matrices

$$\begin{aligned} E_{ii} &= e_i e_i^T, \quad 1 \le i \le n \\ E_{ij} &= \frac{1}{\sqrt{2}}(e_i e_j^T + e_j e_i^T), \quad 1 \le i < j \le n \end{aligned}$$

form an orthonormal basis for $Sym(n, \mathbb{R})$, with respect to the inner product given by (3.6). It is then possible to represent the operators $\mathcal{L}_A$ with respect to this basis, obtaining an $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ matrix.

(ii) For the matrix representation in (i), we have that $\mathcal{L}_A^T = \mathcal{L}_{A^T}$.

(iii) If the eigenvalues of $A \in \mathbb{R}^{n\times n}$ are $\lambda_1, \ldots, \lambda_n$, then the eigenvalues of $\mathcal{L}_A$ are given by $\lambda_i + \lambda_j$ for $1 \le i \le j \le n$ [43]. In particular, $\mathcal{L}_A$ is invertible (in fact all of its eigenvalues lie in the open left half plane) if A is Hurwitz.

(iv) For a Hurwitz matrix $A \in \mathbb{R}^{n \times n}$, the set $\mathcal{P}_A$ is the pre-image of the cone of negative definite matrices under the mapping $\mathcal{L}_A$. So, writing $\mathbf{P}_n$ for the cone of positive definite matrices in $\mathbb{R}^{n \times n}$, we have that

$$\mathcal{P}_A = \mathcal{L}_A^{-1}(-\mathbf{P}_n).$$

Similarity and the Lyapunov equation:

The following simple lemma records a fundamental fact concerning the effect of a similarity transformation on the solutions of the Lyapunov equation (3.1).

Lemma 3.2.1 *Let $A \in \mathbb{R}^{n \times n}$ be Hurwitz and $T \in \mathbb{R}^{n \times n}$ be non-singular, and define $\tilde{A} = T^{-1}AT$. Then,*

$$\begin{aligned} A^T P + PA &< 0 \\ \Leftrightarrow \tilde{A}^T(T^T PT) + (T^T PT)\tilde{A} &< 0. \end{aligned} \tag{3.7}$$

Hence the matrix P is in $\mathcal{P}_A$ if and only if $T^T PT$ is in $\mathcal{P}_{\tilde{A}}$. An immediate consequence of Lemma 3.2.1 is that $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ have a CQLF if and only if $\Sigma_{\tilde{A}_1}, \ldots, \Sigma_{\tilde{A}_k}$ have a CQLF, where $\tilde{A}_i = T^{-1}A_iT$, $1 \leq i \leq k$ for a given non-singular T.

The matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$:

The concept of a matrix pencil, which we introduce now, will play a key role in many of the results to be presented later on. At this point, it should be mentioned that the term matrix pencil has previously been used in the mathematics literature in a slightly different way to that understood here [36].

Let the matrices A_1, A_2 in $\mathbb{R}^{n \times n}$ be given. Then, in keeping with the notation of [133, 129], the matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is defined to be the parameterized family of matrices given by

$$\sigma_{\gamma[0,\infty)}[A_1, A_2] = \{A_1 + \gamma A_2 : \gamma > 0\}. \tag{3.8}$$

We say that the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is non-singular (Hurwitz) if $A_1 + \gamma A_2$ is non-singular (Hurwitz) for all $\gamma > 0$. Note that if A_1 is itself non-singular, then the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is non-singular if and only if the matrix product $A_1^{-1}A_2$ has no negative real eigenvalues. Furthermore, if both A_1 and A_2 are non-singular, then the product $A_1A_2^{-1}$ has no negative real eigenvalues if and only if the product $A_1^{-1}A_2$ has no negative real eigenvalues. In particular, this is true if A_1, A_2 are Hurwitz matrices. For this reason, when we are discussing singularity for matrix pencils associated with Hurwitz matrices A_1, A_2, we shall often use the products $A_1A_2^{-1}$ and $A_1^{-1}A_2$ interchangeably. Note that it also follows that $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is non-singular if and only if $\sigma_{\gamma[0,\infty)}[A_1^{-1}, A_2]$ is non-singular if A_1 and A_2 are both non-singular.

Note that for two Hurwitz matrices A_1, A_2 in $\mathbb{R}^{n\times n}$, the matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is non-singular (Hurwitz) if and only if $(1-\alpha)A_1 + \alpha A_2$ is non-singular (Hurwitz) for $0 \leq \alpha \leq 1$.

CQLFs and stability radii:

Finally, for this section, we note the connection between the CQLF existence problem and the so-called complex stability radius [41]. For any matrix $A \in \mathbb{C}^{n\times n}$, let $\|A\|$ denote the matrix norm induced by the usual Euclidean norm on $\mathbb{C}^n$ [42], and for $r > 0$ let $B_r(A)$ denote the open ball, in $\mathbb{C}^{n\times n}$, centred at A of radius r.

$$B_r(A) = \{X \in \mathbb{C}^{n\times n} : \|X - A\| < r\}$$

Then, given a Hurwitz matrix $A \in \mathbb{C}^{n\times n}$, the unstructured complex stability radius $r_{\mathbb{C}}(A)$ is defined by

$$r_{\mathbb{C}}(A) = \inf\{\|\Delta\| : \Delta \in \mathbb{C}^{n\times n}, \sigma(A+\Delta) \cap \overline{RHP} \text{ is not empty }\}. \qquad (3.9)$$

Here Δ is thought of as a perturbation on the nominal matrix A, and $\overline{RHP}$ denotes the closed right half plane. Then, $r_{\mathbb{C}}(A)$ gives us the open ball of largest radius about A in $\mathbb{C}^{n\times n}$ that consists entirely of Hurwitz matrices. The connection with the CQLF existence problem is given by the fact that $r_{\mathbb{C}}(A)$ is also the maximal value of $r > 0$ such that, for every X in $B_r(A)$, there is some $P = P^* > 0$ in $\mathbb{C}^{n\times n}$ satisfying

$$\begin{aligned} A^*P + PA &< 0 \\ X^*P + PX &< 0. \end{aligned}$$

Having provided some background on the CQLF existence problem, we now move on to discuss the various approaches that have been taken to this problem, beginning in the next section with the numerical approach based on linear matrix inequalities (LMIs).

3.3 Numerical approaches to the CQLF problem - LMIs

The question of whether or not a CQLF exists for a family of LTI systems can be cast as a feasibility problem for a system of linear matrix inequalities (LMIs) [12] [2]. Specifically, there exists a CQLF, $V(x) = x^T Px$, for the family of exponentially stable LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ if and only if the system of LMIs given by

$$\begin{aligned} P = P^T &> 0 \\ A_i^T P + PA_i &< 0 \quad \text{for } 1 \leq i \leq k \end{aligned} \tag{3.10}$$

is feasible. The advantage of looking at the problem in this way is that modern techniques from convex optimization can be applied to test the feasibility of the

[2] If a solution exists to a system of LMIs, the system is said to be feasible; otherwise it is infeasible.

system (3.10) in an efficient manner and to determine if a CQLF exists for the systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$.

Conversely, it is also possible to verify that no CQLF exists for the family of LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ using LMI methods. In fact, it is known [12, 47] that if there exist symmetric matrices $R_1, \ldots, R_k \in \mathbb{R}^{n \times n}$ satisfying

$$R_i > 0, \qquad \sum_{i=1}^{k}(A_i^T R_i + R_i A_i) > 0, \tag{3.11}$$

then $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ do not have a CQLF. Once again, (3.11) defines a system of LMIs in the variables $R_1, \ldots, R_k$. Thus, software packages such as the MATLAB LMI toolbox [34] can be used to demonstrate either the existence or non-existence of a CQLF for a family of LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$. Of course, in recent years the use of LMIs in control has grown considerably [31] and these techniques are now used for far more than testing for the existence of a CQLF.

While LMIs provide a highly effective numerical way of checking for the existence of a CQLF, it is important to point out a number of drawbacks that are associated with this approach.

(i) Examples have been found where known analytic results can be used to show that a CQLF exists for a family of systems (or does not exist), but the commonly used LMI toolbox for MATLAB fails to give a definitive answer to the question. Examples of this kind can be found in [80, 64].

(ii) LMIs provide a "black box" approach to the CQLF existence problem, giving little insight into the relationship between CQLF existence and the stability of switched linear systems. In particular, it is important to stress that the issues raised in the opening section of this chapter cannot be settled within the LMI framework.

In the light of these observations, the importance of developing a more complete theoretical understanding of the CQLF existence problem, as well as of deriving dynamically meaningful analytic conditions for CQLF existence is clear. For the remainder of this chapter, we shall be describing the work done by the numerous authors who have considered the CQLF problem from a more theoretical standpoint.

3.4 Structural results

In this section, we shall describe a number of classes of systems for which particularly simple conditions for CQLF existence are known. In fact, for each of the classes described here, any family of LTI systems, $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$, belonging to the class will have a CQLF if and only if each of the system matrices $A_1, \ldots, A_k$ is Hurwitz. Essentially, for any switched linear system constructed by switching between a family of LTI systems belonging to one of these classes, the exponential stability of each of its constituent LTI systems guarantees the uniform exponential stability of the overall system for arbitrary switching signals. The results to be discussed here take the following straightforward form; if the system matrices $A_1, \ldots, A_k$ have a certain structure, then there is a CQLF for the associated family of LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ if and only if each of the $A_i, 1 \leq i \leq k$ is Hurwitz.

3.4.1 Systems for which the 'Euclidean norm' defines a CQLF

To begin with, we describe a number of system classes that have CQLFs of a particularly simple form. Specifically, note that the usual Euclidean norm $V(x) = x^T x$ will be a CQLF for the exponentially stable LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ if and only if [136]

$$A_i^T + A_i < 0 \quad \text{for all } i \in \{1, \ldots, k\}. \tag{3.12}$$

A number of matrix classes have been identified for which this is the case.

(i) [22] If for each $i \in \{1, \ldots, k\}$, A_i is symmetric and Hurwitz, then A_i is in fact negative definite for $1 \leq i \leq k$ and hence the condition (3.12) holds.

(ii) It is shown in [125] that for a family of matrices $A_1, \ldots, A_k$ in $\mathbb{R}^{2\times 2}$ of the form

$$A_i = \begin{bmatrix} -\mu_i & \omega_i \\ -\omega_i & -\mu_i \end{bmatrix}, \quad \mu_i > 0, \;\; 1 \leq i \leq k, \tag{3.13}$$

condition (3.12) is satisfied. Hence $V(x) = x^T x$ is a CQLF for the associated set of second order LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$.

(iii) Recall that a matrix A in $\mathbb{R}^{n\times n}$ is said to be normal if $AA^T = A^T A$. It follows from a standard result of linear algebra [42] that if $A \in \mathbb{R}^{n\times n}$ is normal and Hurwitz, then there is a real orthogonal matrix O in $\mathbb{R}^{n\times n}$ such that OAO^T is block diagonal of the form

$$\begin{bmatrix} B_1 & 0 & \ldots & 0 \\ 0 & B_2 & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \ldots & B_p \end{bmatrix} \tag{3.14}$$

where each block B_i is either a negative scalar $b_i \in \mathbb{R}$, or a real 2×2 block of the form (3.13). Therefore it follows that $O(A + A^T)O^T < 0$, and by congruence that $A + A^T < 0$. It now follows immediately that any family of normal Hurwitz matrices $A_1, \ldots, A_k$ in $\mathbb{R}^{n\times n}$ satisfies the condition (3.12).

Note that a perturbation result has been derived in [87] that allows the above observations to be extended to larger classes of systems.

3.4.2 Triangular systems

In the previous subsection, we described a number of system classes for which, when a CQLF exists, it can take a particularly simple form. Specifically, for the system classes discussed there, if the matrices $A_1, \ldots, A_k$ were all Hurwitz, then the LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ had a CQLF, $V(x) = x^T P x$, with P given by the $n \times n$ identity matrix. The next system class to which we shall turn our attention is the class of LTI systems whose system matrices are Hurwitz and in upper triangular form [42]. We shall see that families of LTI systems of this form always have a CQLF $V(x) = x^T P x$, and that in this case P can be chosen to be a diagonal positive definite matrix. It should be noted that analogous results to those presented here for upper triangular matrices also hold for lower triangular matrices.

In [125, 132], Shorten and Narendra analysed the stability of switched linear systems constructed by switching between constituent systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ where each $A_i \in \mathbb{R}^{n \times n}$ is Hurwitz and upper triangular for $1 \leq i \leq k$. (Note that this requires that each of the A_i has real eigenvalues.) Firstly, they proved the uniform exponential stability of such systems using a direct argument, showing how these systems can be viewed as cascades of simple first order systems. This result was then extended to cater for upper triangular matrices (in $\mathbb{C}^{n \times n}$) with complex eigenvalues as well as matrices that can be simultaneously transformed by similarity into upper triangular form.

In the same papers, the above stability result was also established by showing that a family of LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$, where the A_i are Hurwitz upper triangular matrices in $\mathbb{R}^{n \times n}$, always has a CQLF, $V(x) = x^T P x$. Moreover, P may be taken to be in the form

$$P = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & p_{22} & \dots & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & \dots & \dots & p_{nn} \end{bmatrix}. \tag{3.15}$$

This result was extended to show that if $A_1, \dots, A_k$ can be simultaneously transformed into upper triangular form by similarity, then the systems $\Sigma_{A_1}, \dots, \Sigma_{A_k}$ have a CQLF. Formally, the following result was derived. Note that the same result has also appeared in [88].

Theorem 3.4.1 *Let $A_1, \dots, A_k$ be Hurwitz matrices in $\mathbb{R}^{n\times n}$. Suppose that there is some non-singular $T \in \mathbb{C}^{n\times n}$ such that $T^{-1}A_iT$ is upper (lower)-triangular for $1 \leq i \leq k$. Then the associated LTI systems $\Sigma_{A_1}, \dots, \Sigma_{A_k}$ have a CQLF.*

From the above discussion, we can see that systems $\Sigma_{A_1}, \dots, \Sigma_{A_k}$, where $A_1, \dots, A_k$ can be simultaneously put into triangular form by similarity, have a number of very appealing properties. Firstly, all that is required for a CQLF to exist for such systems is that each of the system matrices is Hurwitz. Also, as pointed out in [125], triangular systems allow for particularly simple design laws as they can treated as cascades of simple first or second order systems. However, we should point out that the conditions of Theorem 3.4.1 are restrictive and that the property of simultaneous triangularizability is not robust. This point has been partially addressed by Mori, Mori and Kuroe in [88], where Theorem 3.4.1 has been extended slightly using a perturbation argument. Another limitation of Theorem 3.4.1 is that, in order to apply it, we must first determine whether the matrix T exists or not. In general, this is a far from straightforward task [112].

In the context of triangular systems, the results presented in [25] on triangular linear differential inclusions should also be noted. Finally for this section, we note that a similar result to Theorem 3.4.1 has appeared in [22], and that it was established in [5] that, given a triangular Hurwitz matrix $A \in \mathbb{R}^{n \times n}$, there exists a diagonal $P > 0$ such that $A^T P + PA < 0$.

3.5 Necessary and sufficient conditions for CQLF existence

The simplest possible necessary and sufficient condition for a family of LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ to have a CQLF is that the individual matrices $A_1, \ldots, A_k$ are all Hurwitz. For the system classes considered in the previous section, this simple condition was indeed equivalent to the existence of a CQLF. In general however, this is not the case, and in fact, verifiable necessary and sufficient conditions for CQLF existence are known for only a handful of system classes. In this section, we shall describe classes of LTI systems for which checkable necessary and sufficient conditions for CQLF existence are known. The first class to be considered is that of second order systems.

3.5.1 Second order systems

In [133], Shorten and Narendra considered the problem of determining necessary and sufficient conditions for a family of LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$, where $A_i \in \mathbb{R}^{2 \times 2}, 1 \leq i \leq k$, to have a CQLF. They approached the general problem in two stages. First of all, the question of when two second order exponentially stable LTI systems have a CQLF was addressed and the following result was obtained. Note that this result has also appeared, without a proof in [21].

Theorem 3.5.1 *Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{2 \times 2}$. Then the LTI systems*

$\Sigma_{A_1}, \Sigma_{A_2}$ have a CQLF if and only if the matrix products $A_1 A_2^{-1}$ and $A_1 A_2$ have no negative real eigenvalues. An equivalent condition is that the matrix pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ are Hurwitz.

Comments:

The conditions of Theorem 3.5.1 have a number of advantages. First of all, they are easy to check and are appealingly simple. Moreover, the form of these conditions links the existence of a CQLF with the exponential stability of switched linear systems, and provides insight into the conservatism of CQLF existence as a criterion for exponential stability. Specifically, given two Hurwitz matrices A_1, A_2 in $\mathbb{R}^{2\times 2}$, consider the pair of switched linear systems given by

$$\dot{x} = A(t)x \quad A(t) \in \{A_1, A_2\}; \tag{3.16}$$

$$\dot{x} = A(t)x \quad A(t) \in \{A_1, A_2^{-1}\}. \tag{3.17}$$

Then it follows from Theorem 3.5.1 that if there is no CQLF for the systems $\Sigma_{A_1}, \Sigma_{A_2}$, then either:

(i) there exists some $\gamma > 0$ such that $A_1 + \gamma A_2$ has an eigenvalue in the right half plane;

or

(ii) there is some $\gamma > 0$ such that $A_1 + \gamma A_2^{-1}$ has an eigenvalue in the right half plane.

In the first case, it follows from Theorem 2.3.1 that the switched linear system (3.16) is not uniformly exponentially stable under arbitrary switching. Similarly in the case (ii), the system (3.17) is not uniformly exponentially stable under arbitrary switching. Thus, Theorem 3.5.1

provides insight into the question of how conservative CQLF existence is as a criterion for the exponential stability of switched linear systems. In fact, it tells us that if two second order LTI systems Σ_{A_1}, Σ_{A_2} do not have a CQLF, then at least one of the associated switched linear systems (3.16), (3.17) is not uniformly exponentially stable for arbitrary switching signals. We shall be revisiting this issue again in later sections and chapters.

Having dealt with the question of CQLF existence for a pair of second order stable LTI systems, the authors in [133] turned their attention to the problem of determining when an arbitrary finite family of such systems possesses a CQLF. Using Helly's theorem for convex sets [116], they established that if any collection of three systems from the family $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ has a CQLF, then there is a CQLF for the overall family. They then described a method that can be used to test for the existence of a CQLF for three stable second order LTI systems, thereby, in principle, providing a complete answer to the CQLF existence problem in the second order case.

Recently, the problem of CQLF existence for second order systems has been addressed in a different manner in [19]. In this paper, it is shown that the existence of a CQLF for a family of second order LTI systems is equivalent to the positivity of a certain integral. While this integral condition can be checked numerically, it is less transparent than the conditions of Theorem 3.5.1, and it is difficult to see how it relates to the dynamics of the associated switching systems.

3.5.2 Systems in companion form - the Circle Criterion

LTI systems whose system matrices are in companion form have long played an important role within control theory [50, 93, 51], and in the current subsection we turn our attention to the stability of switched linear systems constructed by switching between a pair of such systems. Formally, let A_1, A_2 be Hurwitz matrices in companion form in $\mathbb{R}^{n\times n}$, and consider the system

$$\dot{x} = A(t)x \quad A(t) \in \{A_1, A_2\}. \tag{3.18}$$

We shall describe results below that give necessary and sufficient conditions for the LTI systems Σ_{A_1}, Σ_{A_2} to have a CQLF. It should be pointed out that some of the results discussed here were first discovered in a different context, by authors investigating the problem of absolute stability for the class of non-linear feedback systems known as Lur'e systems [141, 93, 51]. One of the most fundamental results derived in this area is the so-called Kalman-Yakubovic-Popov, or KYP, Lemma which is given below as Theorem 3.5.2. Note that a number of different versions of the KYP Lemma have been derived by various authors [51, 141, 93].

Theorem 3.5.2 *Let a Hurwitz matrix $A \in \mathbb{R}^{n\times n}$, column vectors $b, c \in \mathbb{R}^n$ and $\gamma > 0$ in $\mathbb{R}$ be given. Suppose that the pair (A, b) is completely controllable [118], and that*

$$\frac{\gamma}{2} + \mathrm{Re}\{c^T(j\omega I - A)^{-1}b\} > 0 \quad \text{for all } \omega \in \mathbb{R}. \tag{3.19}$$

Then there is some real $\epsilon > 0$, a positive definite $P = P^T > 0$ in $\mathbb{R}^{n\times n}$ and a real column vector $q \in \mathbb{R}^n$ such that

$$\begin{aligned} A^T P + PA &= -qq^T - \epsilon P \\ Pb - c &= \sqrt{\gamma} q. \end{aligned} \tag{3.20}$$

Theorem 3.5.2 plays a key role in the proof of the celebrated stability criterion known as the Circle Criterion, derived in [97] by Narendra and Goldwyn. While this result originally arose out of an interest in the absolute stability of Lur'e systems, from our point of view the Circle Criterion gives the following condition for CQLF existence for pairs of LTI systems whose system matrices are in companion form.

Theorem 3.5.3 *Let $A, A - bc^T \in \mathbb{R}^{n\times n}$ be Hurwitz matrices in companion form where b, c are column vectors in $\mathbb{R}^n$. Then the LTI systems $\Sigma_A, \Sigma_{A-bc^T}$ have a CQLF if and only if*

$$1 + \mathrm{Re}\{c^T(j\omega I - A)^{-1}b\} > 0 \quad \textit{for all } \omega \in \mathbb{R}. \tag{3.21}$$

The sufficiency of condition (3.21) for CQLF existence was proven in the original paper of Narendra and Goldwyn, while the necessity was proven by Willems in [146] based on the results of [145].

Theorem 3.5.3 describes a frequency domain stability criterion for time-varying systems, related to the classical Nyquist criterion for time-invariant systems, and it allows for a similar graphical interpretation [97, 141]. In a recent paper [128], the problem of CQLF existence for pairs of stable LTI systems in companion form was revisited by Shorten and Narendra, and the following simple time-domain version of the Circle Criterion was derived.

Theorem 3.5.4 *Let $A, A - bc^T \in \mathbb{R}^{n\times n}$ be Hurwitz matrices in companion form where b, c are column vectors in $\mathbb{R}^n$. Then the LTI systems $\Sigma_A, \Sigma_{A-bc^T}$ have a CQLF if and only if the matrix product $A(A - bc^T)$ has no negative real eigenvalues.*

Comments:

As pointed out in [128], Theorem 3.5.4 can be seen as a time-domain version of the Circle Criterion. While the original result requires the positivity of (3.21) to be verified for *every* real value of ω, the time-domain version can be checked by a relatively straightforward eigenvalue calculation. Furthermore, as with the result for second order systems given in Theorem 3.5.1, the form of the above condition can be used to gain insight into the relationship between CQLF existence and stability for switched linear systems using the result of Theorem 2.3.1. It should also be noted that it is considerably easier to prove the necessity of the time-domain condition for CQLF existence than is the case for the original frequency-domain formulation.

3.6 Sufficient Conditions for CQLF existence

In the previous two sections we have seen that by restricting our attention to specific classes of systems, such as second order systems, it is sometimes possible to derive verifiable necessary and sufficient conditions for CQLF existence for families of LTI systems. While no verifiable necessary and sufficient conditions have been found for a general family of LTI systems to have a CQLF, a number of authors have published sufficient conditions for CQLF existence. The aim of this section is to summarize the results of this nature that have appeared in the literature in recent years.

3.6.1 Triangular Systems in Disguise

A number of the sufficient conditions reported over the past number of years are intimately related to the results for triangular systems presented in Section 3.4.2 above. In fact, the system classes covered by some of these results are subclasses of those systems that fall within the framework of Theorem 3.4.1.

Commuting System Matrices:

The following result on CQLF existence for systems with commuting system matrices was established by Narendra and Balakrishnan in [94].

Theorem 3.6.1 *Let $A_1, \ldots, A_k$ be a set of Hurwitz matrices in $\mathbb{R}^{n\times n}$ such that*

$$A_i A_j = A_j A_i \quad \text{for } i, j \in \{1, 2, \ldots, k\}.$$

Then the associated LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ *have a CQLF.*

A standard result of linear algebra [42] states that if the matrices $A_1, \ldots, A_k$ commute pairwise, then there is a single unitary matrix $U \in \mathbb{C}^{n \times n}$ such that $U^* A_i U$ is in upper triangular form for $1 \leq i \leq k$. Thus, the class of LTI systems covered by Theorem 3.6.1 is a subclass of those systems whose system matrices can be simultaneously put into triangular form, and this result can be thought of as a special case of Theorem 3.4.1. However, it should be noted that a direct, and constructive, proof of Theorem 3.6.1 is given in [94]. Related results for non-linear switched systems with commuting vector fields have recently appeared in [120, 74].

Lie-Algebraic Conditions:

Theorem 3.6.1 suggests that the properties of the matrix commutators

$$[A_i, A_j] = A_i A_j - A_j A_i \quad \text{for } i, j \in \{1, \ldots, k\}$$

may play an important role in determining whether or not the LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ have a CQLF. This issue was addressed for discrete-time systems by Gurvits in [37], and we shall now discuss some recent work on Lie-algebraic conditions for CQLF existence [1, 67, 68] which can be seen as a continuation of this line of research.

Before stating the next theorem, we need to introduce two basic concepts from Lie theory. First of all, a real matrix Lie-algebra is a subspace of $\mathbb{R}^{n \times n}$ that is closed with respect to the commutation operator

$$[A, B] = AB - BA.$$

Secondly, the Lie-algebra generated by a set of matrices $\{A_1, A_2, \ldots, A_k\}$ in $\mathbb{R}^{n \times n}$ is defined to be the smallest Lie-algebra in $\mathbb{R}^{n \times n}$ containing each of $A_1, \ldots, A_k$ and is denoted by $\{A_1, \ldots, A_k\}_{LA}$. For background on the theory of Lie-algebras, consult [18, 45].

The following result gives a sufficient condition for the existence of a CQLF for the stable LTI systems $\Sigma_{A_i}, 1 \leq i \leq k$ and was established in [68].

Theorem 3.6.2 *If $A_1, \ldots, A_k$ are Hurwitz matrices in $\mathbb{R}^{n\times n}$ and $\{A_1, \ldots, A_k\}_{LA}$ is a solvable Lie-algebra, then the LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ have a CQLF.*

See [18, 45, 1] for the definition of a solvable Lie-algebra. For our purposes it is enough to note that if the Lie-algebra $\{A_1, \ldots, A_k\}_{LA}$ is solvable, then the system matrices $A_1, \ldots, A_k$ can be simultaneously put into upper triangular form. Thus, as with Theorem 3.6.1, the class of systems covered by this Lie-algebraic result is a subclass of those systems which can be simultaneously put into upper triangular form. However, as pointed out in [68], the Lie algebraic condition of Theorem 3.6.2 is co-ordinate-independent and does not require that a transformation that simultaneously puts $A_1, \ldots, A_k$ into upper triangular form be found. It is also explained in [68] and [66] how to test for the solvability of a Lie-algebra.

The following extension of theorem 3.6.2 was established in [1].

Theorem 3.6.3 *Let $A_1, \ldots, A_k$ be Hurwitz matrices in $\mathbb{R}^{n\times n}$ and let $\mathbf{g} = \{A_1, \ldots, A_k\}_{LA}$. Then if $\mathbf{g} = \mathbf{r} \oplus \mathbf{l}$ with $\mathbf{r}$ a solvable ideal and $\mathbf{l}$ a compact subalgebra of $\mathbf{g}$, the LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ have a CQLF.*

Theorem 3.6.3 extends Theorem 3.6.2 by allowing the Lie-algebra $\{A_1, \ldots, A_k\}_{LA}$ to be the direct sum of a solvable ideal and a *compact* subalgebra. To say that $\mathbf{l}$ is a compact Lie-algebra amounts to requiring that all of the matrices in $\mathbf{l}$ have strictly imaginary eigenvalues.

Comment on the CQLF existence problem and commutators:

At this stage it is worth noting the following points about the link between the question of CQLF existence for a pair of exponentially stable LTI systems, Σ_{A_1},

Σ_{A_2}, and the commutator $[A_1, A_2]$. Firstly, Theorem 3.6.1 shows that if $[A_1, A_2]$ has rank zero, then there is a CQLF for the systems Σ_{A_1}, Σ_{A_2}. In addition, Laffey has shown in [60] that if $[A_1, A_2]$ has rank 1, then A_1 and A_2 can be simultaneously triangularized. Thus, in this case also, the systems Σ_{A_1}, Σ_{A_2} have a CQLF. Now, it is not difficult to find examples of exponentially stable LTI systems Σ_{A_1}, Σ_{A_2} with no CQLF, where $\text{rank}([A_1, A_2]) = 2$. In view of these facts, the problem of deriving conditions for Σ_{A_1}, Σ_{A_2} to have a CQLF when the rank of $[A_1, A_2]$ is 2 arises naturally. At the time of writing, this general problem appears to be open. However, the results of Theorem 3.5.1 and Theorem 3.5.4 give some insights into this question, as they provide necessary and sufficient conditions for CQLF existence for system classes where the rank of the commutator $[A_1, A_2]$ can be equal to 2.

3.6.2 Further sufficient conditions

We now discuss a number of miscellaneous results that give sufficient conditions for a set of exponentially stable LTI systems to have a CQLF. The first conditions that we shall present are expressed in terms of so-called *M-matrices* [43].

M-Matrix-based conditions:

A matrix A in $\mathbb{R}^{n\times n}$ is said to be an *M-matrix* [43, 7] if

(i) The off-diagonal elements of A are non-positive, i.e. $a_{ij} \leq 0$ for $1 \leq i, j \leq n$, $i \neq j$.

(ii) All of the eigenvalues of A are in the open right half plane. (therefore $-A$ is Hurwitz.)

Mori, Mori and Kuroe established in [89] that a family of systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ will have a CQLF if the system matrices $A_1, \ldots, A_k$ satisfy a certain condition stated in terms of M-matrices. In the paper [91], the same authors extended this result as well as deriving a corresponding condition for families of discrete-time systems. In the statement of the next theorem, taken from [91], the following notation is used. Given a family of matrices $A_1, \ldots, A_k$ in $\mathbb{R}^{n \times n}$, $a_{pq}^{(i)}$ denotes the element in the (p, q) position of the matrix A_i, $1 \leq i \leq k$.

Theorem 3.6.4 *Let $A_1, \ldots, A_k$ be Hurwitz matrices in $\mathbb{R}^{n \times n}$. Suppose that there exists a non-singular $T \in \mathbb{C}^{n \times n}$ such that writing $\tilde{A}_i = T^{-1} A_i T$ for $1 \leq i \leq k$, we have that the matrix B given by*

$$\begin{aligned} b_{pp} &= -\max_i \mathrm{Re}(\tilde{a}_{pp}^{(i)}) \quad 1 \leq p \leq n, \\ b_{pq} &= -\max_i |\tilde{a}_{pq}^{(i)}| \quad 1 \leq p, q \leq n, \;\; p \neq q \end{aligned}$$

is an M-matrix. Then there is a CQLF for the systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$.

It is pointed out in [91], that if the matrices $A_1, \ldots, A_k$ can be simultaneously put into triangular form, then the conditions of Theorem 3.6.4 are satisfied. Thus, the class of systems covered by Theorem 3.6.4 is larger than those covered by Theorem 3.4.1.

Lyapunov operator conditions:

We now look at conditions for CQLF existence that are expressed in terms of the Lyapunov operators $\mathcal{L}_A$, introduced in Section 3.2 above. Several results of this general type have been published by Ooba and Funahashi in the sequence of papers ([101], [99], [100]). In [101], two simple sufficient conditions for the existence of a CQLF for a pair of exponentially stable LTI systems Σ_{A_1}, Σ_{A_2} are presented. The key idea behind both of the results in this paper is that Σ_{A_1} and Σ_{A_2} have a CQLF if and only if there is some positive definite $P \in \mathbf{P}_n$ such that $\mathcal{L}_{A_1} \mathcal{L}_{A_2}^{-1}(P) \in \mathbf{P}_n$, where $\mathbf{P}_n$ is the cone of positive definite matrices in $\mathbb{R}^{n \times n}$. Before stating the next result, recall that for a symmetric linear operator L on the space of symmetric matrices $Sym(n, \mathbb{R})$, the notation $L > 0 (< 0)$ means that $\langle L(H), H \rangle > 0 (< 0)$ for all non-zero $H \in Sym(n, \mathbb{R})$, where $\langle H, K\rangle = \text{trace}(H^T K)$ is the inner product on $Sym(n, \mathbb{R})$.

Theorem 3.6.5 *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Hurwitz and suppose that*

$$\mathcal{L}^T_{A_2-A_1}\mathcal{L}_{A_2-A_1} - (\mathcal{L}^T_{A_1}\mathcal{L}_{A_1} + \mathcal{L}^T_{A_2}\mathcal{L}_{A_2}) < 0. \tag{3.22}$$

Then Σ_{A_1}, Σ_{A_2} have a CQLF.

Note that the condition (3.22) is equivalent to

$$\mathcal{L}^T_{A_1}\mathcal{L}_{A_2} + \mathcal{L}^T_{A_2}\mathcal{L}_{A_1} > 0.$$

A second similar condition involving the commutator of $\mathcal{L}_{A_1}$ and $\mathcal{L}_{A_2}$ is also presented in [101]. While both of these conditions can be checked numerically, neither one is constructive. Also, it is important to keep in mind that the conditions are only sufficient for CQLF existence and can only be used as a test for two systems Σ_{A_1}, Σ_{A_2}.

The same authors have derived more general versions of these two results in [100] as well as the following corollary that relates the existence of a CQLF for $\Sigma_{A_1}, \Sigma_{A_2}$ to the norm of the commutator $A_1A_2 - A_2A_1$. In the statement of the next result, for a matrix $A \in \mathbb{R}^{n\times n}$, $\|A\|^2 = \langle A, A\rangle$, while for a linear operator L on $Sym(n, \mathbb{R})$, $\|L\|_s$ denotes the operator norm induced by $\|.\|$. Thus, $\|L\|_s = \sup_{\|A\|=1} \|L(A)\|$.

Corollary 3.6.1 *Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{n\times n}$. Suppose that*

$$\|A_1A_2 - A_2A_1\| < \frac{1}{2\|\mathcal{L}^{-1}_{A_1}\|_s\|\mathcal{L}^{-1}_{A_2}\|_s}. \tag{3.23}$$

Then Σ_{A_1}, Σ_{A_2} have a CQLF.

A different type of sufficient condition, again based on Lyapunov operators, for the family of LTI systems $\Sigma_{A_1}, \dots, \Sigma_{A_k}$ to have a CQLF was derived in [99]. Before we state the main result of that paper, we need to introduce the notion of a semi-positive matrix in $\mathbb{R}^{n\times n}$. Simply put, a semi-positive matrix is a matrix that maps some vector with positive entries to another vector with positive entries. More formally, if we write $\mathbb{R}^n_+$ for the cone of vectors in $\mathbb{R}^n$ all of whose components are positive, then $A \in \mathbb{R}^{n\times n}$ is semi-positive if there is some $x \in \mathbb{R}^n_+$ such that $Ax \in \mathbb{R}^n_+$.

Given the Hurwitz matrices $A_1, \ldots, A_k$ in $\mathbb{R}^{n\times n}$, for each i, j with $1 \le i, j \le k$, define the number μ_{ij} to be

$$\mu_{ij} = \lambda_{min}(\mathcal{L}_{A_i}\mathcal{L}^{-1}_{A_j}(I_n)), \tag{3.24}$$

where $\lambda_{min}(A)$ denotes the minimum eigenvalue of a symmetric matrix A, and I_n is the $n \times n$ identity matrix. We then have the following sufficient condition for the systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ to have a CQLF.

Theorem 3.6.6 *Let $A_1, \ldots, A_k$ be Hurwitz matrices in $\mathbb{R}^{n\times n}$, and define the matrix $M \in \mathbb{R}^{k\times k}$ by*

$$M = [\mu_{ij}]_{1\le i,j\le k} \tag{3.25}$$

where the μ_{ij} are given by (3.24). If M is semi-positive, then $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ have a CQLF.

Comments:

In order to use the condition in Theorem 3.6.6, it is necessary to be able to test a matrix for semi-positivity. While some algebraic conditions are known that guarantee semi-positivity [7], in general it is necessary to perform this test numerically. However, if we can find a vector $x \in \mathbb{R}^n_+$ such that $Mx \in \mathbb{R}^n_+$, then it is explained in [99] how to construct a CQLF for the systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$. We shall encounter the notion of semi-positivity again in a later chapter, and see that there is a strong connection between the properties of semi-positive matrices and those of the Lyapunov operators.

Before finishing this section, it is worth noting that a numerical test for CQLF existence based on Lyapunov operators has recently been published in [20].

3.7 The CQLF problem and convex sets of matrices

Certain convex sets of matrices arise quite naturally in the consideration of the CQLF existence problem for finite families of stable LTI systems, and it is possible to approach the problem through studying the properties of these sets. There are essentially two distinct ways of considering the question of CQLF existence in this light. Specifically, the following two classes of matrix cones have been studied.

(i) For a symmetric matrix $P \in \mathbb{R}^{n\times n}$, define

$$\mathcal{A}_P = \{A \in \mathbb{R}^{n\times n} : A^T P + PA < 0\}. \tag{3.26}$$

We can then recast the CQLF existence problem in the following way.

> *Given the Hurwitz matrices $A_1, \ldots, A_k$ in $\mathbb{R}^{n\times n}$, determine if there exists some positive definite $P \in \mathbb{R}^{n\times n}$ such that $\{A_1, \ldots, A_k\} \subseteq \mathcal{A}_P$.*

For a given symmetric P, the set $\mathcal{A}_P$ is a convex cone of matrices that is closed with respect to matrix inversion. Such sets of matrices, known as convex invertible cones (CICs), have been studied by Cohen and Lewkowicz in [22, 23, 65], and a number of their key properties have been identified. In the paper [3], Ando has also investigated the properties of these cones, and derived an abstract characterization of sets of the form $\mathcal{A}_P$. In fact, he describes necessary and sufficient conditions for a given set of matrices to be of the form $\mathcal{A}_P$ for some positive definite matrix P. However, to date the results obtained through this line of research have been largely abstract, albeit very interesting, in nature, and have not led to easily verifiable conditions for CQLF existence. The sets $\mathcal{A}_P$ are also discussed in the survey [40] of the role played by cones in questions of matrix stability.

(ii) An alternative, and more direct, approach to the CQLF problem, that is also based on studying convex sets of matrices, is to study the sets $\mathcal{P}_A$ introduced in Section 3.2. In terms of these sets, the CQLF existence problem can be stated as follows.

Given the Hurwitz matrices $A_1, \ldots, A_k$ in $\mathbb{R}^{n\times n}$, determine if the intersection

$$\mathcal{P}_{A_1} \cap \mathcal{P}_{A_2} \cdots \cap \mathcal{P}_{A_k} \tag{3.27}$$

is non-empty.

This is the essence of the approach taken by Shorten and Narendra to the CQLF problem for second order systems in [133], and most of the work that we shall be presenting in this thesis arises from considering the CQLF problem in this way. The question of determining when the above intersection (3.27) is non-empty has also been addressed in [22, 24] and in [19]. In this last paper, the author provides a characterization of the sets $\mathcal{P}_A$ that is essentially based on the observation that a matrix $P = P^T > 0$ in $\mathbb{R}^{n\times n}$ satisfies $A^TP + PA < 0$ if and only if there is some orthogonal matrix T in $\mathbb{R}^{n\times n}$ such that $\tilde{P} = T^TPT$ is diagonal and $(T^TAT)^T\tilde{P} + \tilde{P}(T^TAT) < 0$. Based on this characterization, it is shown that the existence of a CQLF for a family of LTI systems is equivalent to a related multi-variable integral being positive. The resulting condition appears to be difficult, if not impossible to check however, and no example of its use is given in [19]. A more tractable condition for second order systems is derived in the same paper. This has already been discussed in Section 3.5.1 above.

We note that related work on geometrical properties associated with the Lyapunov equation has been reported in the papers [30, 150].

3.8 CQLFs - The discrete-time case

Our discussion thus far has exclusively focussed on the case of continuous-time switched linear systems and, consequently on the CQLF existence problem for families of continuous-time LTI systems. Of course, similar issues to those discussed can also arise in discrete-time, and we shall be presenting results for discrete-time systems as well as continuous-time systems. In this, the final section of this chapter, we introduce the CQLF existence problem for families of discrete-time LTI systems, indicate the relevance of this problem in the context of discrete-time switched linear systems, and give a brief overview of the literature on the CQLF existence problem in discrete-time.

3.8.1 Background on the CQLF problem in discrete-time

Given a matrix $A \in \mathbb{R}^{n\times n}$, it is well-known that the associated discrete-time LTI system

$$\Sigma_A^d : x(j+1) = Ax(j), \quad x(j_0) = x_0 \tag{3.28}$$

is exponentially stable if and only if all of the eigenvalues of A lie inside the unit circle in $\mathbb{C}$ [118, 50] [3]. Such matrices are known as *Schur* matrices. As is the case with continuous-time systems, the situation becomes considerably more complicated when we consider switched systems [2, 37] of the form

$$\Sigma^d_{\mathcal{A}} : x(j+1) = A(j)x(j), \quad A(j) \in \{A_1, \ldots, A_k\}. \tag{3.29}$$

Here the switched system is considered to be the family of all time-varying systems obtained as we allow the matrix-valued function $A(.)$ to vary over all piecewise constant mappings from the integers into the set $\mathcal{A} = \{A_1, \ldots, A_k\}$ (where switching can take place at the times $j_0, j_0+1, j_0+2, \ldots$). The various types of stability as well as the concepts of switching signals and switching sequences for (3.29) can be defined analogously to the continuous-time case. Note that the stability questions for discrete-time switched linear systems can also be treated within the framework of linear difference inclusions of the form [37]

$$x(j+1) \in \{A_1 x(j), \ldots, A_k x(j)\}. \tag{3.30}$$

As with switched linear systems in continuous time, the system (3.29) may not be uniformly exponentially stable for all switching signals, even if all of the individual system matrices $A_1, \ldots, A_k$ are Schur. However, if a common Lyapunov function exists for its constituent systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$, then the system (3.29) will be uniformly exponentially stable for all switching signals. In particular, if a common quadratic Lyapunov function (CQLF), $V(x) = x^T P x$, exists for $\Sigma^d_{A_1}, \ldots, \Sigma^d_{A_k}$, then the exponential stability of the associated switched linear system (3.29) is guaranteed.

Formally, $V(x) = x^T P x$ is a CQLF for the stable discrete-time LTI systems $\Sigma^d_{A_1}, \ldots, \Sigma^d_{A_k}$ if and only if $P = P^T > 0$ and

$$A_i^T P A_i - P < 0 \quad \text{for } 1 \leq i \leq k. \tag{3.31}$$

[3] When discussing discrete-time systems, we shall use the superscript 'd' in order to avoid confusion with the continuous-time case.

The Stein inequality:

For a single matrix A in $\mathbb{R}^{n\times n}$, the inequality

$$A^T PA - P < 0 \tag{3.32}$$

is known as the Stein inequality [35], and has a number of properties analogous to those of the continuous-time Lyapunov equation (3.1). In particular, the following result holds [118, 50].

Theorem 3.8.1 *Let $A \in \mathbb{R}^{n\times n}$ be given. There exists a positive definite $P = P^T > 0$ in $\mathbb{R}^{n\times n}$ satisfying*

$$A^T PA - P = Q \tag{3.33}$$

with $Q = Q^T < 0$ if and only if A is Schur. Moreover, if A is Schur, then there is a unique symmetric solution P to (3.33) for every symmetric Q, and P will be positive definite if and only if Q is negative definite.

The bilinear or Cayley transform:

As indicated above, there is an intimate relationship between the Lyapunov equation (3.1) in continuous-time and the Stein equation (3.33) in discrete-time. This relationship is made mathematically formal through the bilinear transform [35, 50] defined by

$$C(A) = (A - I)(A + I)^{-1}. \tag{3.34}$$

Specifically, if A is a Schur matrix, then P is a solution of the Stein equation (3.33), if and only if

$$C(A)^T P + PC(A) = Q'$$

with $Q' = 2(A+I)^{-T}Q(A+I)^{-1}$. Thus in particular, it follows that $A^T PA - P < 0$ if and only if $C(A)^T P + PC(A) < 0$. The inverse of the bilinear transform is given by $C^{-1}(A) = (I + A)(I - A)^{-1}$ for $A \in \mathbb{R}^{n\times n}$.

3.8.2 Results on the discrete-time CQLF problem

Given the strong connections that exist between the Lyapunov and Stein matrix equations, it is no surprise that many of the results for the continuous-time CQLF problem have analogues in the discrete-time case. First of all, we note that testing whether or not a family of discrete-time LTI systems has a CQLF can be handled numerically in the same way as for continuous-time systems.

Numerical approaches - LMIs:

The conditions (3.31) for $V(x) = x^T Px$ to be a CQLF for the family of discrete-time LTI systems $\Sigma_{A_1}^d, \ldots, \Sigma_{A_k}^d$ define a system of linear matrix inequalities (LMIs) in the variable P. As with the CQLF problem for continuous-time systems, this means that modern optimization routines can be used to test for the existence of a CQLF in discrete-time [12]. However, the issues associated with using LMIs in continuous-time (discussed in Section 3.3) still apply, and the determination of dynamically meaningful, and verifiable, conditions for CQLF existence is as important in the discrete-time case as it was for continuous-time systems.

Necessary and sufficient conditions in discrete-time:

Necessary and sufficient conditions for CQLF existence are known for a number of classes of discrete-time LTI systems. As in the continuous-time case, if the matrices $A_1, \ldots, A_k$ are known to have certain structures, then there will be a CQLF for $\Sigma_{A_1}^d, \ldots, \Sigma_{A_k}^d$ provided that A_i is Schur for $1 \leq i \leq k$. Specifically, the following results are known.

(i) Given the symmetric matrices $A_1, \ldots, A_k$ in $\mathbb{R}^{n \times n}$, the discrete-time LTI systems $\Sigma_{A_1}^d, \ldots, \Sigma_{A_k}^d$ have a CQLF if and only if A_i is Schur for $1 \leq i \leq k$.

(ii) Given the normal matrices $A_1, \ldots, A_k$ in $\mathbb{R}^{n \times n}$, the discrete-time LTI systems $\Sigma_{A_1}^d, \ldots, \Sigma_{A_k}^d$ have a CQLF if and only if A_i is Schur for $1 \leq i \leq k$.

In both of the above cases, when a CQLF exists, it can be taken to be the usual Euclidean norm on $\mathbb{R}^n$, $V(x) = x^T x$.

The following discrete-time version of Theorem 3.4.1 for systems that can be simultaneously put into triangular form was shown by Mori, Mori and Kuroe in [90].

Theorem 3.8.2 *Let $A_1, \ldots, A_k$ be Schur matrices in $\mathbb{R}^{n\times n}$. Suppose that there is a single non-singular $T \in \mathbb{C}^{n\times n}$ such that $T^{-1}A_iT$ is upper (lower)-triangular for $1 \leq i \leq k$. Then the associated discrete-time LTI systems $\Sigma^d_{A_1}, \ldots, \Sigma^d_{A_k}$ have a CQLF.*

Comments:

> Of course, it follows from Theorem 3.8.2 that if the matrices $A_1, \ldots, A_k$ commute pairwise, then there is a CQLF for $\Sigma^d_{A_1}, \ldots, \Sigma^d_{A_k}$. A similar statement applies to the case where $A_1, \ldots, A_k$ generate a solvable Lie-algebra [37].

Necessary and sufficient conditions for a pair of second order discrete-time LTI systems to have a CQLF have been derived by Akar and Narendra in [2]. These conditions are expressed in terms of matrix pencils in the spirit of the continuous-time results presented in [127, 133].

Convex sets of matrices:

Of course, it is also possible to approach the CQLF existence problem for discrete-time systems through studying analogous sets of matrices to those used in the investigation of the continuous-time problem. In this connection, it should be noted that Ando has obtained a characterization of sets of the form

$$\mathcal{A}^d_P = \{A : A^T PA - P < 0\}$$

for some (hidden) $P = P^T > 0$ in the paper [3]. Furthermore, in analogy with the continuous-time case, for a Schur matrix A in $\mathbb{R}^{n\times n}$, we can define the set $\mathcal{P}_A^d$ as

$$\mathcal{P}_A^d = \{P = P^T > 0 : A^T PA - P < 0\}.$$

The properties of these sets for Schur matrices in $\mathbb{R}^{2\times 2}$ are studied in the paper [148], where a necessary and sufficient condition for a family of second order discrete-time LTI systems to have a CQLF is derived. The approach and the conditions presented in [148] are closely related to the work on continuous-time systems in [19].

3.9 Concluding remarks

In this chapter, we have introduced the problem of CQLF existence for families of exponentially stable LTI systems, and surveyed the results that have appeared on this problem in the mathematics and engineering literatures. We have also highlighted the need for a greater understanding of the theoretical aspects of the CQLF existence problem than is currently available. In particular, we have pointed out the importance of understanding how conservative CQLF existence is as a criterion for exponential stability for switched linear systems, and of identifying classes of switched linear systems for which CQLF existence is not conservative for stability. The issues of deriving verifiable, dynamically meaningful conditions for CQLF existence, and of gaining insights into the link between CQLF existence and the exponential stability of switched linear systems, shall be major themes in the forthcoming chapters. We have also provided background and a brief literature review on the CQLF existence problem for discrete-time systems.

Chapter 4

Two results on the CQLF existence problem

In this chapter, we describe a novel approach to the CQLF existence problem for a pair of LTI systems. The essence of this approach is to consider the marginal situation of a pair of LTI systems Σ_{A_1}, Σ_{A_2}, that are on the 'boundary' of having a CQLF in the sense that there is no CQLF for Σ_{A_1}, Σ_{A_2}, but there is a positive semi-definite $P = P^T \geq 0$ with

$$A_i^T P + PA_i \leq 0 \quad i = 1, 2.$$

We present a result that, under a mild additional assumption, describes simple algebraic conditions that must be satisfied in this situation. The implications of this result for the relationship between CQLF existence and the stability of switched linear systems are discussed. Using similar techniques, a corresponding result is derived for the CQLF existence problem for discrete-time systems also.

4.1 Introductory remarks

Despite the considerable amount of work that has been done on the question of CQLF existence, our understanding of the problem is still very far from complete. In fact, apart from the Circle Criterion and the conditions derived for second order systems, most of the theoretical results currently available are either too abstract to be of practical use, or else merely give sufficient conditions for a CQLF to exist for a family of LTI systems. Moreover, as discussed in the previous chapter, the numerical approaches to the problem provide little or no insight into the CQLF existence problem and its relationship with the stability of switched linear systems. Clearly, it would be desirable to have a general framework within which it was possible to derive further results giving verifiable necessary and sufficient conditions for CQLF existence that can be interpreted dynamically. While it is likely to be extremely difficult, if not impossible, to derive such conditions for general families of LTI systems, it may be possible to obtain useful results for practically significant system classes.

Motivated by the above considerations, in this chapter we describe a novel way of approaching the CQLF existence problem, based on the theory of convex sets of matrices, and in particular on the existence of separating hyperplanes for non-intersecting convex sets. This approach is very general in nature, and for certain system classes can be used to derive dynamically meaningful conditions that are necessary and sufficient for CQLF existence. In fact, we shall see in later chapters how the approach developed here can be used to derive attractive conditions for CQLF existence for important system classes, and how it provides a unifying framework within which to view some of the major results previously presented in the literature. It is worth noting that the approach described in this chapter can be applied to the CQLF existence problem for continuous-time systems and discrete-time systems in an identical fashion. In fact, essentially the same argument is used to prove the two main results of this chapter, one concerning the continuous-time CQLF existence problem and the other the corresponding problem for discrete-time systems. A key point about these results is that they provide, in theory, a means of identifying system classes for which simple, meaningful conditions for CQLF existence can be obtained. Moreover, they also give insight into the conservatism of CQLF existence as a criterion for the stability of switched linear systems. We shall see in later chapters how similar ideas to those described here can be used to study questions related to other types of Lyapunov functions also.

4.2 A novel approach to CQLF existence - the underlying ideas

Our objective in this section is to describe, in an informal way, the principal ideas behind the approach to the CQLF existence problem that is presented in this chapter. While the discussion here mainly focusses on the continuous-time case, all of the remarks of this section can be easily translated to apply to the CQLF existence problem for discrete-time systems also.

Recall that in Section 3.7, it was described how the CQLF existence problem for two or more LTI systems can be cast in terms of the cones

$$\mathcal{P}_A = \{P = P^T > 0 : A^T P + PA < 0\}.$$

First of all, given the Hurwitz matrices A_1, A_2 in $\mathbb{R}^{n\times n}$, there exists a CQLF for Σ_{A_1} and Σ_{A_2} if the two cones $\mathcal{P}_{A_1}$, $\mathcal{P}_{A_2}$ have a non-empty intersection. This is illustrated in Figure 4.1.

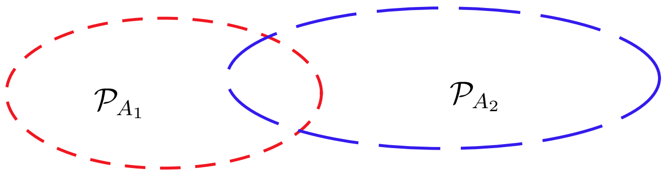

Figure 4.1: CQLF exists for Σ_{A_1}, Σ_{A_2} - $\mathcal{P}_{A_1} \cap \mathcal{P}_{A_2}$ is non-empty

On the other hand, if $\mathcal{P}_{A_1}$, $\mathcal{P}_{A_2}$ do not intersect, then there is no CQLF for the systems Σ_{A_1}, Σ_{A_2}. This situation is illustrated in Figure 4.2.

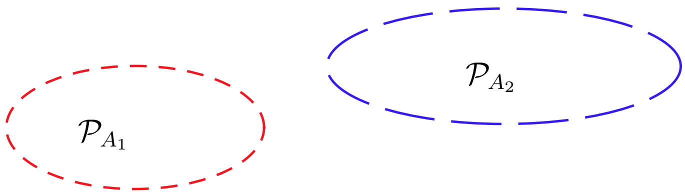

Figure 4.2: CQLF does not exist for Σ_{A_1}, Σ_{A_2} - $\mathcal{P}_{A_1} \cap \mathcal{P}_{A_2}$ is empty

The key idea underlying our approach to the CQLF existence problem for two LTI systems is to consider the limiting case of two systems Σ_{A_1}, Σ_{A_2} that are on the "borderline" between the scenarios depicted in Figure 4.1 and Figure 4.2. This limiting case is also the situation with which the main results of this chapter are concerned. Here, the closures (with respect to the topology on $Sym(n, \mathbb{R})$ induced by the usual inner product (3.6)) of $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$ have a non-trivial intersection (meaning that there is a non-zero element in the intersection) but the sets themselves are disjoint. This is depicted in Figure 4.3.

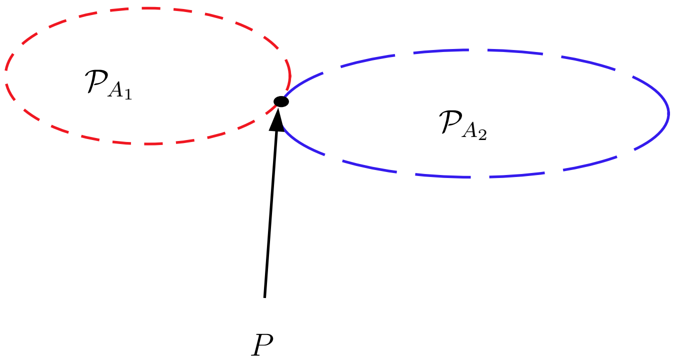

Figure 4.3: CQLF existence - a limiting case

Formally, the situation depicted in Figure 4.3 can be described as follows. There is no CQLF for the systems Σ_{A_1}, Σ_{A_2}, but there does exist a non-zero positive semi-definite $P = P^T \geq 0$ such that $A_i^T P + P A_i \leq 0$ for $i \in \{1, 2\}$. The main results of this chapter show that in this situation, under mild additional assumptions, the system matrices A_1, A_2 satisfy simple algebraic conditions that can be interpreted dynamically. Furthermore, in the next chapter, we shall show how the results of this chapter can be used to derive dynamically significant necessary and sufficient conditions for CQLF existence for certain system classes.

Before finishing this section, it is important to emphasize that all of the above discussion applies to the CQLF existence question for discrete-time systems also. In fact, in discrete-time we would consider the limiting case of two exponentially stable discrete-time LTI systems $\Sigma^d_{A_1}$, $\Sigma^d_{A_2}$ that have no CQLF, but for which there is a matrix $P = P^T \geq 0$ satisfying

$$A_i^T P A_i - P \leq 0 \qquad \text{for } i = 1, 2.$$

Once again we shall see that, under mild additional assumptions, in this situation the system matrices A_1, A_2 satisfy verifiable and dynamically meaningful algebraic conditions. Before moving on to deriving in detail the principal results of this chapter, in the next section we present a number of preliminary technical facts that will be needed later on.

4.3 Some mathematical preliminaries

In this section we describe a number of preliminary results concerning the CQLF existence problem in continuous-time and discrete-time. We shall also present some facts about parameterizations of hyperplanes in the space of symmetric matrices, $Sym(n, \mathbb{R})$, that are crucial for much of the later work of the thesis. To begin with, we state a number of basic lemmas on the CQLF existence problem for continuous-time systems.

Continuous-time preliminaries:

The following well known lemma [133, 129] provides simple necessary conditions for CQLF existence for a pair of LTI systems.

Lemma 4.3.1 *Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{n\times n}$. Suppose that the systems Σ_{A_1}, Σ_{A_2} have a CQLF. Then both of the matrix pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ are Hurwitz, and hence non-singular. Also, the matrix products $A_1A_2^{-1}$ and A_1A_2 have no negative real eigenvalues.*

Proof: Let $V(x) = x^T P x$ be a CQLF for Σ_{A_1}, Σ_{A_2}. Then, using Theorem 3.2.2 we have that

$$A_1^T P + PA_1 < 0, \quad A_2^T P + PA_2 < 0, \quad A_2^{-T} P + PA_2^{-1} < 0.$$

But then, for any $\gamma > 0$, it follows that $(A_1 + \gamma A_2)^T P + P(A_1 + \gamma A_2) < 0$ and $(A_1 + \gamma A_2^{-1})^T P + P(A_1 + \gamma A_2^{-1}) < 0$, and hence, from Theorem 3.2.1, $A_1 + \gamma A_2$ and $A_1 + \gamma A_2^{-1}$ are both Hurwitz for all $\gamma > 0$.

Now $A_1 + \gamma A_2$ is non-singular for all $\gamma > 0$ if and only if

$$\det(A_1 + \gamma A_2) = \det(A_1 A_2^{-1} + \gamma I)\det(A_2) \neq 0,$$

for all $\gamma > 0$. But as A_2 is Hurwitz, and hence non-singular, this is equivalent to $\det(A_1 A_2^{-1} + \gamma I) \neq 0$ for all $\gamma > 0$, which amounts to saying that $A_1 A_2^{-1}$ has no negative real eigenvalues. An identical argument shows that $A_1 + \gamma A_2^{-1}$ is non-singular for all $\gamma > 0$ if and only if $A_1 A_2$ has no negative real eigenvalues. This completes the proof.

Comments:

> Lemma 4.3.1 says that in the situation depicted in Figure 4.1, both of the matrix pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ are Hurwitz, and hence non-singular, and that both of the products $A_1 A_2^{-1}$ and $A_1 A_2$ have no negative real eigenvalues. This is closely connected to the result of Theorem 2.3.1. In fact, it follows from Theorem 2.3.1 that if the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is non-Hurwitz, then the switching system constructed by switching between Σ_{A_1} and Σ_{A_2} is not exponentially stable for arbitrary switching signals, and hence there could not be a CQLF for this pair of LTI systems or for the related pair Σ_{A_1}, $\Sigma_{A_2^{-1}}$. Following similar reasoning we can show using Theorem 2.3.1 that if the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is not Hurwitz, then there is no CQLF for the pair of systems Σ_{A_1}, Σ_{A_2} or for the pair Σ_{A_1}, $\Sigma_{A_2^{-1}}$.

Next consider the situation of two LTI systems Σ_{A_1}, Σ_{A_2} that do not have a CQLF (Figure 4.2). The following lemma describes two ways of adjusting the system matrix A_2 so as to construct a related pair of systems that do have a CQLF. The facts described in this lemma will be used to apply the main results of this chapter to derive necessary and sufficient conditions for CQLF existence for various system classes.

Lemma 4.3.2 *Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{n\times n}$, and suppose that there is no CQLF for the systems Σ_{A_1}, Σ_{A_2}. Let $B = A_2 - A_1$. Then:*

(i) for $\alpha > 0$ sufficiently large, Σ_{A_1} and $\Sigma_{A_2-\alpha I}$ have a CQLF;

(ii) there is some k with $0 < k < 1$ such that Σ_{A_1}, Σ_{A_1+kB} have a CQLF.

Proof:

(i) Choose some $P = P^T > 0$ such that $A_1^T P + PA_1 < 0$ and consider $Q_2 = A_2^T P + PA_2$. Let λ_{max} be the maximal (real) eigenvalue of Q_2, and μ_{min} be the minimal real eigenvalue of P. Then, by assumption, $\mu_{min} > 0$ and $\lambda_{max} \geq 0$. It now follows that for any $\alpha > \frac{\lambda_{max}}{\mu_{min}}$, we have

$$(A_2 - \alpha I)^T P + P(A_2 - \alpha I) < 0.$$

Thus, $V(x) = x^T P x$ is a CQLF for Σ_{A_1}, $\Sigma_{A_2-\alpha I}$. This proves the assertion in (i).

(ii) As in (i), choose some $P = P^T > 0$ such that $A_1^T P + PA_1 < 0$. A simple argument, based on the continuous dependence of the eigenvalues of a matrix on its entries shows that there is some $\delta > 0$ such that for $0 < k < \delta$,

$$(A_1 + kB)^T P + P(A_1 + kB) < 0.$$

Hence, provided $0 < k < \delta$, there is a CQLF for Σ_{A_1} and Σ_{A_1+kB}. This proves (ii).

Comments:

> Lemma 4.3.2 tells us that, given two systems Σ_{A_1}, Σ_{A_2} that do not have a CQLF (Figure 4.2), it is possible to construct related systems that do have a CQLF (Figure 4.1) either (i) by moving the eigenvalues of A_2 sufficiently far into the left half plane or (ii) by replacing A_2 with a suitable convex combination of A_1 and A_2.

Both of these actions can be interpreted geometrically in terms of the sets $\mathcal{P}_{A_1}$, $\mathcal{P}_{A_2}$. In the first case, increasing α moves the eigenvalues of $A_2 - \alpha I$ further into the left half plane and has the effect of making the set $\mathcal{P}_{A_2-\alpha I}$ larger until it eventually intersects with $\mathcal{P}_{A_1}$. In the second case, by choosing smaller values of $k > 0$, we are moving $\mathcal{P}_{A_1+kB}$ closer to $\mathcal{P}_{A_1}$ until they intersect. Eventually, for $k = 0$, the two sets actually coincide.

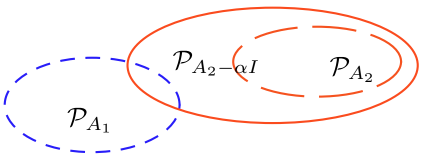

Figure 4.4: Increasing size of $\mathcal{P}_{A_2-\alpha I}$

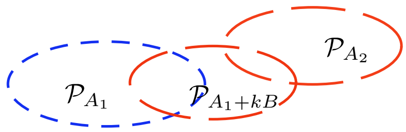

Figure 4.5: Moving $\mathcal{P}_{A_1+kB}$ closer to $\mathcal{P}_{A_1}$

Discrete-time preliminaries:

It is possible to derive versions of Lemmas 4.3.1 and 4.3.2 for the discrete-time CQLF existence problem also. Before we state these discrete-time results, recall from Section 3.8 that the bilinear (or Cayley) transform is defined as $C(A) = (A - I)(A+I)^{-1}$, with the inverse transformation given by $C^{-1}(B) = (I+B)(I-B)^{-1}$, and that $A^T PA - P < 0$ if and only if $C(A)^T P + PC(A) < 0$. The following necessary conditions for two discrete-time LTI systems to have a CQLF correspond to those given by Lemma 4.3.1 for the continuous-time case.

Lemma 4.3.3 *Let A_1, A_2 be Schur matrices in $\mathbb{R}^{n\times n}$. Suppose that the discrete-time LTI systems $\Sigma^d_{A_1}$, $\Sigma^d_{A_2}$ have a CQLF. Then the matrix pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$, $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ are Hurwitz, and hence non-singular, and the matrix products $C(A_1)C(A_2)^{-1}$, $C(A_1)C(A_2)$ have no negative real eigenvalues.*

Proof: If $\Sigma^d_{A_1}$, $\Sigma^d_{A_2}$ have a CQLF, then so do the continuous-time LTI systems

$\Sigma_{C(A_1)}$, $\Sigma_{C(A_2)}$. The result now follows directly from Lemma 4.3.1.

It is also possible to combine the properties of the bilinear transform with Lemma 4.3.2 to obtain the following result.

Lemma 4.3.4 *Let A_1, A_2 be Schur matrices in $\mathbb{R}^{n\times n}$, and let $B = A_2 - A_1$. Suppose that there is no CQLF for the discrete-time LTI systems $\Sigma^d_{A_1}$, $\Sigma^d_{A_2}$. Then;*

(i) For $\alpha > 0$ sufficiently large, $\Sigma^d_{A_1}$ and $\Sigma^d_{C^{-1}(C(A_2)-\alpha I)}$ have a CQLF,

(ii) There is some k with $0 < k < 1$ such that $\Sigma^d_{A_1}$ and $\Sigma^d_{A_1+kB}$ have a CQLF.

As in the continuous-time case, both of the techniques described in Lemma 4.3.4 can be interpreted in terms of the sets $\mathcal{P}^d_A$.

Parametrization of hyperplanes:

Finally for this section, we state the following two technical lemmas. These results, particularly Lemma 4.3.6, are crucial for the work of the next two sections. As the proofs of Lemma 4.3.5 and Lemma 4.3.6 are technical and quite long, we do not include them at this point but present them in the appendices.

Lemma 4.3.5 *Let u, v, x, y be any four non-zero vectors in $\mathbb{R}^n$. There exists a non-singular $T \in \mathbb{R}^{n\times n}$ such that each component of the vectors Tu, Tv, Tx, Ty is non-zero.*

Lemma 4.3.6 *Let x, y, u, v be non-zero vectors in $\mathbb{R}^n$. Suppose that there is some $k > 0$ such that for all symmetric matrices $P \in Sym(n, \mathbb{R})$*

$$x^T P y = -k u^T P v.$$

Then either

$$x = \alpha u \text{ for some real scalar } \alpha, \text{ and } y = -(\frac{k}{\alpha})v$$

or

$$x = \beta v \text{ for some real scalar } \beta \text{ and } y = -(\frac{k}{\beta})u.$$

Comments:

For fixed $x, y \in \mathbb{R}^n$, the mapping $P \to x^T P y$ is a linear functional on $Sym(n, \mathbb{R})$, and the set of P in $Sym(n, \mathbb{R})$ such that $x^T P y = 0$ is then a hyperplane through the origin. The hypotheses of Lemma 4.3.6 imply that the conditions $x^T P y = 0$ and $u^T P v = 0$ define the same hyperplane in $Sym(n, \mathbb{R})$. Moreover, as the constant k is positive, it follows that the half-space where $x^T P y$ is negative corresponds with the half space where $u^T P v$ is positive. This is illustrated in Figure 4.6.

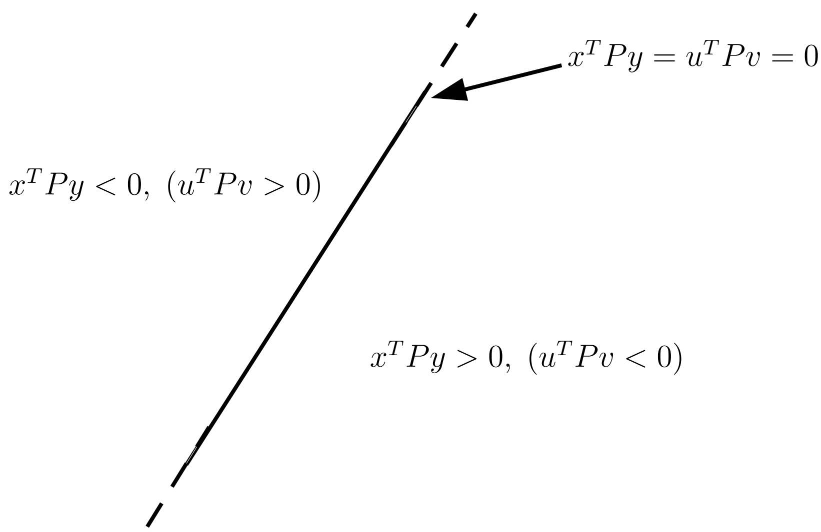

Figure 4.6: Situation considered in Lemma 4.3.6

4.4 The main results

The main results of the current chapter are derived in this section. First of all, in Section 4.3.1, we consider a pair of continuous-time LTI systems Σ_{A_1}, Σ_{A_2} in the situation depicted in Figure 4.3. We shall see that, under a mild additional assumption, it is possible to characterize such pairs of systems with simple algebraic conditions on the system matrices A_1, A_2. Then, in Section 4.3.2, an identical analysis is carried out for discrete-time systems and a corresponding, and closely related, result is derived using the same techniques as are employed in the continuous-time case.

4.4.1 Continuous-time case

In Theorem 4.4.1 below, we consider a pair of continuous-time LTI systems $\Sigma_{A_1}, \Sigma_{A_2}$ such that;

(i) there does not exist a CQLF for Σ_{A_1}, Σ_{A_2},

(ii) there exists some non-zero positive semi-definite $P = P^T \geq 0$ such that $A_i^T P + PA_i = Q_i \leq 0$, $\text{rank}(Q_i) = n - 1$, for $i \in \{1, 2\}$.

It is appropriate at this point to comment on the assumption that the ranks of the matrices Q_1, Q_2 are both $n-1$. Extensive numerical testing with the LMI toolbox for MATLAB has indicated that, while it is not entirely generic, this rank condition is satisfied by a substantial number of pairs of systems in the situation of Figure 4.3. Furthermore, we shall show in a later chapter, when we come to discuss the boundary structure of the cones $\mathcal{P}_A$, that those matrices P on the boundary of $\mathcal{P}_A$ for which the rank of $A^T P + PA$ is $n-1$ are dense in the boundary. This partially accounts for how often the conditions of Theorem 4.4.1 have been satisfied in numerically generated examples.

Theorem 4.4.1 *[129, 122] Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{n\times n}$ such that Σ_{A_1}, Σ_{A_2} do not have a CQLF. Furthermore, suppose that there is some $P = P^T \geq 0$ such that*

$$A_i^T P + PA_i = Q_i \leq 0, \ i \in \{1, 2\} \tag{4.1}$$

for some negative semi-definite matrices Q_1, Q_2 in $\mathbb{R}^{n\times n}$, both of rank $n-1$. Under these conditions, at least one of the pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$, $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is singular. Equivalently, at least one of the matrix products A_1A_2 and $A_1A_2^{-1}$ has a negative real eigenvalue.

Comments:

As the proof of Theorem 4.4.1 is fairly long, it is worthwhile outlining the principal steps involved before beginning.

(a) As Q_1 and Q_2 are of rank $n-1$, vectors x_1, $x_2 \in \mathbb{R}^{n\times n}$ exist such that $Q_1x_1 = 0$ and $Q_2x_2 = 0$. Furthermore, these vectors are unique up to scalar multiples.

(b) Consider the two hyperplanes $\mathcal{H}_1$ and $\mathcal{H}_2$ in $Sym(n, \mathbb{R})$ defined by:

$$\begin{aligned}\mathcal{H}_1 &= \{H \in Sym(n, \mathbb{R}) : x_1^T HA_1x_1 = 0\},\\ \mathcal{H}_2 &= \{H \in Sym(n, \mathbb{R}) : x_2^T HA_2x_2 = 0\}.\end{aligned}$$

We shall show that, under the hypotheses of the theorem, $\mathcal{H}_1$ and $\mathcal{H}_2$ define one and the same hyperplane.

(c) It then follows that there is some real $k > 0$ with $x_1^T HA_1x_1 = -kx_2^T HA_2x_2$, for all H in $Sym(n, \mathbb{R})$.

(d) Finally, we apply Lemma 4.3.6 to deduce the result of the theorem.

Proof: As Q_1 and Q_2 are of rank $n-1$, there are non-zero vectors x_1, x_2 in $\mathbb{R}^n$ such that

$$\begin{aligned} x_1^T Q_1 x_1 &= 2x_1^T P A_1 x_1 = 0, \\ x_2^T Q_2 x_2 &= 2x_2^T P A_2 x_2 = 0. \end{aligned} \tag{4.2}$$

The proof of Theorem 4.4.1 is split into two main stages.

Stage 1:

First of all, we shall show that if there exists $\overline{P}$ in $Sym(n, \mathbb{R})$ satisfying

$$x_1^T \overline{P} A_1 x_1 < 0, \quad x_2^T \overline{P} A_2 x_2 < 0 \tag{4.3}$$

then a CQLF exists for Σ_{A_1} and Σ_{A_2}.

Suppose that there is some $\overline{P}$ satisfying (4.3). We shall show that by choosing $\delta_1 > 0$ sufficiently small, it is possible to guarantee that $A_1^T(P + \delta_1 \overline{P}) + (P + \delta_1 \overline{P})A_1$ is negative definite.

Consider the set

$$\Omega_1 = \{x \in \mathbb{R}^n : x^T x = 1 \text{ and } x^T \overline{P} A_1 x \geq 0\}.$$

Note that if Ω_1 was empty, then any positive constant $\delta_1 > 0$ would make $A_1^T(P + \delta_1 \overline{P}) + (P + \delta_1 \overline{P})A_1$ negative definite. Now assume that Ω_1 is non-empty.

The function that takes x to $x^T \overline{P} A_1 x$ is continuous. Thus Ω_1 is closed and bounded, hence compact. Furthermore x_1 (or any non-zero multiple of x_1) is not in Ω_1 and thus $x^T P A_1 x < 0$ for all x in Ω_1.

Let M_1 be the maximum value of $x^T \overline{P} A_1 x$ on Ω_1, and let M_2 be the maximum value of $x^T P A_1 x$ on Ω_1. Then by the final remark in the previous paragraph, $M_2 < 0$. Choose any constant $\delta_1 > 0$ such that

$$\delta_1 < \frac{|M_2|}{M_1 + 1} = C_1$$

and consider the symmetric matrix

$$P + \delta_1 \overline{P}.$$

By separately considering the cases $x \in \Omega_1$ and $x \notin \Omega_1$, $x^T x = 1$, it follows that for all non-zero vectors x of Euclidean norm 1

$$x^T(A_1^T(P + \delta_1\overline{P}) + (P + \delta_1\overline{P})A_1)x < 0$$

provided $0 < \delta_1 < \frac{|M_2|}{M_1+1}$. Since the above inequality is unchanged if we scale x by any non-zero real number, it follows that $A_1^T(P + \delta_1\overline{P}) + (P + \delta_1\overline{P})A_1$ is negative definite. It now follows from Theorem 3.2.1 that $P + \delta_1\overline{P}$ is positive definite.

The identical argument can now be used to show that there is some positive constant $C_2 > 0$ such that

$$x^T(A_2^T(P + \delta_1\overline{P}) + (P + \delta_1\overline{P})A_2)x < 0$$

for all non-zero x in $\mathbb{R}^n$, provided $0 < \delta_1 < C_2$. Now choose any $\delta > 0$ such that $\delta < \min\{C_1, C_2\}$ and consider the positive definite matrix

$$P_1 = P + \delta\overline{P}.$$

Then $V(x) = x^T P_1 x$ is a CQLF for Σ_{A_1} and Σ_{A_2}.

Stage 2:

So under the hypotheses of the theorem, there is no $\overline{P}$ in $Sym(n, \mathbb{R})$ satisfying the conditions (4.3). We next show that this implies that one of the two pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$, $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ must be singular.

As there is no $\overline{P}$ satisfying (4.3), any symmetric $\overline{P}$ that makes the expression $x_1^T\overline{P}A_1x_1$ negative will make the expression $x_2^T\overline{P}A_2x_2$ positive. More formally

$$x_1^T\overline{P}A_1x_1 < 0 \iff x_2^T\overline{P}A_2x_2 > 0 \tag{4.4}$$

for $\overline{P} \in Sym(n, \mathbb{R})$. This implies that

$$x_1^T\overline{P}A_1x_1 = 0 \iff x_2^T\overline{P}A_2x_2 = 0.$$

The mappings $\overline{P} \to x_1^T \overline{P} A_1 x_1$ and $\overline{P} \to x_2^T \overline{P} A_2 x_2$ define linear functionals on the space $Sym(n, \mathbb{R})$. Moreover, we have seen that the null sets of these functionals are identical. Thus, they must be scalar multiples of each other. Furthermore, (4.4) implies that they are *negative* multiples of each other. Therefore there is some constant $k > 0$ such that

$$x_1^T \overline{P} A_1 x_1 = -k x_2^T \overline{P} A_2 x_2 \tag{4.5}$$

for all $\overline{P} \in Sym(n, \mathbb{R})$.

Now Lemma 4.3.6 implies that either $x_1 = \alpha x_2$ with $A_1 x_1 = -(\frac{k}{\alpha}) A_2 x_2$ for some real α, or $x_1 = \beta A_2 x_2$ and $A_1 x_1 = -(\frac{k}{\beta}) x_2$ for some real β. Consider the former situation to begin with. Then we have

$$\begin{aligned} A_1(\alpha x_2) &= -(\frac{k}{\alpha}) A_2 x_2 \\ \implies (A_1 + (\frac{k}{\alpha^2}) A_2) x_2 &= 0 \end{aligned}$$

and thus the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is singular and the matrix $A_1 A_2^{-1}$ has a negative real eigenvalue.

On the other hand, in the latter situation, we have that

$$x_2 = \frac{1}{\beta} A_2^{-1} x_1$$

Thus

$$\begin{aligned} A_1 x_1 &= -(\frac{k}{\beta^2}) A_2^{-1} x_1 \\ \implies (A_1 + (\frac{k}{\beta^2}) A_2^{-1}) x_1 &= 0 \end{aligned}$$

Thus, in this case the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is singular and the corresponding matrix product $A_1 A_2$ has a negative real eigenvalue. This completes the proof of Theorem 4.4.1.

<u>**Comments:**</u>

A key factor in the proof of Theorem 4.4.1 is that there exists a separating hyperplane between the non-intersecting convex cones $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$ [116]. In fact, such a hyperplane would exist even without the extra assumption on the rank of the matrices Q_1, Q_2. However, the assumption that both of these matrices have rank $n-1$ means that the separating hyperplane is effectively unique, and it is this fact that leads to the conclusions of the theorem. We shall say more of this and similar issues when we come to discuss the boundary structure of the sets $\mathcal{P}_A$ in a later chapter. In particular, we shall show why the separating hyperplane is unique under the assumptions of Theorem 4.4.1.

While Theorem 4.4.1 may appear somewhat theoretical, in the next chapter we shall describe how it can be used to derive concrete, and applicable, conditions for CQLF existence for a pair of LTI systems. In fact, we shall see that it unifies, in a certain sense, two of the most significant results previously derived on the CQLF existence problem.

It is important to appreciate that in the situation of Figure 4.3, there will generally be many matrices P in the intersection of the boundaries of $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$. However, the hypotheses of Theorem 4.4.1 require the existence of only *one* positive semi-definite matrix P in this intersection such that $A_i^T P + PA_i$ has rank $n-1$ for $i = 1, 2$. It is not necessary for all matrices common to both boundaries to have this property in order for the theorem to apply.

When the hypotheses of Theorem 4.4.1 are satisfied, the result says that one of the matrix pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$, $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ has an eigenvalue in the closed right half-plane. As has been remarked before, Theorem 2.3.1 relates conditions of this sort to the dynamics of the pair of switched systems

$$\begin{aligned}\dot{x} &= A(t)x \quad A(t) \in \{A_1, A_2\}\\ \dot{x} &= A(t)x \quad A(t) \in \{A_1, A_2^{-1}\}.\end{aligned}$$

In fact, it follows from Theorem 2.3.1 that any pair of LTI systems satisfying the hypotheses of Theorem 4.4.1 is, on the one hand, on the 'border' of those pairs of systems that have a CQLF, while at the same time, one of the above associated switched linear systems is not exponentially stable under arbitrary switching. This is in itself noteworthy in view of the commonly held opinion that CQLF existence is an overly conservative criterion for switched system stability. Loosely speaking, it shows that there are pairs of systems arbitrarily close to having a CQLF, for which one of the associated switching systems is not exponentially stable for all switching signals. Furthermore, it suggests that if we have a system class to which Theorem 4.4.1 applies, it may be possible to derive dynamically interpretable necessary and sufficient conditions for CQLF existence for pairs of systems belonging to that class. This leads naturally to the important question of how to identify system classes that satisfy the hypotheses of the theorem. This issue is addressed in the next chapter where we shall describe two significant classes of systems for which elegant and powerful conditions for CQLF existence can be derived with the help of Theorem 4.4.1.

4.4.2 Discrete-time case

It has been pointed out in Section 3.8.2 that it is also possible to approach the CQLF existence problem for discrete-time systems in terms of convex sets of matrices. In this section, we shall apply the analysis techniques of the last section to derive a similar result for the CQLF existence problem for discrete-time systems. The approach adopted is the same, with the Stein equation playing the role that the Lyapunov equation played in the continuous-time case. In fact, a major attraction of the methods described in this chapter is their generality and adaptability, and we shall see here and in later chapters how the basic ideas of this chapter can be applied in a variety of different situations to derive novel results and provide new insights into various questions relating to Lyapunov functions and stability.

Theorem 4.4.2 below considers the case of two discrete-time LTI systems $\Sigma^d_{A_1}$, $\Sigma^d_{A_2}$ in the situation of Figure 4.3. Under the same assumptions as in the continuous-time case, we shall see that similarly simple algebraic conditions on the system matrices A_1, A_2 are satisfied. The comments made prior to Theorem 4.4.1 regarding the rank $n-1$ assumption on the matrices Q_i apply in the discrete-time case also.

Theorem 4.4.2 *[78, 77] Let A_1, A_2 be Schur matrices in $\mathbb{R}^{n\times n}$ such that the discrete-time LTI systems $\Sigma^d_{A_1}$, $\Sigma^d_{A_2}$ do not have a CQLF. Furthermore, suppose that there is some $P = P^T \geq 0$ such that*

$$A_1^T P A_1 - P = Q_1 \leq 0, \tag{4.6}$$

$$A_2^T P A_2 - P = Q_2 \leq 0, \tag{4.7}$$

for some negative semi-definite matrices Q_1, Q_2 both of rank $n-1$. Under these conditions, at least one of the matrix pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$, $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ is singular, and equivalently, at least one of the matrix products $C(A_1)C(A_2)$ and $C(A_1)C(A_2)^{-1}$ has a negative real eigenvalue.

<u>**Comments:**</u>

While the proof of Theorem 4.4.2 is again quite long, the steps involved are essentially the same as those in the proof of Theorem 4.4.1.

(a) There exist vectors x_1, x_2 in $\mathbb{R}^n$, unique up to scalar multiples, such that $Q_1x_1 = 0$, $Q_2x_2 = 0$.

(b) Next, we show that the two hyperplanes $\mathcal{H}_1$, $\mathcal{H}_2$, in $Sym(n, \mathbb{R})$ defined by

$$\begin{aligned} \mathcal{H}_1 &= \{H \in Sym(n, \mathbb{R}) : x_1^T(A_1^THA_1 - A_1)x_1 = 0\}, \\ \mathcal{H}_2 &= \{H \in Sym(n, \mathbb{R}) : x_2^T(A_2^THA_2 - A_2)x_2 = 0\} \end{aligned}$$

coincide.

(c) It then follows that there is some $k > 0$ such that $x_1^T(A_1^THA_1 - A_1)x_1 = -kx_2^T(A_2^THA_2 - A_2)x_2$ for all $H \in Sym(n, \mathbb{R})$.

(d) Slightly more algebraic manipulation than in the continuous-time case is then required to apply Lemma 4.3.6 to complete the proof.

Proof: As Q_1 and Q_2 are of rank $n-1$, there are non-zero vectors x_1, x_2 such that

$$x_i^TQ_ix_i = 0, \quad \text{for } i = 1, 2. \tag{4.8}$$

As with Theorem 4.4.1, there are two stages to the proof.

Stage 1:

We first show that if there exists some $\overline{P}$ in $Sym(n, \mathbb{R})$ such that

$$\begin{aligned} x_1^T(A_1^T\overline{P}A_1 - \overline{P})x_1 &< 0, \\ x_2^T(A_2^T\overline{P}A_2 - \overline{P})x_2 &< 0, \end{aligned} \tag{4.9}$$

then $\Sigma_{A_1}^d$, $\Sigma_{A_2}^d$ have a CQLF.

So assume that there is some $\overline{P}$ satisfying (4.9), and consider the set

$$\Omega_1 = \{x \in \mathbb{R}^n : xx^T = 1, x^T(A_1^T\overline{P}A_1 - \overline{P})x \geq 0\}.$$

We shall show that there is a positive constant $C_1 > 0$ such that $A_1^T(P + \delta_1\overline{P})A_1 - (P + \delta_1\overline{P}) < 0$ provided that $0 < \delta_1 < C_1$. Note that if Ω_1 was empty, then $A_1^T(P + \delta_1\overline{P})A_1 - (P + \delta_1\overline{P}) < 0$ for all $\delta_1 > 0$.

So assume that the set Ω_1 is non-empty. The function that takes x to $x^T(A_1^T\overline{P}A_1 - \overline{P})x$ is continuous. Then Ω_1 is closed and bounded, hence compact. Furthermore x_1 (or any non-zero multiple of x_1) is not in Ω_1 and thus $x^T(A_1^TPA_1 - P)x < 0$ for all x in Ω_1.

Let M_1 be the maximum value of $x^T(A_1^T\overline{P}A_1 - \overline{P})x$ on Ω_1, and let M_2 be the maximum value of $x^T(A_1^TPA_1 - P)x$ on Ω_1. Then by the final remark in the previous paragraph, $M_2 < 0$. Choose any constant $\delta_1 > 0$ such that

$$\delta_1 < \frac{|M_2|}{M_1 + 1}$$

and consider the matrix

$$P + \delta_1\overline{P}.$$

By separately considering the cases $x \in \Omega_1$ and $x \notin \Omega_1$, $x^Tx = 1$, it is easy to see that for all non-zero vectors x of Euclidean norm 1

$$x^T(A_1^T(P + \delta_1\overline{P})A_1 - (P + \delta_1\overline{P}))x < 0$$

provided $0 < \delta_1 < \frac{|M_2|}{M_1+1}$. Let C_1 denote the value $\frac{|M_2|}{M_1+1} > 0$. Then for any δ_1 with $0 < \delta_1 < C_1$, $A_1^T(P + \delta_1\overline{P})A_1 - (P + \delta_1\overline{P}) < 0$ as claimed.

The same argument can now be used to guarantee the existence of a positive constant C_2 such that

$$x^T(A_2^T(P + \delta_1\overline{P})A_2 - (P + \delta_1\overline{P}))x < 0$$

for all non-zero x provided we choose $0 < \delta_1 < C_2$. Then, choose $\delta > 0$ less than the minimum of C_1, C_2, and consider the matrix

$$P_1 = P + \delta\overline{P}.$$

It follows from Theorem 3.8.1 that $P_1 > 0$ and thus $V(x) = x^TP_1x$ would be a CQLF for $\Sigma_{A_1}^d$, $\Sigma_{A_2}^d$.

Stage 2:

So under the hypotheses of the theorem, there is no symmetric matrix $\overline{P}$ such that

$$x_1^T(A_1^T\overline{P}A_1 - \overline{P})x_1 < 0 \tag{4.10}$$

$$x_2^T(A_2^T\overline{P}A_2 - \overline{P})x_2 < 0. \tag{4.11}$$

Thus, the two linear functionals defined on the space $Sym(n,\mathbb{R})$ by

$$\overline{P} \rightarrow x_i^T(A_i^T\overline{P}A_i - \overline{P})x_i, \quad i \in \{1,2\},$$

have the same kernel. As in the proof of Theorem 4.4.1, we can combine this with the fact that there is no $\overline{P}$ satisfying (4.10), (4.11), to conclude that there is some positive constant k such that

$$x_1^T(A_1^T\overline{P}A_1 - \overline{P})x_1 = -kx_2^T(A_2^T\overline{P}A_2 - \overline{P})x_2 \tag{4.12}$$

for all real symmetric matrices $\overline{P}$.

Expanding the expression

$$(A_ix_i - x_i)^T\overline{P}(A_ix_i + x_i)$$

and noting that, for symmetric $\overline{P}$,

$$x_i^TA_i^T\overline{P}x_i - x_i^T\overline{P}A_ix_i = 0,$$

we see that, for $i = 1, 2$

$$x_i^T(A_i^T\overline{P}A_i - \overline{P})x_i = (A_ix_i - x_i)^T\overline{P}(A_ix_i + x_i) \tag{4.13}$$

for all $\overline{P}$ in $Sym(n,\mathbb{R})$.

Combining this fact with (4.12) and applying Lemma 4.3.6 now shows that either

$$\begin{aligned}(A_1x_1 + x_1) &= \alpha(A_2x_2 + x_2), \qquad (4.14)\\ (A_1x_1 - x_1) &= -\frac{k}{\alpha}(A_2x_2 - x_2)\end{aligned}$$

or

$$\begin{aligned}(A_1x_1 + x_1) &= \alpha(A_2x_2 - x_2), \\ (A_1x_1 - x_1) &= -\frac{k}{\alpha}(A_2x_2 + x_2).\end{aligned} \tag{4.15}$$

In the first case (4.14), we have

$$x_1 = \alpha(A_1 + I)^{-1}(A_2 + I)x_2$$

and substituting this into the second identity in (4.14) yields

$$(A_1 - I)(A_1 + I)^{-1}(A_2 + I)x_2 = -\frac{k}{\alpha^2}(A_2 - I)x_2$$

Letting $y = (A_2 + I)x_2$ we see that

$$(C(A_1) + \frac{k}{\alpha^2}C(A_2))y = 0$$

and hence the pencil $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$ is singular and the product $C(A_1)C(A_2)^{-1}$ has a negative real eigenvalue. A similar argument shows that in the case (4.15), the pencil $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ is singular and the product $C(A_1)C(A_2)$ has a negative eigenvalue. This completes the proof of Theorem 4.4.2.

Comments:

As with Theorem 4.4.1, it is only necessary for there to be one P such that $A_i^TPA_i - P$ is of rank $n-1$ for $i = 1, 2$, for the hypotheses of Theorem 4.4.2 to be satisfied.

Note that it is possible to give an alternative proof of Theorem 4.4.2 by combining the properties of the bilinear transform with the result of Theorem 4.4.1 for continuous-time systems. However, the proof that we have presented here provides a further illustration of the key ideas of this chapter as well as demonstrating how the techniques used to prove Theorem 4.4.1 can be adapted to derive similar results in different contexts. Several more such illustrations of the adaptability of these techniques will be encountered in later chapters.

Finally, it is of interest to note that the bilinear transform, relating the discrete-time and continuous-time CQLF existence problems, arises naturally in the course of the proof of Theorem 4.4.2.

4.5 An illustrative example

In this section, we present a numerical example to illustrate the result of Theorem 4.4.1. We describe an example for the continuous-time case only, as the ideas behind the discrete-time result of Theorem 4.4.2 are essentially the same. Specifically, in Example 4.5.1 we consider pairs of LTI systems that are converging towards the situation covered by Theorem 4.4.1, and observe the behaviour of the eigenvalues of the two matrix products mentioned in the statement of the theorem. We shall see that as two systems Σ_{A_1}, Σ_{A_2} get closer to the scenario of Theorem 4.4.1, some pair of complex conjugate eigenvalues of one of the matrices A_1A_2, $A_1A_2^{-1}$ collapses onto a single negative real eigenvalue of algebraic multiplicity two.

Example 4.5.1 *Consider the matrices A and B in $\mathbb{R}^{3\times 3}$ given by*

$$A = \begin{pmatrix} 4.3876 & -13.3912 & 40.4673 \\ -4.3483 & -85.7644 & -47.3620 \\ -12.6313 & 37.5234 & -87.6206 \end{pmatrix}$$

$$B = \begin{pmatrix} -44.6189 & 17.8573 & 8.9612 \\ -15.3243 & 46.7799 & 92.4043 \\ 7.9629 & 11.5449 & 66.3210 \end{pmatrix},$$

and for $t > 0$ define $A(t) = A + tB$. Then, using the LMI toolbox in MATLAB, it *is possible to verify that the LTI systems Σ_A, $\Sigma_{A(t)}$ have a CQLF for values of t up to 0.807. Furthermore, there is no CQLF for Σ_A, $\Sigma_{A(t)}$ when $t = 0.808$. Thus, by continuity the marginal situation depicted in Figure 4.3 occurs for some value of the parameter t between 0.807 and 0.808. We still need to investigate whether or not the additional 'rank $n-1$' assumption of the theorem is satisfied in the limit.*

With this in mind, we consider the systems Σ_A and $\Sigma_{A(t)}$ for several values of t tending towards 0.808. For each of these values, we use the LMI toolbox to find a positive definite matrix P_t such that $x^T P_t x$ is a CQLF for Σ_A and $\Sigma_{A(t)}$. We then calculate the eigenvalues of the symmetric matrices $A^T P_t + P_t A$ and $A(t)^T P_t + P_t A(t)$ and compare the largest and second largest eigenvalues of each matrix. The results of this comparison are illustrated in Figure 4.7 and Figure 4.8 below. We can see from these plots that the largest eigenvalues of $A^T P_t + P_t A$ and $A(t)^T P_t + P_t A(t)$ are tending towards zero as we increase t, while the second largest eigenvalues are not. Thus, the systems Σ_A and $\Sigma_{A(t)}$ are tending towards the situation of Theorem 4.4.1.

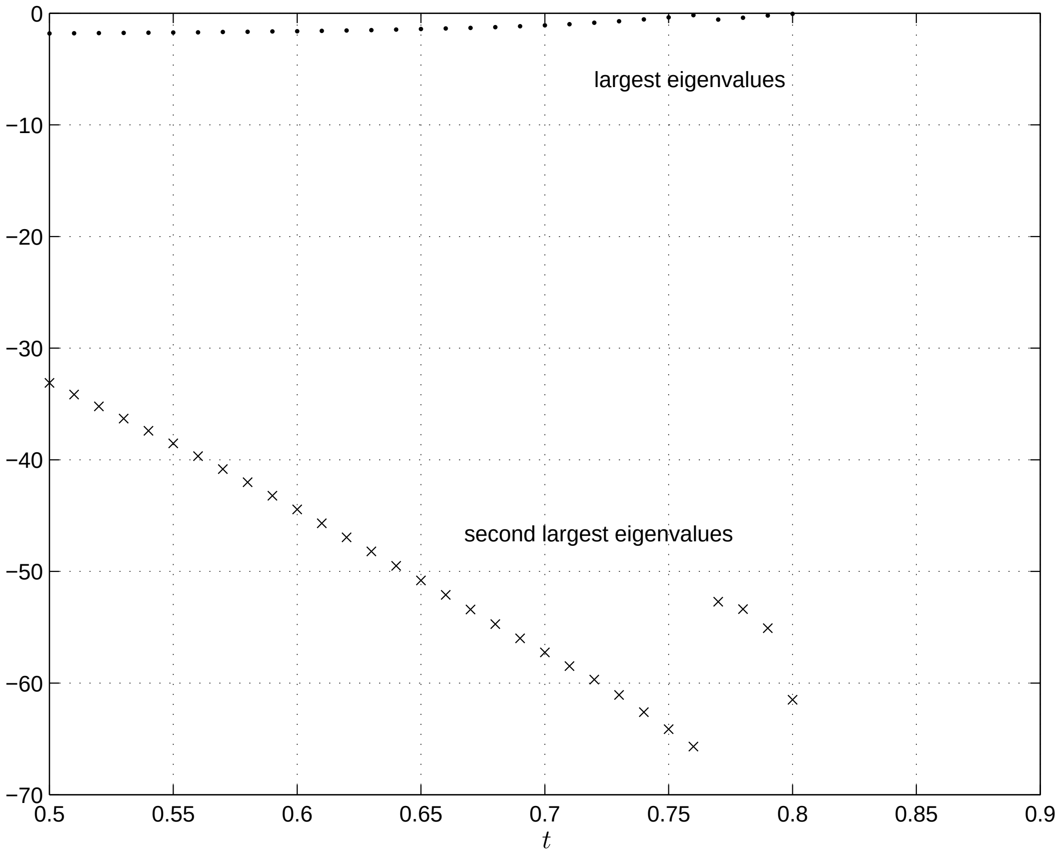

Figure 4.7: Largest and second largest eigenvalues of $A^T P_t + P_t A$

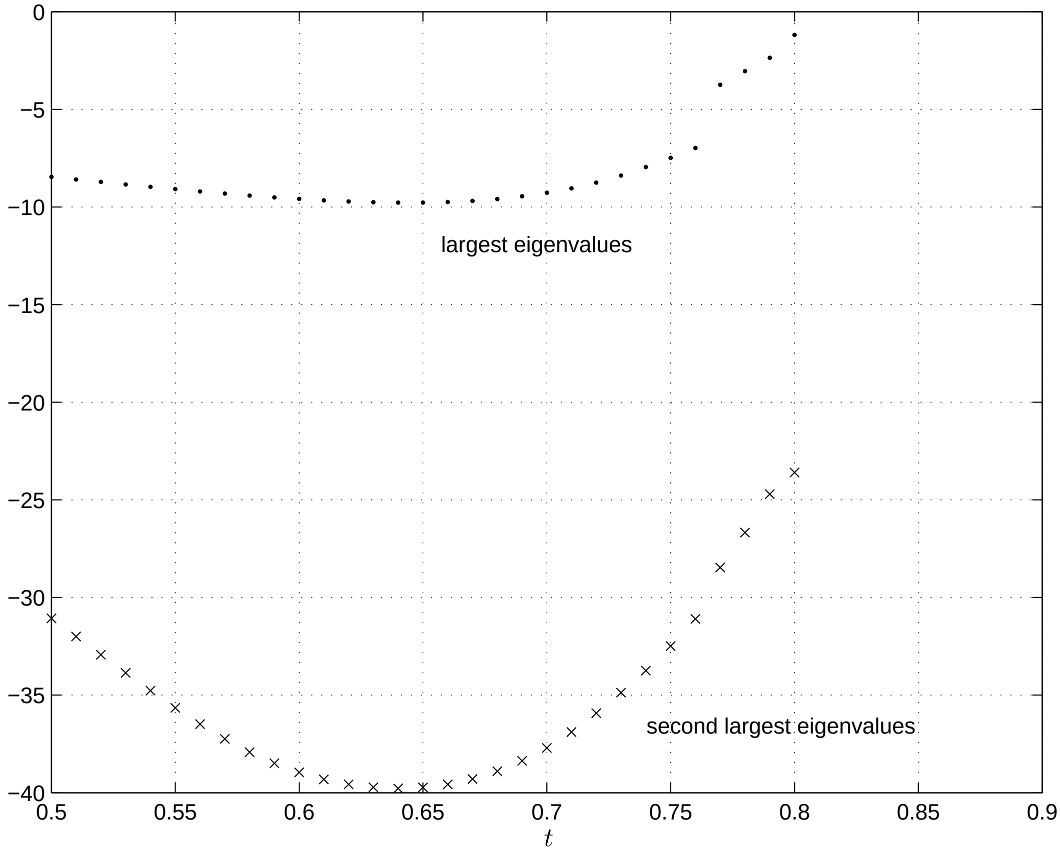

Figure 4.8: Largest and second largest eigenvalues of $A(t)^T P_t + P_t A(t)$

Now, plotting the real and imaginary parts of the eigenvalues of the matrix products $AA(t)$ for values of t tending towards 0.808*, we obtain Figure 4.9 below. Of course, for as long as Σ_A and $\Sigma_{A(t)}$ have a CQLF, the product $AA(t)$ has no negative eigenvalue (Lemma 4.3.1). However as indicated in the Figure, as t approaches* 0.808*, one pair of complex conjugate eigenvalues converges towards a single real negative eigenvalue of algebraic multiplicity two. This is typical of what happens for systems converging towards the scenario of Theorem 4.4.1.*

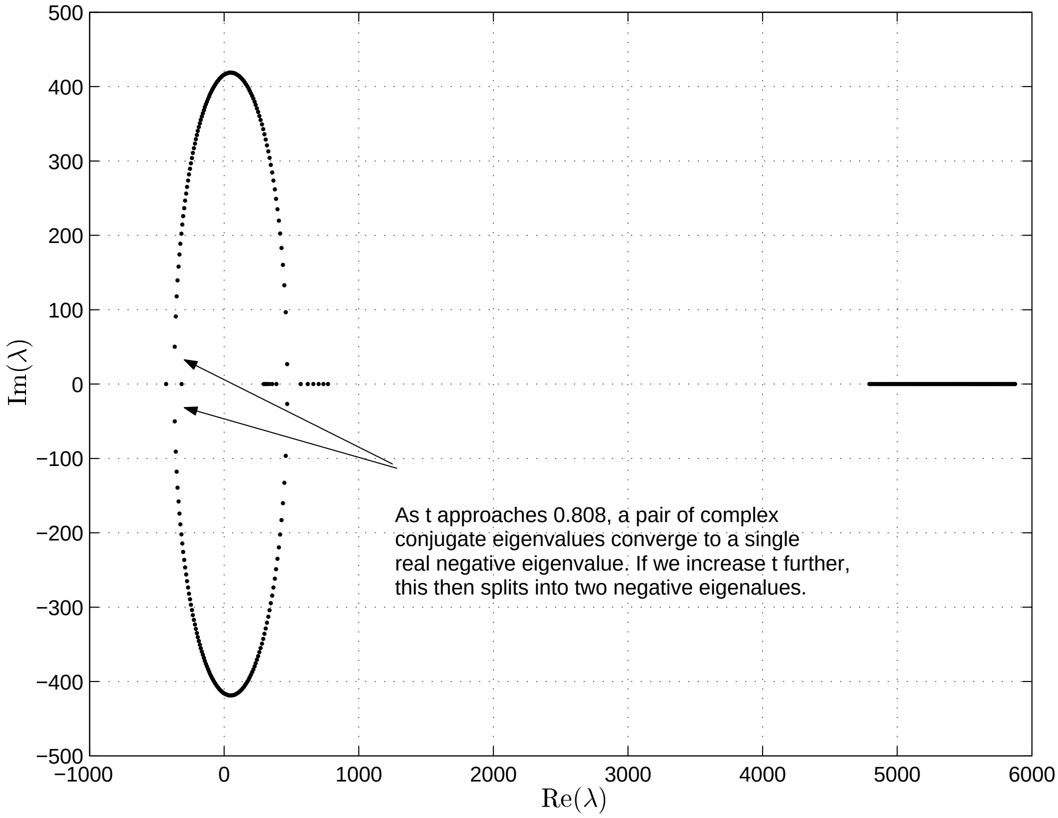

Figure 4.9: Eigenvalues of $AA(t)$ for t tending to 0.808

4.6 Concluding remarks

In this chapter we have described a novel approach to the CQLF existence problem for a pair of exponentially stable LTI systems, based on analysing certain convex cones of matrices. The key idea underlying this approach was to consider the marginal situation of a pair of LTI systems $\Sigma_{A_1} : \dot{x} = A_1 x$, $\Sigma_{A_2} : \dot{x} = A_2 x$, that are on the 'boundary' of possessing a CQLF in the following sense. There is no CQLF for Σ_{A_1}, Σ_{A_2}, but there is a positive semi-definite $P = P^T \geq 0$ with

$$A_i^T P + P A_i \leq 0 \quad i = 1, 2.$$

In Theorem 4.4.1, we showed that in this situation, under an additional assumption, the system matrices A_1, A_2 satisfy simple algebraic conditions. The implications of this result for the connection between CQLF existence and the exponential stability of switched linear systems were discussed in the text. In particular, it was noted that Theorem 4.4.1 shows that there are pairs of LTI systems arbitrarily close to having a CQLF for which the associated switched linear system is not exponentially stable. Furthermore, it was observed that for system classes to which the theorem can be applied, it may be possible to obtain conditions for CQLF existence which are easily verifiable and are also relevant to the dynamics of switched linear systems. The same approach was also applied to the CQLF existence problem for discrete-time systems, and a corresponding result was derived. In the next chapter, we shall see how to use the results of this chapter to obtain necessary and sufficient conditions for CQLF existence for certain system classes.

Chapter 5

Second order systems and the Circle Criterion revisited

We show that the results of the last chapter provide a unifying framework for the SISO Circle Criterion and the conditions for CQLF existence obtained for pairs of second order systems. Also, a recent result giving a necessary and sufficient condition for CQLF existence for pairs of LTI systems whose system matrices are in companion form is extended to pairs of systems whose system matrices differ by a general rank one matrix. The key role played by Theorem 4.4.1 in determining simple conditions for CQLF existence is highlighted, and the implications of our results for the connection between CQLF existence and the stability of switched linear systems are described. In particular, classes of switched linear systems are identified for which CQLF existence is not a conservative way of establishing stability.

5.1 Introductory remarks

In the previous chapter we described a novel way of approaching the CQLF existence problem for a pair of LTI systems based on the theory of convex sets of matrices. Following this approach, we derived two results on CQLF existence, one in continuous-time and the other in discrete-time. While these results may appear to be primarily theoretical in nature, they still provide insights into the question of CQLF existence and its relationship to the stability issues associated with switched linear systems. In particular, we have seen how a pair of systems that are right on the 'boundary', in a certain sense, of having a CQLF can still give rise to an unstable switching system. In this chapter, we shall develop the ideas of the previous chapter further, showing how the two major results therein can be used to derive necessary and sufficient conditions for CQLF existence for certain system classes, and indicating how Theorem 4.4.1 unifies, in a certain sense, two of the most powerful results on CQLF existence for continuous-time systems obtained thus far. In particular, we shall show how that theorem can be used to give new, and insightful, proofs of the necessary and sufficient conditions previously derived for second order systems in continuous-time and discrete-time, and to obtain a time-domain version of the classical Circle Criterion. We shall also point out a number of advantages of this time-domain formulation of the result, and illustrate the results obtained through numerical examples.

5.2 Second order switched linear systems

In this section, we shall use Theorem 4.4.1 and Theorem 4.4.2 to derive necessary and sufficient conditions for CQLF existence for pairs of second order LTI systems in continuous-time and discrete-time. While such conditions have already been obtained in [133, 2], the proofs that we give here are more intuitive and explain why the conditions for CQLF existence for pairs of second order systems can take a particularly simple form. Further, the proofs given here show how the CQLF existence results for second order systems fit into the general framework of the previous chapter and show how to use the results and ideas of that chapter to derive conditions for CQLF existence. The fact that it is possible to place the second order results within a general context raises the possibility of extending the method to other, more complex classes of system and obtaining similar conditions. The arguments given here follow those in [129, 78, 122, 77].

5.2.1 Continuous-time systems

Consider a pair of second order exponentially stable LTI systems Σ_{A_1}, Σ_{A_2} in continuous-time. Thus, the system matrices A_1, A_2 are Hurwitz matrices in $\mathbb{R}^{2\times 2}$. We shall show how to use Theorem 4.4.1 to derive necessary and sufficient conditions for Σ_{A_1} and Σ_{A_2} to have a CQLF.

To begin with, we note the following two simple facts.

(i) It follows from Lemma 4.3.1 that if Σ_{A_1}, Σ_{A_2} have a CQLF, then the two matrix pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ are Hurwitz, and hence non-singular. Furthermore, it also follows that the matrix products $A_1 A_2^{-1}$ and $A_1 A_2$ have no negative real eigenvalues. This establishes that $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ must be Hurwitz if a CQLF exists for Σ_{A_1} and Σ_{A_2}.

(ii) On the other hand, if Σ_{A_1}, Σ_{A_2} do not have a CQLF, then it follows from Lemma 4.3.2 that there is some $\alpha > 0$ such that $\Sigma_{A_1 - \alpha I}$, Σ_{A_2} have a CQLF.

Now suppose that there is no CQLF for Σ_{A_1}, Σ_{A_2}. Then, from (ii), there is some $\alpha > 0$ such that $\Sigma_{A_1 - \alpha I}$, Σ_{A_2} have a CQLF. (See Figure 5.1.)

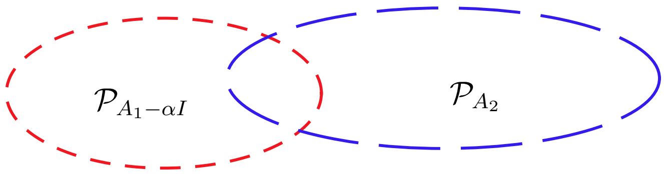

Figure 5.1: CQLF exists for $\Sigma_{A_1-\alpha I}$, Σ_{A_2}

If we now reduce α, this reduces the size of the set $\mathcal{P}_{A_1-\alpha I}$, and the size of the intersection $\mathcal{P}_{A_2} \cap \mathcal{P}_{A_1-\alpha I}$, until the intersection becomes empty and we arrive at the marginal situation shown in Figure 5.2.

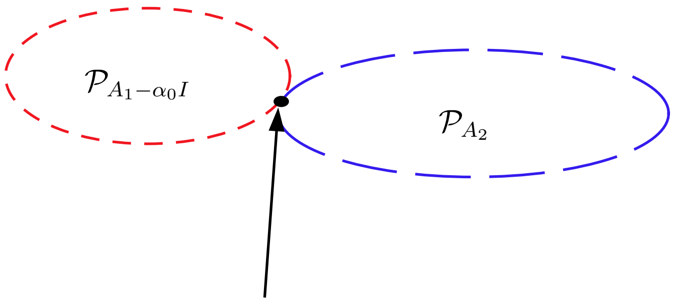

Figure 5.2: Reducing α to arrive at the limiting case

More formally, if we define

$$\alpha_0 = \inf\{\alpha > 0 : \Sigma_{A_1-\alpha I}, \Sigma_{A_2} \text{ have a CQLF } \},$$

then it follows that $\alpha_0 \geq 0$ and that the two systems $\Sigma_{A_1-\alpha_0 I}$, Σ_{A_2} satisfy the hypotheses of Theorem 4.4.1. Loosely speaking, as we increase α, α_0 is the value occurring *immediately* before $\Sigma_{A_1-\alpha I}$ and Σ_{A_2} have a CQLF. Essentially, we are observing that at this point, Theorem 4.4.1 applies.

It now follows immediately from Theorem 4.4.1 that one of the matrix pencils $\sigma_{\gamma[0,\infty)}[A_1-\alpha_0 I, A_2]$ or $\sigma_{\gamma[0,\infty)}[A_1-\alpha_0 I, A_2^{-1}]$ is singular. In the first case, there is some $\gamma > 0$ and some $x \neq 0$ in $\mathbb{R}^2$ such that $(A_1+\gamma A_2)x = \alpha_0 x$. Hence the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is not Hurwitz in this case. In the second case, there is some $\gamma > 0$ and some non-zero vector x in $\mathbb{R}^2$ such that $(A_1+\gamma A_2^{-1})x = \alpha_0 x$. Hence in this case, the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is not Hurwitz.

To summarize the conclusions of the above discussion, we have shown that for two Hurwitz matrices A_1, A_2 in $\mathbb{R}^{2\times 2}$:

(a) if $\Sigma_{A_1}, \Sigma_{A_2}$ have a CQLF then both of the pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ are Hurwitz;

(b) if Σ_{A_1}, Σ_{A_2} do not have a CQLF, then at least one of $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is not Hurwitz.

This then establishes the following necessary and sufficient conditions for two second order LTI systems to have a CQLF.

Theorem 5.2.1 *Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{2\times 2}$. Then Σ_{A_1}, Σ_{A_2} have a CQLF if and only if both of the matrix pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$, $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ are Hurwitz.*

Moreover, it has been shown in [133] that for A_1, A_2 Hurwitz and in $\mathbb{R}^{2\times 2}$, the matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is Hurwitz if and only if the matrix product $A_1 A_2^{-1}$ has no negative real eigenvalues. This leads immediately to the following corollary to Theorem 5.2.1. The form of the conditions given in Corollary 5.2.1 below are more readily checked than the matrix pencil conditions of Theorem 5.2.1.

Corollary 5.2.1 *Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{2\times 2}$. Then Σ_{A_1}, Σ_{A_2} have a CQLF if and only if the matrix products $A_1A_2^{-1}$ and A_1A_2 have no negative real eigenvalues.*

Comments:

(i) In [133], as well as the matrix product conditions, an alternative algebraic method of checking whether or not two or more second order systems have a CQLF is also described. However, the geometrical approach that we have taken, based on Theorem 4.4.1, leads directly to the simple and meaningful conditions for CQLF existence given in Theorem 5.2.1.

(ii) The argument of this section highlights the key factor that determines that the conditions for CQLF existence for second order systems take the simple form that they do. For two second order LTI systems Σ_{A_1}, Σ_{A_2} in the marginal situation of Figure 5.2, the rank of $A_i^TP + PA_i \leq 0$ must be 1 for $i = 1, 2$, and Theorem 4.4.1 applies. Any system class to which Theorem 4.4.1 can be applied in a similar manner may admit similarly simple conditions for CQLF existence.

(iii) The above results give us two ways to check whether or not a CQLF exists for two LTI systems Σ_{A_1}, Σ_{A_2}. Firstly, we can plot the eigenvalues of $(1-\alpha)A_1 + \alpha A_2$ and $(1-\alpha)A_1 + \alpha A_2^{-1}$ for $0 \leq \alpha \leq 1$, and check if both of the eigenvalue loci lie entirely within the open left half plane. Alternatively, we can use Corollary 5.2.1 and calculate the eigenvalues of the matrix products A_1A_2, $A_1A_2^{-1}$, checking that neither product has a negative real eigenvalue.

5.2.2 Discrete-time systems

We shall next use Theorem 4.4.2 to derive necessary and sufficient conditions for a pair of second order discrete-time LTI systems to have a CQLF. The technique used to obtain these conditions is virtually identical to that employed above in the continuous-time case.

Consider a pair of Schur matrices A_1, A_2 in $\mathbb{R}^{2\times 2}$ and the associated discrete-time LTI systems $\Sigma^d_{A_1}$, $\Sigma^d_{A_2}$. Throughout the following argument, for a Schur matrix A, $C(A)$ denotes the bilinear transform of A given by (3.34). To begin with, we note the following simple facts.

(i) It follows from Lemma 4.3.3 that if $\Sigma^d_{A_1}$, $\Sigma^d_{A_2}$ have a CQLF, then the two matrix pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$ and $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ are Hurwitz, and hence non-singular. Furthermore, it also follows that the matrix products $C(A_1)C(A_2)^{-1}$ and $C(A_1)C(A_2)$ have no real negative eigenvalues.

(ii) On the other hand, if $\Sigma^d_{A_1}$, $\Sigma^d_{A_2}$ do not have a CQLF, then it follows from Lemma 4.3.4 that there is some $\alpha > 0$ such that $\Sigma^d_{C^{-1}(C(A_1)-\alpha I)}$, $\Sigma^d_{A_2}$ have a CQLF.

Now suppose that there is no CQLF for $\Sigma^d_{A_1}$, $\Sigma^d_{A_2}$. Then it follows from (ii) above that there is some $\alpha > 0$ such that there is a CQLF for $\Sigma^d_{C^{-1}(C(A_1)-\alpha I)}$ and $\Sigma^d_{A_2}$. For the same reasons as outlined in the continuous-time case, if we define

$$\alpha_0 = \inf\{\alpha > 0 : \Sigma^d_{C^{-1}(C(A_1)-\alpha I)}, \Sigma^d_{A_2} \text{ have a CQLF } \},$$

then it follows that $\Sigma^d_{C^{-1}(C(A_1)-\alpha_0 I)}$ and $\Sigma^d_{A_2}$ satisfy the hypotheses of Theorem 4.4.2. Thus either $\sigma_{\gamma[0,\infty)}[C(A_1)-\alpha_0 I, C(A_2)]$ is singular or $\sigma_{\gamma[0,\infty)}[C(A_1)-\alpha_0 I, C(A_2)^{-1}]$ is singular. In the first case, it follows that the matrix pencil $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$ is not Hurwitz, while in the second case we have that the pencil $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$

is not Hurwitz. Summarizing the above discussion we have the following result on CQLF existence for pairs of discrete-time LTI systems.

Theorem 5.2.2 *Let A_1, A_2 be Schur matrices in $\mathbb{R}^{2\times 2}$. Then the discrete-time LTI systems $\Sigma^d_{A_1}$, $\Sigma^d_{A_2}$ have a CQLF if and only if both of the matrix pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$, $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ are Hurwitz.*

As in the continuous-time case, we may combine this result with the fact that for two Hurwitz matrices A_1, A_2 in $\mathbb{R}^{2\times 2}$, the matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is Hurwitz if and only if the matrix product $A_1A_2^{-1}$ has no negative real eigenvalues and obtain the following corollary.

Corollary 5.2.2 *Let A_1, A_2 be Schur matrices in $\mathbb{R}^{2\times 2}$. Then the discrete-time LTI systems $\Sigma^d_{A_1}$, $\Sigma^d_{A_2}$ have a CQLF if and only if both of the matrix products $C(A_1)C(A_2)$, $C(A_1)C(A_2)^{-1}$ have no negative real eigenvalues.*

Comments:

> Necessary and sufficient conditions for a pair of second order discrete-time LTI systems to have a CQLF have been previously reported in [2]. The conditions given there are also stated in terms of matrix pencils, and can be checked either by plotting a root locus, or by direct algebraic calculations. As for continuous-time systems, through the use of Theorem 4.4.2 we have been able to derive conditions for CQLF existence in a geometric and intuitive manner. Furthermore, the proof that we have given here indicates why the simple conditions of Theorem 5.2.2 hold. As with continuous-time systems, the key point is that in the marginal situation of Figure 5.2 the rank of $A_i^TPA_i - P$ must be 1 for $i = 1, 2$, thereby allowing us to apply Theorem 4.4.2.

5.2.3 Examples

Finally, for this section, we present some numerical examples to illustrate the results derived above.

Two systems with a CQLF:

Example 5.2.1 *Consider the second order LTI systems* Σ_{A_1}, Σ_{A_2} *where*

$$A_1 = \begin{pmatrix} 16.5603 & -31.8756 \\ 36.2990 & -25.4635 \end{pmatrix}, A_2 = \begin{pmatrix} -1.1144 & -3.1126 \\ -0.9722 & -3.9614 \end{pmatrix}.$$

The eigenvalue loci of $(1-\alpha)A_1 + \alpha A_2$ *and* $(1-\alpha)A_1 + \alpha A_2^{-1}$ *for* $0 \leq \alpha \leq 1$ *are given in figures 5.3 and 5.4 below.*

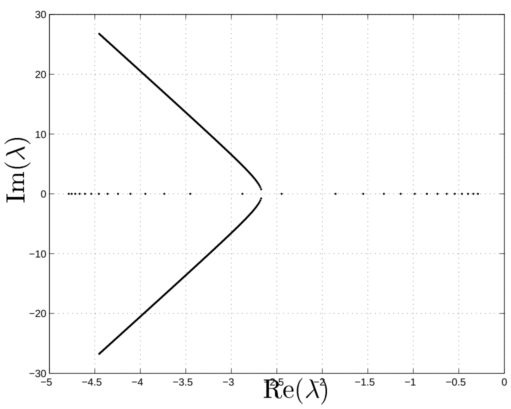

Figure 5.3: Eigenvalue locus of $(1-\alpha)A_1 + \alpha A_2$

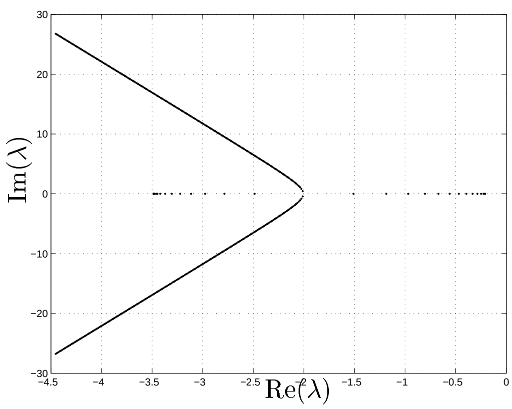

Figure 5.4: Eigenvalue locus of $(1-\alpha)A_1 + \alpha A_2^{-1}$

From the above Figures, it is clear that neither locus crosses into the right half plane. Hence, it follows from Theorem 5.2.1 that there is a CQLF for Σ_{A_1} *and* Σ_{A_2}.

Alternatively, if we calculate the eigenvalues of the products A_1A_2, $A_1A_2^{-1}$, *we find that*

$$\begin{aligned} \sigma(A_1A_2) &= \{0.2113 + 31.9527j,\ 0.2113 - 31.9527j\} \\ \sigma(A_1A_2^{-1}) &= \{16.1217 + 16.4233j,\ 16.1217 - 16.4233j\}. \end{aligned}$$

Here, $\sigma(A)$ represents the spectrum of the matrix A. It follows from Corollary 5.2.1 that Σ_{A_1}, Σ_{A_2} have a CQLF.

Two systems with no CQLF:

Example 5.2.2 *Consider the second order LTI systems Σ_{A_1}, Σ_{A_2} where*

$$A_1 = \begin{pmatrix} 8.2012 & 21.5591 \\ -16.1443 & -8.8490 \end{pmatrix}, A_2 = \begin{pmatrix} -48.8335 & 7.6504 \\ -8.2792 & -1.5512 \end{pmatrix}.$$

The eigenvalue loci of $(1-\alpha)A_1 + \alpha A_2$ and $(1-\alpha)A_1 + \alpha A_2^{-1}$ for $0 \leq \alpha \leq 1$ are given in figures 5.5 and 5.6 below.

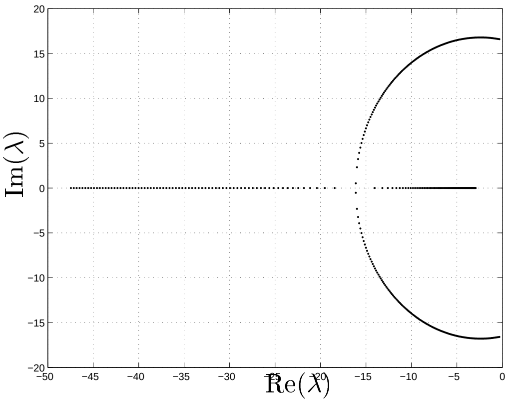

Figure 5.5: Eigenvalue locus of $(1-\alpha)A_1 + \alpha A_2$

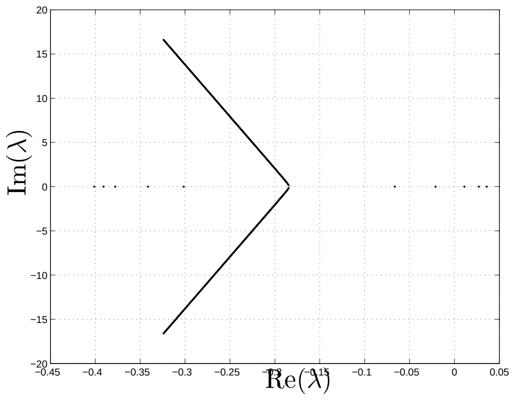

Figure 5.6: Eigenvalue locus of $(1-\alpha)A_1 + \alpha A_2^{-1}$

From Figure 5.6, we can see that $(1-\alpha)A_1 + \alpha A_2^{-1}$ has eigenvalues in the right half plane for some values of α between 0 and 1. Thus it follows from Theorem 5.2.1 that Σ_{A_1}, Σ_{A_2} do not have a CQLF.

Alternatively, if we calculate the eigenvalues of the products A_1A_2, $A_1A_2^{-1}$, we find that

$$\begin{aligned} \sigma(A_1A_2) &= \{-627.7259, -61.0403\} \\ \sigma(A_1A_2^{-1}) &= \{0.4151, 4.7716\}. \end{aligned}$$

Thus, $A_1 A_2$ has two negative real eigenvalues, and it follows from Corollary 5.2.1 that Σ_{A_1}, Σ_{A_2} do not have a CQLF.

5.3 Pairs of systems differing by rank one

The two most significant classes of systems for which necessary and sufficient conditions for CQLF existence are known are given by second order systems, considered in the previous section, and systems whose system matrices are in companion form. For the latter case, the classical SISO Circle Criterion [97] provides a necessary and sufficient condition for CQLF existence, and recently in [128], a novel time-domain formulation of this result has been developed. This time-domain version of the condition was discussed in Chapter 3 where it was presented as Theorem 3.5.4. Our aim in this section is to show that the result of Theorem 3.5.4 extends to pairs of exponentially stable LTI systems Σ_{A_1}, Σ_{A_2} where $A_2 - A_1$ is a general rank one matrix, thereby relaxing the assumption that the matrices must be in companion form. In later sections, we shall see how this result can also be treated within the framework of the previous chapter and how Theorem 4.4.1 casts some light on why the conditions for CQLF existence take a simple form in this case.

5.3.1 Necessary and sufficient conditions for CQLF existence

Consider two Hurwitz matrices A, $A - bc^T$ in $\mathbb{R}^{n \times n}$, where b, c are column vectors in $\mathbb{R}^n$. If both of the matrices are in companion form, then Theorem 3.5.4, due to Shorten and Narendra, establishes that a necessary and sufficient condition for a CQLF to exist for the associated pair of LTI systems, Σ_A, Σ_{A-bc^T} is that the product $A(A - bc^T)$ has no negative real eigenvalues. As previously remarked, this result can be thought of as an alternative time-domain formulation of the classical SISO Circle Criterion. Furthermore, the time-domain version of the result is easier to check than its frequency-domain equivalent, and provides insight into the relationship between CQLF existence and the stability of switched linear systems. In Theorem 5.3.2 below, we use Meyer's extended form of the KYP Lemma [84] to show that the result of Theorem 3.5.4 also holds in the case of two LTI systems whose system matrices differ by a general rank one perturbation. The time-domain condition thus obtained is more compact than the equivalent frequency-domain condition for CQLF existence, and, moreover, the necessity of the condition for CQLF existence is immediate from the time-domain form of the result.

Recall from Section 3.5.2 that the Kalman Yakubovic Popov (KYP) Lemma [51, 93], stated above as Theorem 3.5.2, established that for a Hurwitz matrix A in $\mathbb{R}^{n\times n}$, and vectors b, c in $\mathbb{R}^n$ with the pair (A, b) completely controllable, the condition

$$\frac{\gamma}{2} + \mathrm{Re}\{c^T(j\omega I - A)^{-1}b\} > 0 \quad \text{for all } \omega \in \mathbb{R},$$

was sufficient for the existence of a positive definite solution P to the constrained Lyapunov equations

$$\begin{aligned} A^T P + PA &= -qq^T - \epsilon P \\ Pb - c &= \sqrt{\gamma} q, \end{aligned}$$

for a given positive ϵ and vector $q \in \mathbb{R}^n$. Using this result, in [97] Narendra and Goldwyn derived the SISO Circle Criterion, giving a frequency domain sufficient condition for a CQLF to exist for the systems Σ_A, Σ_{A-bc^T}. As stated above, Willems later established the necessity of the Circle Criterion for CQLF existence in [146]. Following on from this work, Meyer extended the KYP Lemma in [84], removing the assumption that the pair (A, b) is completely controllable. Combining the approach of Narendra and Goldwyn [97] with Meyer's results [84] yields the following sufficient condition for CQLF existence for systems differing by rank one.

Theorem 5.3.1 *Let A, $A - bc^T$ be Hurwitz matrices in $\mathbb{R}^{n\times n}$, where $b, c \in \mathbb{R}^n$. Suppose that*

$$1 + \mathrm{Re}\{c^T(j\omega I - A)^{-1}b\} > 0 \text{ for all } \omega \in \mathbb{R}. \tag{5.1}$$

Then there is a CQLF for the pair of LTI systems Σ_A, Σ_{A-bc^T}.

We shall show that the condition (5.1) is equivalent to the matrix $A(A-bc^T)$ having no negative real eigenvalues. In order to do this, we shall need the following technical lemma. For details, see [50].

Lemma 5.3.1 *Let A, $A - bc^T$ be Hurwitz matrices in $\mathbb{R}^{n\times n}$, where $b, c \in \mathbb{R}^n$. Then for any complex number s,*

$$\det(c^T(sI - A)^{-1}b) = \frac{\det(sI - (A - bc^T)) - \det(sI - A)}{\det(sI - A)}.$$

Theorem 5.3.2 *[123, 124] Let A, $A - bc^T$ be Hurwitz matrices in $\mathbb{R}^{n\times n}$, where $b, c \in \mathbb{R}^n$. Then*

$$1 + \mathrm{Re}\{c^T(j\omega I - A)^{-1}b\} > 0 \text{ for all } \omega \in \mathbb{R}$$

if and only if the matrix product $A(A - bc^T)$ has no negative real eigenvalues.

Proof: As A and $A - bc^T$ are both Hurwitz, their determinants have the same sign, and hence $\det(A(A - bc^T)) > 0$. It follows that $A(A - bc^T)$ has no negative real eigenvalues if and only if

$$\det(\lambda I + (A - bc^T)A) = \det(\lambda I + A^2 - bc^T A) > 0 \text{ for all } \lambda > 0.$$

Applying an appropriate similarity transformation if necessary, we may assume without loss of generality that the rank one matrix bc^T is in one the Jordan canonical forms

$$\text{(i) } bc^T = \begin{pmatrix} \mu & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & & & \\ 0 & \dots & \dots & 0 \end{pmatrix}, \text{ (ii) } bc^T = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 1 & \dots & \dots & 0 \\ \vdots & & & \\ 0 & \dots & \dots & 0 \end{pmatrix}. \tag{5.2}$$

Now, for bc^T in one of the above forms, it can verified by direct computation that

$$\det(\lambda I + A^2 - bc^T A) = \text{Re}\{\det(\lambda I + A^2 - bc^T A - \sqrt{\lambda} j bc^T)\},$$

for all $\lambda > 0$. Writing $\lambda = \omega^2$ it follows that, for all real ω,

$$\text{Re}\{\det(\omega^2 I + A^2 - bc^T A - j\omega bc^T)\} > 0. \tag{5.3}$$

Noting that as A is Hurwitz, $\det(\omega^2 I + A^2)$ is positive for all real ω, it follows by direct calculation that for all $\omega \in \mathbb{R}$,

$$\begin{aligned}
& \text{Re}\{\det(\omega^2 I + A^2 - bc^T A - j\omega bc^T)\} > 0 \\
\iff \quad & \text{Re}\{\det(\omega^2 I + A^2) + \det(\omega^2 I + A^2 - bc^T A - j\omega bc^T) - \det(\omega^2 I + A^2)\} > 0 \\
\iff \quad & \frac{\text{Re}\{\det(\omega^2 I + A^2) + \det(\omega^2 I + A^2 - bc^T A - j\omega bc^T) - \det(\omega^2 I + A^2)\}}{\det(\omega^2 I + A^2)} > 0 \\
\iff \quad & 1 + \frac{\text{Re}\{\det(\omega^2 I + A^2 - bc^T A - j\omega bc^T) - \det(\omega^2 I + A^2)\}}{\det(\omega^2 I + A^2)} > 0. \qquad (5.4)
\end{aligned}$$

Next, note that

$$\begin{aligned}
\det(\omega^2 I + A^2 - bc^T A - j\omega bc^T) &= \det(j\omega I - (A - bc^T))\det(-j\omega I - A) \\
\det(\omega^2 I + A^2) &= \det(j\omega I - A)\det(-j\omega I - A). \qquad (5.5)
\end{aligned}$$

Using (5.5), we see that the condition (5.4) is equivalent to

$$1 + \text{Re}\{\frac{\det(j\omega I - (A - bc^T)) - \det(j\omega I - A)}{\det(j\omega I - A)}\} > 0 \tag{5.6}$$

for all $\omega \in \mathbb{R}$. Finally Lemma 5.3.1 shows that this is equivalent to

$$1 + \text{Re}\{c^T(j\omega I - A)^{-1} b\} > 0$$

for all $\omega \in \mathbb{R}$ as claimed.

It is now relatively straightforward to combine Theorem 5.3.2 with Theorem 5.3.1 to obtain the following necessary and sufficient condition for a CQLF to exist for a pair of LTI systems Σ_{A_1}, Σ_{A_2} with $\text{rank}(A_2 - A_1) = 1$.

Theorem 5.3.3 *Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{n\times n}$ with* $\text{rank}(A_2 - A_1) = 1$. *Then the LTI systems Σ_{A_1}, Σ_{A_2} have a CQLF if and only if the matrix product A_1A_2 has no negative real eigenvalue or, equivalently, the matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is non-singular.*

Proof: If Σ_{A_1}, Σ_{A_2} have a CQLF, then it follows from Lemma 4.3.1 that the product A_1A_2 has no negative real eigenvalues. Conversely, assume that A_1A_2 has no negative real eigenvalue. Then, writing $A_1 - A_2 = bc^T$ Theorem 5.3.2 implies that $1 + \text{Re}\{c^T(j\omega I - A)^{-1}b\} > 0$ for all real ω. It then follows immediately from Theorem 5.3.1 that the systems Σ_{A_1}, Σ_{A_2} have a CQLF.

Comments:

> We have shown that the matrix $A(A - bc^T)$ having no negative real eigenvalues is equivalent to the condition (5.1), for a general rank one perturbation bc^T. While (5.1) has been known as a sufficient condition for CQLF existence for some time, it is far from straightforward to show directly that it is also necessary for CQLF existence. However, when the condition is expressed in the equivalent matrix product form, it follows immediately from Lemma 4.3.1 that it is necessary for CQLF existence. Furthermore, while the condition (5.1) requires us to check that $1 + \text{Re}\{c^T(j\omega I - A)^{-1}b\}$ is positive for *every* real value of ω, the condition of Theorem 5.3.3 only requires a single eigenvalue computation, and is thus more readily tested.

It has been mentioned before that Theorem 3.2.2, due to Loewy, may be used to extend a result on CQLF existence for a family of systems to any related family obtained by replacing various system matrices with their inverses. In the present case, we can immediately write down the following corollary to Theorem 5.3.3.

Corollary 5.3.1 *Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{n\times n}$ with $\mathrm{rank}(A_2 - A_1^{-1}) = 1$. Then the following three statements are equivalent.*

(i) The matrix product $A_1^{-1}A_2$ has no negative real eigenvalues.

(ii) The switched linear system

$$\dot{x}(t) = A(t)x(t) \quad A(t) \in \{A_1, A_2\} \tag{5.7}$$

is uniformly exponentially stable for arbitrary switching signals.

(iii) The systems Σ_{A_1}, Σ_{A_2} have a CQLF.

Proof: (i) $\Leftrightarrow$ (iii): From Theorem 3.2.2, Σ_{A_1}, Σ_{A_2} have a CQLF if and only if $\Sigma_{A_1^{-1}}$, Σ_{A_2} have a CQLF, and hence from Theorem 5.3.3, if and only if the matrix product $A_1^{-1}A_2$ has no negative real eigenvalues.

(ii) $\Leftrightarrow$ (iii): It is immediate that (iii) implies (ii). On the other hand, if there is no CQLF for Σ_{A_1}, Σ_{A_2}, it follows from the above argument that $A_1^{-1}A_2$ has a negative real eigenvalue. Hence, there is some $\gamma > 0$ such that $A_1 + \gamma A_2$ has an eigenvalue in the closed right half plane. Thus from Theorem 2.3.1, the switched linear system (5.7) is not uniformly exponentially stable under arbitrary switching. This completes the proof.

Comments:

It is important to underline that Corollary 5.3.1 provides an example of an entire class of switched linear systems for which requiring the existence of a CQLF is not a conservative criterion for exponential stability.

In fact, for two Hurwitz matrices A_1, A_2 with $\text{rank}(A_1 - A_2^{-1}) = 1$, the existence of a CQLF for Σ_{A_1}, Σ_{A_2} is equivalent to the uniform exponential stability of the associated switched system (5.7) under arbitrary switching.

Finally for this section, we note that the result of Theorem 5.3.3 can be extended slightly to apply to a larger class of systems in the following way. The key observation behind the following result is that two systems Σ_{A_1}, Σ_{A_2} have a CQLF if and only if the systems Σ_{A_1}, Σ_{cA_2} have a CQLF for any positive constant c.

Corollary 5.3.2 *Let A_1, A_2 be two Hurwitz matrices in $\mathbb{R}^{n\times n}$. Suppose that there is some $c > 0$ such that* $\text{rank}(A_2 - cA_1) = 1$. *Then a necessary and sufficient condition for there to be a CQLF for the systems Σ_{A_1}, Σ_{A_2} is that the matrix product A_1A_2 has no negative real eigenvalue.*

Proof: The necessity of the condition follows immediately from Lemma 4.3.1. Conversely, suppose that A_1A_2 has no negative real eigenvalues. Then $(cA_1)A_2$ has no negative eigenvalues either and $\text{rank}(A_2 - cA_1) = 1$. Hence from Theorem 5.3.3, Σ_{cA_1}, Σ_{A_2} have a CQLF, and thus so do Σ_{A_1}, Σ_{A_2} as claimed.

Comments:

Corollary 5.3.2 provides another example of a class of systems for which simple necessary and sufficient conditions for CQLF existence can be given. Of course, a difficulty with using the corollary is that it is necessary to check whether or not a positive constant c exists such that $\text{rank}(A_2 - cA_1) = 1$. One way of testing for the existence of such a constant is to calculate the eigenvalues of $A_1^{-1}A_2$. If this matrix has a positive real eigenvalue of geometric multiplicity $n-1$, then there is such a constant and Corollary 5.3.2 may be applied. To see this, note that there is some $c > 0$ such that $\text{rank}(A_2 - cA_1) = 1$ if and only if, for this c, $A_1^{-1}A_2 - cI$ also has rank one. This is then equivalent to c being an eigenvalue of $A_1^{-1}A_2$ of geometric multiplicity $n-1$.

5.3.2 Examples

We shall now present two numerical examples to illustrate the ideas and results of the previous subsection.

Example 5.3.1 *Consider the stable third order LTI systems Σ_A, Σ_{A-bc^T} where*

$$A = \begin{pmatrix} -47.3992 & 33.3965 & 7.7548 \\ -18.7324 & 10.5304 & -35.5976 \\ 15.3888 & 30.8884 & -22.9334 \end{pmatrix}, b = \begin{pmatrix} 0.4608 \\ 0.4574 \\ 0.4507 \end{pmatrix}, c = \begin{pmatrix} 0.4122 \\ 0.9016 \\ 0.0056 \end{pmatrix}.$$

If we calculate the eigenvalues of $A(A - bc^T)$, we find that

$$\sigma(A(A - bc^T)) = \{2668.4, -1488.7 + 327.7j, -1488.7 - 327.7j\}.$$

Thus, it follows from Theorem 5.3.3 that the systems Σ_A, Σ_{A-bc^T} have a CQLF.

Example 5.3.2 *Consider the two third order LTI systems Σ_{A_1}, Σ_{A_2} where*

$$A_1 = \begin{pmatrix} -31.0609 & -6.0655 & -1.1295 \\ 2.7560 & -43.4734 & -19.0071 \\ -9.7284 & -22.8422 & -12.3802 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -62.1219 & -12.1310 & -2.2590 \\ 5.5120 & -86.9469 & -38.0142 \\ -18.9567 & -44.9843 & -24.8604 \end{pmatrix}.$$

Then note that in this case $\mathrm{rank}(A_2 - A_1) = 3$. *However, it can be easily checked in MATLAB that the eigenvalues of* $A_1^{-1}A_2$ *are given by* $\{2.1294, 2, 2\}$, *and that the eigenspace corresponding to the eigenvalue* 2 *has dimension* 2. *Thus by the comment after Corollary 5.3.2, it follows that there is some positive constant c such that* $\mathrm{rank}(A_2 - cA_1) = 1$. *In fact,*

$$A_2 - 2A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.5 & 0.7 & -0.1 \end{pmatrix}.$$

Hence, we can apply Corollary 5.3.2 to test for a CQLF. The eigenvalues of A_1A_2 *are given by*

$$\sigma(A_1A_2) = \{12.8, 1820.3, 5884\},$$

and hence it follows from Corollary 5.3.2 that Σ_{A_1}, Σ_{A_2} *have a CQLF.*

A practical application of Theorem 5.3.3:

Note that in the recent paper [142], the result of Theorem 5.3.3 has been used to analyse the robust stability of a controller designed for four-wheel steering vehicles. Essentially, in [142] the authors consider the problem of designing a controller that manipulates the front and rear steering angles of the vehicle to track reference signals for yaw-rate and side-slip angle. The controller presented in the paper is based on the so-called single-track model of the lateral dynamics of the vehicle, and relies on the assumption of constant longitudinal speed. A key factor that has to be taken into account in the design process is that the steering angle on the rear tyres is constrained to lie within a relatively small range, effectively introducing a saturation effect into the corresponding control action. After an appropriate transformation, in order to establish the input-output stability of the overall system in [142], it is sufficient to demonstrate the asymptotic stability of the closed loop system depicted in Figure 5.7 below.

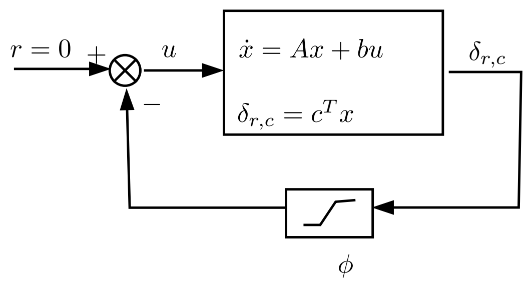

Figure 5.7: Closed loop system analysed in [142]

The entries of the matrix A in $\mathbb{R}^{11\times 11}$, and of the vectors b, c in $\mathbb{R}^{11\times 1}$ are determined by various physical parameters associated with the vehicle. It should be noted that the values of some of these entries depend on the constant longitudinal speed of the vehicle, and that all of the underlying physical parameters are subject to uncertainty. The output $\delta_{r,c}$ of the forward path is the rear steering angle that is demanded by the controller. As mentioned above, this angle is constrained to lie within a limited range and this fact is captured by the saturation non-linearity ϕ in the feedback path. The state equations for the closed loop system in Figure 5.7 can be written in the form:

$$\dot{x} = (A - k(x)bc^T)x, \tag{5.8}$$

where $k(x)$ is a non-linear function satisfying $0 < k(x) \leq 1$ for all x. It now follows that the existence of a CQLF for the two bounding LTI systems Σ_A, Σ_{A-bc^T} would guarantee the asymptotic stability of the system (5.8). But from Theorem 5.3.3, we know that if the matrices A and $A - bc^T$ are Hurwitz, then there is a CQLF for Σ_A and Σ_{A-bc^T} if and only if the matrix product $A(A - bc^T)$ has no negative real eigenvalues.

In [142], for three different fixed values of the longitudinal speed, the matrices A and $A - bc^T$ were calculated for a large number of possible values of the underlying physical parameters, and in each case:

(i) it was verified that A and $A - bc^T$ were Hurwitz;

(ii) the eigenvalue of the product $A(A - bc^T)$ closest to the negative real axis was calculated.

All of the eigenvalues calculated in (ii) were then plotted, in what can be thought of as a root-locus for the non-linear system (5.8). This plot is reproduced below in Figure 5.8 with the kind permission of M. Vilaplana.

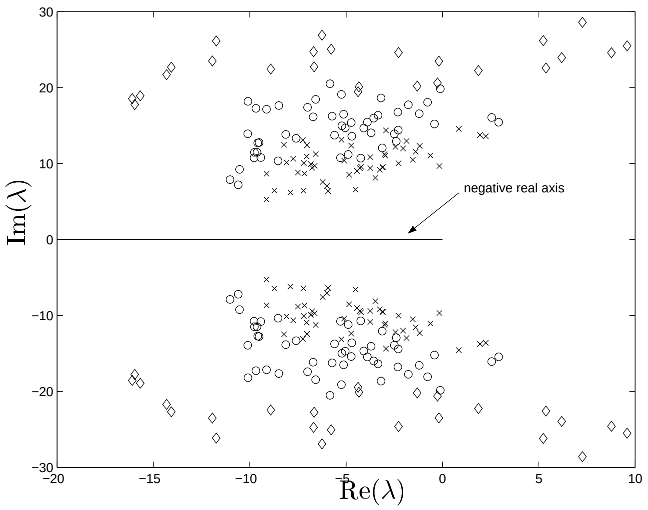

Figure 5.8: Eigenvalue plot taken from [142]

It is clear that none of the eigenvalues plotted in Figure 5.8 lie on the negative real axis. Thus, the authors of [142] were able to conclude that the system (5.8) was asymptotically stable using Theorem 5.3.3.

5.4 A new perspective on the Circle Criterion

In Section 5.2, we saw how Theorem 4.4.1 could be applied to derive necessary and sufficient conditions for CQLF existence for pairs of second order LTI systems. In this section, we shall revisit the problem of determining necessary and sufficient conditions for a CQLF to exist for a pair of LTI systems Σ_A, Σ_{A-bc^T}, whose system matrices differ by rank one. It may appear that we are merely re-deriving, in a somewhat circuitous manner, the results already established in Theorem 3.5.4 and Theorem 5.3.3. However, the presentation here is based on Theorem 4.4.1 and the methods described in the previous chapter, as opposed to the purely algebraic proof given in the last section. In fact, our main goal in this section is to show that Theorem 4.4.1 provides a general framework within which both the results for second order systems, and the SISO Circle Criterion can be unified, and to provide insight into why the condition for CQLF existence for pairs of LTI systems differing by rank one takes the simple form given in Theorem 5.3.3. We have already stated that where Theorem 4.4.1 can be applied, simple conditions for CQLF existence are likely to hold. As such, we wish to show the role played by Theorem 4.4.1 in determining conditions for CQLF existence for those classes of systems covered by the SISO Circle Criterion and Theorem 5.3.3. Moreover, we wish to provide another example of a significant system class to which Theorem 4.4.1 can be applied, and provide a further illustration of how this theorem may be used to find necessary and sufficient conditions for CQLF existence for various system classes. Incidentally, it is worth noting that for both of the system classes to which Theorem 4.4.1 is applied in this chapter, the rank of the commutator $[A_1, A_2] = A_1A_2 - A_2A_1$ is at most two. In the light of this observation, it is natural to ask whether or not it is possible to identify further system classes where the rank of $[A_1, A_2]$ is at most two, to which Theorem 4.4.1 can be applied in a similar fashion.

It is important to emphasize that while some of the proofs of this section are quite lengthy, the arguments presented provide insight into the result of Theorem 5.3.3. In fact, they indicate why the matrix product conditions for CQLF existence derived in Theorem 5.3.3 hold. As in the case of second order systems, a key role is played by the rank of the matrices Q_i appearing in the statement of Theorem 4.4.1.

5.4.1 Two Lemmas on rank-one perturbed matrices

In this short subsection, we present two technical lemmas concerned with pairs of Hurwitz matrices that differ by rank one. Both of these are needed in order to show the role played by Theorem 4.4.1 in determining necessary and sufficient conditions for CQLF existence for pairs of systems whose system matrices differ by rank one.

On a number of occasions so far, we have seen that the properties of the matrix pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$, $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ can play an important role in determining whether or not a CQLF exists for Σ_{A_1}, Σ_{A_2}. The following lemma, which is taken from [61], shows that if $\mathrm{rank}(A_2 - A_1) = 1$, then one of these pencils cannot be singular.

Lemma 5.4.1 *Let $A, A - bc^T$ be Hurwitz matrices in $\mathbb{R}^{n\times n}$. Then the matrix product $A^{-1}(A - bc^T)$ has no negative real eigenvalues. Equivalently, the matrix pencil $\sigma_{\gamma[0,\infty)}[A, A - bc^T]$ is non-singular.*

Lemma 5.4.2 *[123] Let $A, A - bc^T$ be Hurwitz matrices in $\mathbb{R}^{n\times n}$ with b, c in $\mathbb{R}^n$. Suppose that for some $\lambda_0 > 0$, the matrix product $A(A - \lambda_0 bc^T)$ has a negative real eigenvalue (the pencil $\sigma_{\gamma[0,\infty)}[A^{-1}, A - \lambda_0 bc^T]$ is singular). Then for all real $\lambda > \lambda_0$, the product $A(A - \lambda bc^T)$ also has a negative real eigenvalue (the pencil $\sigma_{\gamma[0,\infty)}[A^{-1}, A - \lambda bc^T]$ is singular).*

<u>**Comments:**</u>

From Lemma 4.3.2, it follows that for sufficiently small values of $\lambda > 0$, the matrix product $A(A-\lambda bc^T)$ will not have a negative real eigenvalue. Lemma 5.4.2 states that as we increase λ, if at any point $A(A-\lambda bc^T)$ has a negative real eigenvalue, then it will continue to do so for all larger values of λ also. The proof of Lemma 5.4.2 is given in Appendix B.

5.4.2 Systems in companion form

To begin with, we assume that we are given two Hurwitz matrices A, $A - bc^T$ in $\mathbb{R}^{n\times n}$ in companion form. Thus we may write

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, c = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{pmatrix}. \tag{5.9}$$

The following preliminary points are important for the work of this section.

(i) It follows from the Circle Criterion (Theorem 3.5.3) that there is a CQLF for Σ_A, Σ_{A-bc^T} if and only if

$$\Gamma(\omega) = 1 + \mathrm{Re}\{c^T(j\omega I - A)^{-1}b\} > 0 \tag{5.10}$$

for all $\omega \in \mathbb{R}$.

(ii) Furthermore, if we define

$$\lambda_c = \sup\{\lambda > 0 : \Sigma_A \text{ and } \Sigma_{A-\lambda bc^T} \text{ have a CQLF }\},$$

then (provided $\lambda_c < \infty$) it follows by continuity that

$$\Gamma_c(\omega) = 1 + \mathrm{Re}\{\lambda_c c^T(j\omega I - A)^{-1}b\} \geq 0,$$

for all real ω and $\Gamma_c(\omega_0) = 0$ for at least one $\omega_0 \in \mathbb{R}$.

(iii) Later on, we shall need to know how the coefficients of the numerator polynomial of the rational function $\Gamma(.)$ are related to the entries of A and b. It has been pointed out in [51], and in Chapter 10 of [118], that for A, b, c as in (5.9),

$$c^T(sI - A)^{-1}b = \frac{c_0 + c_1 s + \ldots + c_{n-1}s^{n-1}}{\det(sI - A)}, \tag{5.11}$$

for all s in $\mathbb{C}$. From this it follows that we can write

$$\Gamma(\omega) = \frac{p(\omega)}{\det(\omega^2 I + A^2)}$$

where p is a monic even polynomial [1] in ω of degree $2n$. Furthermore, as $\det(sI - A) = a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n$, we may write

$$p(\omega) = \det(\omega^2 I + A^2) + p_1(\omega)$$

where

$$\begin{aligned} p_1(\omega) &= c_0 a_0 + (-c_0 a_2 + c_1 a_1 - c_2 a_0)\omega^2 \\ &+ (c_0 a_4 - c_1 a_3 + c_2 a_2 - c_3 a_1 + c_4 a_0)\omega^4 + \ldots \\ &+ (-c_{n-2} + c_{n-1}a_{n-1})w^{2n-2}. \end{aligned} \tag{5.12}$$

Note that we can identify any monic even polynomial of degree $2n$ with the vector in $\mathbb{R}^n$ formed by the coefficients of $\omega^0, \omega^2, \ldots, \omega^{2n-2}$. If we do this, it then follows from (5.12) that for a given $A \in \mathbb{R}^{n\times n}$ in companion form, the relationship between the entries of the vector c and the polynomial p is described by the affine mapping (from $\mathbb{R}^n$ to $\mathbb{R}^n$)

$$T(c) = \Theta(A) + L(A)c \tag{5.13}$$

[1] A polynomial is monic if its leading coefficient is equal to one. An even polynomial in ω only contains even powers of ω.

where $\Theta(A)$ is a vector that depends on the entries of A and $L(A)$ is the linear map given by the matrix (in $\mathbb{R}^{n\times n}$)

$$L(A) = \begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -a_2 & a_1 & -a_0 & 0 & 0 & \dots & 0 & 0 \\ a_4 & -a_3 & a_2 & -a_1 & a_0 & \dots & 0 & 0 \\ \vdots & \vdots & & & & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & a_{n-1} \end{pmatrix} \tag{5.14}$$

(iv) It is important to note that the determinant of $L(A)$ is not independent of the entries of A. For instance, the product term $a_0a_1a_2\dots a_{n-1}$ only appears once in the expression for the determinant. For this reason, if we consider $\det(L(A))$ as a polynomial in $a_0,\dots,a_{n-1}$, it is not uniformly zero. Thus, for any companion matrix A in $\mathbb{R}^{n\times n}$ such that $L(A)$ is singular, it is possible to find another matrix A', also in companion form, arbitrarily close to A with $L(A')$ invertible, simply by perturbing the entries $a_0, a_1, \dots, a_{n-1}$ appropriately.

The case of $L(A)$ invertible:

We first consider a pair of Hurwitz matrices A, $A - bc^T$ in companion form such that $L(A)$ is invertible. From point (iv) above, it is clear that this is not an overly restrictive assumption to make. Furthermore, we shall later see how to remove this assumption and derive a result for general pairs of Hurwitz companion matrices.

In Theorem 5.4.1 below, we show the relevance of Theorem 4.4.1 in the present context. Specifically, we consider a pair of LTI systems in the marginal situation depicted in Figure 4.3, and demonstrate that the hypotheses of Theorem 4.4.1 are generically satisfied under our current assumptions.

Theorem 5.4.1 *[123] Let $A, A - bc^T$ be Hurwitz matrices in $\mathbb{R}^{n\times n}$, where $A, b, c \in \mathbb{R}^n$ are in the form of (5.9), and $L(A)$ is invertible. Assume that there is no CQLF* *for Σ_A and Σ_{A-bc^T}. Furthermore, suppose that there is a CQLF for Σ_A and $\Sigma_{A-\lambda bc^T}$ for all real λ with $0 < \lambda < 1$. Then given any $\epsilon > 0$, there is some vector $c' \in \mathbb{R}^n$ with $\|c - c'\| < \epsilon$ for which there exists a matrix $P = P^T \geq 0$ satisfying*

$$\begin{aligned} A^T P + PA &= Q_1 \leq 0 \quad \mathrm{rank}(Q_1) = n - 1 \\ (A - bc'^T)^T P + P(A - bc'^T) &= Q_2 \leq 0 \quad \mathrm{rank}(Q_2) = n - 1. \end{aligned}$$

The proof of this result is extremely long and technical so, rather than including it at this point, it is presented in Appendix B.

Comments:

> The previous result indicates that, in a sense, Theorem 4.4.1 applies generically in the present context. Specifically, suppose that we have two LTI systems Σ_A, Σ_{A-bc^T}, where A, $A - bc^T$ are Hurwitz, and A, b, c are in the form (5.9). From Lemma 4.3.2, it follows that for small enough values of t, Σ_A and Σ_{A-tbc^T} have a CQLF. Consider the smallest value t_0 of t for which the systems Σ_A, $\Sigma_{A-t_0bc^T}$ do not have a CQLF. We can think of this as being the point at which a CQLF fails to exist for the systems as we vary t. Then Theorem 5.4.1 shows that even if the hypotheses of Theorem 4.4.1 are not satisfied at this point, there exists a matrix A' and a vector c', arbitrarily close to the original A and c, such that when a CQLF fails to exist for $\Sigma_{A'}$, $\Sigma_{A'-tbc'^T}$ (as we vary t), Theorem 4.4.1 can be applied.

We are now in a position to derive a necessary and sufficient condition for a pair of LTI systems Σ_A, Σ_{A-bc^T} to have a CQLF, where A and $A - bc^T$ are in companion form and the matrix $L(A)$ is invertible.

Theorem 5.4.2 *[123] Let A, $A - bc^T$ be two Hurwitz matrices in companion form in $\mathbb{R}^{n\times n}$ where b, c are column vectors in $\mathbb{R}^n$. Assume that the matrix $L(A)$ defined* *by (5.14) is non-singular. Then a necessary and sufficient condition for a CQLF to exist for the systems Σ_A, Σ_{A-bc^T} is that the matrix product $A(A-bc^T)$ has no negative real eigenvalues or equivalently, that the matrix pencil $\sigma_{\gamma[0,\infty)}[A^{-1}, A-bc^T]$ is non-singular.*

Proof: If there is a CQLF for the systems Σ_A, Σ_{A-bc^T}, then it follows from Lemma 4.3.1 that the product $A(A-bc^T)$ has no negative real eigenvalue.

Conversely, suppose there is no CQLF for Σ_A, Σ_{A-bc^T}. Then it follows from Lemma 4.3.2 that for small enough values of $\lambda > 0$, the systems Σ_A, $\Sigma_{A-\lambda bc^T}$ will have a CQLF. Define $\lambda_c = \sup\{\lambda > 0 : \Sigma_A$ and $\Sigma_{A-\lambda bc^T}$ have a CQLF $\}$. Then $\lambda_c \leq 1$ and Σ_A and $\Sigma_{A-\lambda_c bc^T}$ satisfy the conditions of Theorem 5.4.1.

It now follows from Theorem 5.4.1, Theorem 4.4.1 and Lemma 5.4.1 that for any $\epsilon > 0$ there exists a vector c' in $\mathbb{R}^n$ such that:

(i) $\|\lambda_c c - c'\| < \epsilon$;

(ii) $A(A - bc'^T)$ has a negative real eigenvalue.

It now follows from the continuous dependence of the eigenvalues of a matrix on its entries that, in the limit as ϵ tends to zero, $A(A-\lambda_c bc^T)$ has a negative real eigenvalue. Finally, Lemma 5.4.2 implies that $A(A-bc^T)$ also has a negative real eigenvalue.

Extension to the case of (potentially) singular $L(A)$:

We now show how to extend Theorem 5.4.2 to the case where the matrix $L(A)$ may be singular. With this aim in mind, let A, $A-bc^T$ in $\mathbb{R}^{n\times n}$ be Hurwitz matrices in companion form such that $L(A)$ is singular. It follows immediately from Lemma 4.3.1 that if the systems Σ_A, Σ_{A-bc^T} have a CQLF, the matrix product $A(A-bc^T)$ has no negative real eigenvalues.

Conversely, suppose that there is no CQLF for Σ_A, Σ_{A-bc^T}. Then as mentioned in the comments at the beginning of this subsection, it is possible to find an arbitrarily small perturbation, δA, on the entries of A that will make $L(A+\delta A)$ non-singular. More formally, it is possible to find a vector δc in $\mathbb{R}^n$, of arbitrarily small norm, such that for the companion matrix $A+b(\delta c)^T$, $L(A+b(\delta c)^T)$ is non-singular. Thus, the polynomial in t given by $\det(L(A+tb(\delta c)^T))$ is not uniformly zero. In fact, there can be at most finitely many values of t for which $L(A+tb(\delta c)^T)$ is singular. It follows from this that there exists some real number $r>0$ such that for $0<|t|<r$, $A+tb(\delta c)^T$ is Hurwitz and the matrix $L(A+tb(\delta c)^T)$ is non-singular.

Now, for any t, if Σ_{A-bc^T} and $\Sigma_{A+tb(\delta c)^T}$ have a CQLF, then Σ_{A-bc^T} and $\Sigma_{A-tb(\delta c)^T}$ will definitely *not* have a CQLF. Thus, it must happen that either Σ_{A-bc^T} and $\Sigma_{A+tb(\delta c)^T}$ have no CQLF or that Σ_{A-bc^T} and $\Sigma_{A-tb(\delta c)^T}$ have no CQLF.

It follows from the above argument that there exists a matrix A' in $\mathbb{R}^{n\times n}$ arbitrarily close to A such that:

(i) A' is Hurwitz, in companion form, and $L(A')$ is invertible;

(ii) There is no CQLF for the systems $\Sigma_{A'}$ and Σ_{A-bc^T}.

But we can then apply Theorem 5.4.2 to deduce that $A'(A-bc^T)$ has a negative eigenvalue. As such a matrix A' can be found arbitrarily close to the original A, it follows from the continuous dependence of the eigenvalues of a matrix upon its entries that $A(A-bc^T)$ has a negative real eigenvalue also. This then establishes that the conclusion of Theorem 5.4.2 also holds when the matrix $L(A)$ is singular.

5.4.3 Extension to general systems differing by rank one

We have now given a proof based on Theorem 4.4.1 that $A(A-bc^T)$ having no negative real eigenvalues is necessary and sufficient for CQLF existence for Σ_A, Σ_{A-bc^T}

when A, $A - bc^T$ are Hurwitz and in companion form, indicating the role played by Theorem 4.4.1 in determining the simple conditions for CQLF existence in this case. A key factor in our derivation was that the hypotheses of Theorem 4.4.1 were satisfied by a significant class of pairs of matrices in companion form. To finish this chapter, we show how to extend the above work to show that the condition is also necessary and sufficient for CQLF existence in the case of a general pair of Hurwitz matrices differing by rank one. To do this, we shall make use of the following standard lemmas.

Lemma 5.4.3 *Let A, B be Hurwitz matrices in $\mathbb{R}^{n\times n}$. Suppose that*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix},$$

where A_1, B_1 are in $\mathbb{R}^{p\times p}$, and A_3, B_3 are in $\mathbb{R}^{(n-p)\times(n-p)}$ for some p, $1 \le p < n$. If both pairs of systems (Σ_{A_1}, Σ_{B_1}) and $(\Sigma_{A_3}, \Sigma_{B_3})$ have a CQLF, then Σ_A and Σ_B have a CQLF.

Lemma 5.4.4 *[84] Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz and let b be a vector in $\mathbb{R}^n$. Then there exists a non-singular S in $\mathbb{R}^{n\times n}$ such that*

$$SAS^{-1} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad Sb = \begin{pmatrix} b_1 \\ 0 \end{pmatrix},$$

where $A_1 \in \mathbb{R}^{p\times p}$, $A_2 \in \mathbb{R}^{n-p\times p}$, $A_3 \in \mathbb{R}^{(n-p)\times(n-p)}$, $b_1 \in \mathbb{R}^p$ and the pair (A_1, b_1) is completely controllable ([118]).

As the pair A_1, b_1 in Lemma 5.4.4 is completely controllable, it follows that for any $c_1 \in \mathbb{R}^p$, it is possible to simultaneously transform A_1, $A_1 - b_1c_1^T$ simultaneously into companion form ([51]). It therefore follows from Lemma 5.4.4 that given a pair of

Hurwitz matrices A, $A - bc^T$ in $\mathbb{R}^{n\times n}$, it is possible to find a non-singular S in $\mathbb{R}^{n\times n}$ such that

$$SAS^{-1} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad S(A - bc^T)S^{-1} = \begin{pmatrix} B_1 & B_2 \\ 0 & A_3 \end{pmatrix}, \tag{5.15}$$

where A_1, and B_1 are both in companion form.

Now suppose that A, $A - bc^T$ are Hurwitz matrices in $\mathbb{R}^{n\times n}$ such that $A(A - bc^T)$ has no negative real eigenvalues. Choose some non-singular S such that SAS^{-1}, $S(A - bc^T)S^{-1}$ are in the form (5.15). Then it follows by direct computation that A_1B_1 has no negative real eigenvalues. Thus, from the arguments of the last subsection, we can conclude that there exist positive definite matrices $P_1 \in \mathbb{R}^{p\times p}$ and $P_2 \in \mathbb{R}^{(n-p)\times(n-p)}$ such that

$$\begin{aligned} A_1^T P_1 + P_1 A_1 &< 0 \\ B_1^T P_1 + P_1 B_1 &< 0 \\ A_3^T P_2 + P_2 A_3 &< 0. \end{aligned}$$

It now follows immediately from Lemma 5.4.3 that there is a CQLF for Σ_A, Σ_{A-bc^T}. This shows that $A(A - bc^T)$ having no negative real eigenvalues is a sufficient condition for Σ_A, Σ_{A-bc^T} to have a CQLF. The necessity of the condition is immediate from Lemma 4.3.1.

5.5 Concluding remarks

In this chapter, we have demonstrated how the ideas and results of Chapter 4 can be used to derive necessary and sufficient conditions for CQLF existence for certain system classes. In addition, we have shown that the key result of Theorem 4.4.1 provides a unifying framework for the known conditions for CQLF existence for pairs of second order LTI systems and the classical SISO Circle Criterion. The main contributions of the chapter are now listed.

- Insightful proofs were given for the known necessary and sufficient conditions for CQLF existence for pairs of second order LTI systems in both continuous-time and discrete-time. The proofs given here indicated why the conditions take a simple form in the second order case, with the fact that Theorem 4.4.1 can be applied playing a crucial role.

- A recently obtained result [128] giving verifiable conditions for CQLF existence for pairs of LTI systems in companion form was extended to the case of a pair of exponentially stable LTI systems Σ_{A_1}, Σ_{A_2}, whose system matrices differ by a general rank one matrix. Specifically, we have shown that for two such systems, there is a CQLF if and only if the product $A_1 A_2$ has no negative real eigenvalues.

- The previous result was further extended to the case of a pair of LTI systems Σ_{A_1}, Σ_{A_2} for which there is some positive constant c with $\text{rank}(A_1 - cA_2) = 1$. A simple means of testing for the existence of such a constant c was also described.

- Based on these results, a class of switched linear systems for which CQLF existence is not a conservative way of establishing exponential stability under arbitrary switching was identified.

- A large part of the work of the chapter was taken up with highlighting the role played by Theorem 4.4.1 in determining the simple matrix product conditions for CQLF existence for pairs of LTI systems with system matrices differing by rank one. This work illustrates how Theorem 4.4.1 unifies two of the major results previously available on the CQLF existence problem in the literature, and indicates that Theorem 4.4.1 has a key role to play in determining system classes for which simple conditions for CQLF existence can be found.

Chapter 6

Positive switched linear systems and their stability

We consider several problems related to the stability of positive switched linear systems. First of all, a number of results on CQLF existence for positive LTI systems are given, and a further class of switched linear systems for which CQLF existence is equivalent to uniform exponential stability is described. The problem of common diagonal Lyapunov function (CDLF) existence is then considered, and some simple sufficient conditions for CDLF existence are presented. Using similar techniques to those developed in Chapters 4 and 5, we derive a necessary and sufficient condition for a generic pair of n-dimensional positive LTI systems to have a CDLF. The final problem considered in the chapter is that of copositive Lyapunov function existence. A simple result on common quadratic copositive Lyapunov function existence is presented. We also derive simple, verifiable necessary and sufficient conditions for a pair of second order stable positive LTI systems to have a common linear *copositive Lyapunov function, and derive a theoretical condition that is necessary and sufficient for the existence of such a function for a general pair of stable positive LTI systems (of arbitrary dimension).*

6.1 Introductory remarks

There are many examples of practically important dynamical systems where negative values of their state variables are physically meaningless and for which, effectively, the state vector of the system can only take on non-negative values. Systems of this type arise in ecology, demographics, economics, hydrology, pharmaceutics, biology, and in the study of communication systems [32, 134, 6, 39, 83]. In fact, in any situation where the state measures such quantities as population sizes, chemical concentrations, buffer or window sizes, or commodity prices, it is clear that only non-negative values are meaningful. A similar, and very important, example of this type is provided by stochastic dynamical systems, where the quantities that evolve over time are probabilities. In each of these cases, the state dynamics are constrained, in the sense that any trajectory originating in the non-negative orthant cannot leave it.

A system of the above type, where any state trajectory starting from non-negative initial conditions must remain non-negative for all subsequent times, is known as a *positive system*. The theory of positive linear time-invariant systems is by now well-developed [32, 72], with close connections to the theory of non-negative matrices. Unsurprisingly, given their close ties with non-negative matrices, positive LTI systems have a number of properties that set them apart as a subclass of LTI systems worthy of independent study, and a number of these properties have important implications for the stability analysis of these systems. However, recent applications in areas such as congestion control of the Internet [121], and formation flying [46], have highlighted the need to extend the theory of positive LTI systems to time-varying positive linear systems and, in particular, to positive switched linear systems, where by a *positive* switched linear system, we mean one all of whose constituent systems are positive LTI systems. As with general switched linear systems, several basic questions relating to positive switched linear systems and, in particular, their stability, are as yet unresolved. In this chapter, we shall focus on several problems that arise in the consideration of the stability issue for positive switched linear systems, and present a number of initial results on the stability of these systems.

To begin with, we review the most relevant results from the theory of positive LTI systems, and show that for systems constructed by switching between a number of positive LTI systems, stability is once again a critical issue whose resolution appears to be far from straightforward. As in previous chapters, our approach to the stability analysis of these systems is based on the search for common Lyapunov functions. Firstly, we present some results on the CQLF existence problem for positive LTI systems, indicating where possible the precise link between CQLF existence and exponential stability for positive switched linear systems.

While our analysis thus far has been almost exclusively based on CQLFs, some of the properties of positive LTI systems naturally give rise to the consideration of other types of common Lyapunov function when investigating the stability of positive switched linear systems. One such property is that a positive LTI system is exponentially stable if and only if it has a quadratic Lyapunov function that is given by a *diagonal* quadratic form. We shall refer to such a function as a diagonal Lyapunov function from now on. In the light of this fact, it is possible to obtain stability conditions for positive switched linear systems through investigating the existence of common diagonal Lyapunov functions (CDLFs) for families of positive LTI systems. Hence, the question of determining conditions for two or more exponentially stable positive LTI systems to have a CDLF arises naturally in this context. Much of the work of the present chapter is concerned with this problem. In particular, we first present results that give simple, verifiable sufficient conditions for CDLF existence for families of positive LTI systems in both continuous-time and discrete-time. Furthermore, we shall show how the ideas and techniques described in Chapters 4 and 5 can again be successfully applied to derive an algebraic condition that is necessary and sufficient for CDLF existence for pairs of n-dimensional positive LTI systems. To the best of this author's knowledge, this is the first result giving necessary and sufficient conditions for CDLF existence for n-dimensional positive systems. That the same techniques can again be applied in this context is further evidence of their power in gaining insights into the question of the existence of common Lyapunov functions.

Another class of Lyapunov functions that arises when considering positive switched linear systems is that of the so-called *copositive Lyapunov functions*, which we shall discuss in the final section of the chapter. The main reason for considering these functions is that, given that the trajectories of a positive system are constrained to remain within the non-negative orthant, the global requirements of traditional Lyapunov functions may be overly restrictive. For copositive Lyapunov functions, the usual properties of a Lyapunov function are only imposed for values of the state vector within the non-negative orthant. The existence of a common copositive Lyapunov function for a family of exponentially stable positive LTI systems is sufficient for the exponential stability of the associated positive switched linear system. In the final section of this chapter, we present a number of results on the existence of common copositive Lyapunov functions for positive LTI systems. In particular, we shall consider both linear and quadratic copositive Lyapunov functions, and shall derive a necessary and sufficient condition for a pair of exponentially stable positive LTI systems to have a common linear copositive Lyapunov function. Once again, the techniques previously applied to the CQLF existence problem shall be used to investigate this question.

6.2 Background on positive linear systems

Our main aim in this section is to describe those parts of the theory of positive LTI systems and non-negative matrices that are most relevant to the work of the remainder of the chapter. To begin with, we shall set some notation. For a vector $x \in \mathbb{R}^n$, and a matrix A in $\mathbb{R}^{n \times n}$, x_i denotes the i^{th} component of x and a_{ij} is the (i, j) entry of A. Throughout this chapter, the following notation is adopted:

(i) for $x \in \mathbb{R}^n$, $x \succ 0$ ($x \succeq 0$) means that $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$;

(ii) for $A \in \mathbb{R}^{n \times n}$, $A \succ 0$ ($A \succeq 0$) means that $a_{ij} > 0$ ($a_{ij} \geq 0$) for $1 \leq i, j \leq n$;

(iii) for $x, y \in \mathbb{R}^n$, $x \succ y$ ($x \succeq y$) means that $x - y \succ 0$ ($x - y \succeq 0$);

(iv) finally for $A, B \in \mathbb{R}^{n \times n}$, $A \succ B$ ($A \succeq B$) means $A - B \succ 0$ ($A - B \succeq 0$).

A matrix $A \in \mathbb{R}^{n \times n}$ is said to be non-negative if $A \succeq 0$, and positive if $A \succ 0$. It is important to appreciate that when we say that a matrix is non-negative (positive), this refers to the entries of the matrix, and that it is a different concept to the property of non-negative (positive) definiteness for symmetric matrices.

Also, for a matrix A in $\mathbb{R}^{n \times n}$, $\rho(A)$ denotes the spectral radius of A and $\mu(A)$ denotes the maximal real part of any eigenvalue of A. Note that A is Hurwitz if and only if $\mu(A) < 0$.

Positive LTI systems, non-negative and Metzler matrices:

We now introduce the mathematical definitions of continuous-time and discrete-time positive LTI systems.

Definition 6.2.1 *The LTI system*

$$\Sigma_A : \dot{x} = Ax \quad x(t_0) = x_0$$

is positive if $x_0 \succeq 0 \Rightarrow x(t) \succeq 0$ *for all* $t \geq t_0$.

Comments:

It is well-known [32, 72] that the LTI system Σ_A is positive if and only if all of the off-diagonal elements of the matrix A are non-negative. Such matrices are known as *Metzler* matrices. Furthermore, the LTI system Σ_A is positive if and only if the matrix exponential e^{At} is non-negative for all $t \geq 0$ [49]. Thus we have that A is a Metzler matrix if and only if e^{At} is non-negative for all $t \geq 0$. Note also that a matrix A is Metzler if and only if its transpose is also Metzler.

It is important to realize that the property of positivity is co-ordinate dependent and hence is not invariant under similarity transformations of the system matrix A.

The corresponding definition of positive LTI systems in discrete-time is now given.

Definition 6.2.2 *The discrete-time LTI system*

$$\Sigma_A^d : x(j+1) = Ax(j) \quad x(j_0) = x_0$$

is positive if $x_0 \succeq 0 \Rightarrow x(j) \succeq 0$ *for all* $j \geq j_0$.

Comments:

It is easy to see that the discrete-time LTI system Σ_A^d is positive if and only if the matrix A is non-negative. As with continuous-time systems, it should be kept in mind that the property of a system being positive is not co-ordinate independent.

Before moving on, we note that there is a close connection between Metzler matrices and non-negative matrices. Specifically, any Metzler matrix A in $\mathbb{R}^{n\times n}$ can be written (non-uniquely of course) as $A = N - \alpha I$, where $N \succeq 0$ and I is the identity matrix in $\mathbb{R}^{n\times n}$. Moreover, $\mu(A) = \rho(N) - \alpha$, and A is Hurwitz if and only if $\alpha > \rho(N)$.

Irreducibility, Perron eigenvalues and eigenvectors:

Positive and non-negative matrices, and their generalizations, have been studied extensively within the mathematics literature for some time, and a number of textbooks giving background on this area are now available [7, 119]. Perhaps the most fundamental, and classical, fact about matrices with non-negative entries is the result that the spectral radius of a non-negative matrix is itself an eigenvalue of the matrix. In fact it is possible to say slightly more than this for positive matrices and *irreducible* non-negative matrices. This notion of irreducibility, which will play an important role later in this chapter, will now be defined.

A permutation matrix $P \in \mathbb{R}^{n\times n}$ is a matrix with exactly one entry in each row and column equal to one and all other entries zero. The reason for calling such matrices permutation matrices is that the effect of multiplying a vector by such a matrix is to permute the entries of the vector among themselves. Furthermore, for such matrices $P^T = P^{-1}$ and the effect of the similarity transformation $A \to PAP^T$ is to permute the rows and columns of the matrix A in the same manner. In particular, the diagonal elements of A are permuted among themselves, and thus, for any diagonal matrix D, and permutation matrix P, PDP^T is also diagonal.

Now, a matrix $A \in \mathbb{R}^{n\times n}$ is said to be *reducible* [42] if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ and some r with $1 \leq r < n$ such that PAP^T has the form

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \tag{6.1}$$

where $A_{11} \in \mathbb{R}^{r\times r}, A_{22} \in \mathbb{R}^{(n-r)\times(n-r)}, A_{12} \in \mathbb{R}^{r\times(n-r)}$ and 0 is the zero matrix in $\mathbb{R}^{(n-r)\times r}$. If a matrix is not reducible, then it is *irreducible*. It should be noted that it often occurs in the theory of non-negative matrices that results established for positive matrices can be extended to irreducible non-negative matrices also. The following well-known result, which was originally derived for positive matrices, is an example of this phenomenon [42].

Theorem 6.2.1 *Suppose $A \in \mathbb{R}^{n\times n}$ is irreducible and non-negative. Then*

(i) $\rho(A) > 0$ is an eigenvalue of A of algebraic multiplicity one;

(ii) there is some vector $x \succ 0$ in $\mathbb{R}^n$ such that $Ax = \rho(A)x$.

This theorem guarantees that the eigenspace of A corresponding to $\rho(A)$ is one-dimensional, and that there is an eigenvector x, corresponding to $\rho(A)$, each component of which is positive.

Given their close relationship with non-negative matrices, it is not surprising that Metzler matrices have a number of properties analogous to those given by Theorem 6.2.1. Before stating the next result, note that the matrix A is irreducible if and only if $A - \alpha I$ is irreducible for all $\alpha > 0$.

Theorem 6.2.2 *Let $A = N - \alpha I \in \mathbb{R}^{n\times n}$ be Metzler and irreducible. Then*

(i) $\mu(A) = \rho(N) - \alpha$ is an eigenvalue of A of algebraic (and geometric) multiplicity one;

(ii) there is an eigenvector $x \succ 0$ with $Ax = \mu(A)x$.

Semi-positivity and diagonal Lyapunov functions:

Ultimately, our goal is to derive stability criteria for switched linear systems that arise as a result of switching between families of stable positive LTI systems. For this reason, we shall mostly be concerned with positive LTI systems whose system matrices are both Metzler and Hurwitz. Matrices of this form are intimately related to the class of so-called M-matrices [7, 43]; in fact, a matrix A is Metzler and Hurwitz if and only if $-A$ is an M-matrix. Corresponding to similar conditions for M-matrices, there are a large number of equivalent conditions for a Metzler matrix to be Hurwitz. Some of the most relevant of these for our purposes are listed in the following theorem. For details, see [7, 43].

Theorem 6.2.3 *Let $A \in \mathbb{R}^{n \times n}$ be Metzler. Then the following statements are equivalent.*

(i) *A is Hurwitz.*

(ii) *There is some vector $v \succ 0$ in $\mathbb{R}^n$ such that $-Av \succ 0$.*

(iii) *$-A^{-1}$ is non-negative.*

(iv) *There is some positive definite diagonal matrix D such that $A^T D + DA < 0$. Moreover, it has been shown in [139] that if v, w are vectors in $\mathbb{R}^n$ such that*

(a) $v, w \succ 0$

(b) $-Av \succ 0, -A^T w \succ 0,$

then the positive definite diagonal matrix,

$$D = diag\{w_1/v_1, w_2/v_2, \ldots, w_n/v_n\},$$

satisfies

$$A^T D + DA < 0.$$

<u>**Comments:**</u>

Item (ii) in Theorem 6.2.3 establishes that for any Metzler, Hurwitz matrix A in $\mathbb{R}^{n\times n}$, there is some vector v in the positive orthant of $\mathbb{R}^n$ such that $(-A)v$ is also in the positive orthant. Matrices with this property are known as *semi-positive* matrices [7]. Thus, the above result shows that a Metzler matrix A is Hurwitz if and only if $-A$ is semi-positive. Later in the chapter, when we come to consider common diagonal Lyapunov functions and common linear copositive Lyapunov functions, the question of determining when $-A_1,\dots,-A_k$ are jointly, or simultaneously, semi-positive, for Metzler, Hurwitz matrices $A_1,\dots,A_k$ arises. Formally, this problem amounts to asking whether there exists a single vector $v \succ 0$ such that $-A_i v \succ 0$ for $1 \le i \le k$.

The matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$:

In several of the earlier chapters we have seen that the matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ often plays a key role in the question of CQLF existence for the LTI systems Σ_{A_1}, Σ_{A_2}. Furthermore, the conditions for CQLF existence for pairs of second order systems were simplified owing to the fact that, in that case, the non-singularity of the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ was equivalent to its Hurwitz stability (provided A_1, A_2 are both Hurwitz). The next lemma records the fact that a similar result holds for Metzler Hurwitz matrices, and has been pointed out in [43].

Lemma 6.2.1 *Let A_1, A_2 be Hurwitz Metzler matrices in $\mathbb{R}^{n\times n}$. Then the matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is Hurwitz if and only if it is non-singular. Equivalently $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is Hurwitz if and only if $A_1 A_2^{-1}$ has no negative real eigenvalues.*

Proof: It is immediate that if $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is Hurwitz, then it is non-singular also. Conversely, suppose that for some $\gamma > 0$, $A_1 + \gamma A_2$ is not Hurwitz. Then for this γ, $\mu(A_1 + \gamma A_2) \ge 0$. By continuity, there must be some $\gamma_0 > 0$ for which

$\mu(A_1 + \gamma_0 A_2) = 0$, and thus, as $A_1 + \gamma_0 A_2$ is Metzler, it has 0 as an eigenvalue, and the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is singular. This completes the proof of the lemma.

Switched linear systems and stability:

Before proceeding to discuss a variety of Lyapunov-based techniques for establishing the stability of positive switched linear systems, it should first be shown that stability is again an issue for such systems. Specifically, it is important to demonstrate in the present context that it is possible for an unstable trajectory to result from switching between stable positive LTI systems, given *non-negative* initial conditions. The following example shows that this is indeed the case.

Example 6.2.1 *Consider the two stable second order positive LTI systems* $\Sigma_{A_1}, \Sigma_{A_2}$ *where*

$$A_1 = \begin{pmatrix} -0.2472 & 0.0580 \\ 0.7983 & -0.3354 \end{pmatrix}, A_2 = \begin{pmatrix} -0.4892 & 0.6525 \\ 0.4252 & -0.7118 \end{pmatrix}.$$

Now if we start from the initial state vector $x(0) = (0.5608, 0.828)^T$, *and apply the switching rule that* $A(t) = A_2$ *for* $2j \leq t < 2j+1$, $A(t) = A_1$ *for* $2j+1 \leq t < 2(j+1)$ *for* $j = 0, 1, \ldots$, *we obtain the divergent trajectory shown in Figure 6.1 below. (Here we are switching every second.)*

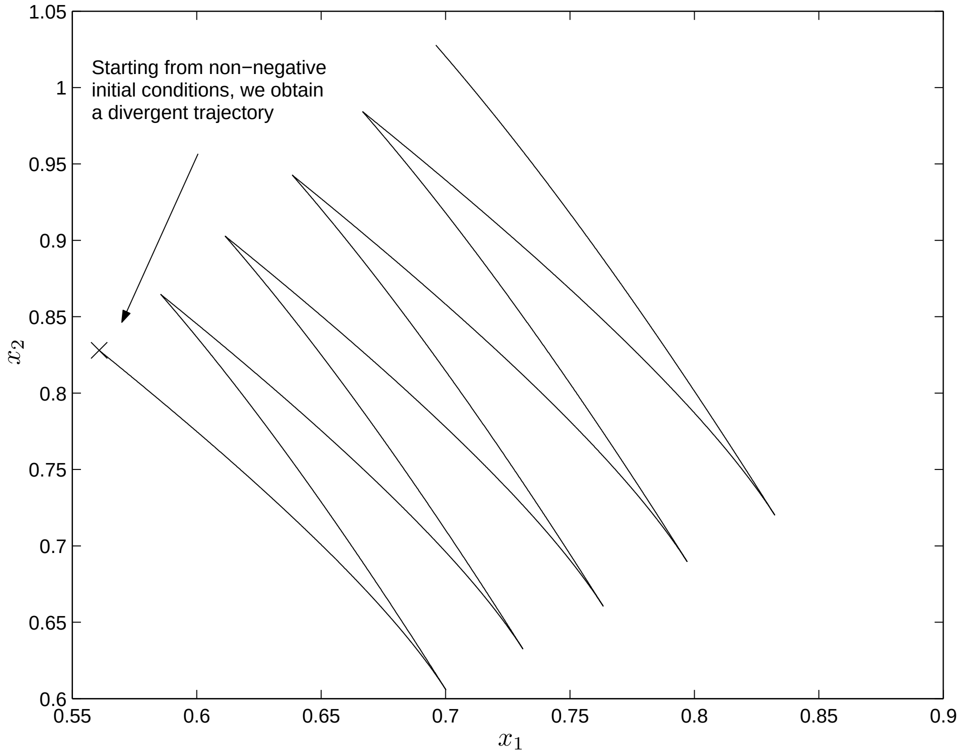

Figure 6.1: Switching and instability for positive switched linear systems

It is also possible to show analytically that this switching rule together with these initial conditions together lead to a trajectory that diverges to infinity. In fact, using MATLAB, it can be shown that $(0.5608, 0.828)^T$ *is an eigenvector of the matrix* $e^{A_1}e^{A_2}$ *with corresponding eigenvalue* 1.0442*. Thus, with the initial conditions and switching sequence defined above, after two seconds the state of the system will be* $(1.0442)x(0)$*, and after* $2k$ *seconds (k switching periods) the state vector will be* $(1.0442)^k x(0)$*. (Over each switching period the state is 'stretched' by a factor of* 1.0442*.) Hence the norm of the state vector diverges to infinity as claimed.*

6.3 Positive linear systems and CQLFs

In the example presented at the end of the previous section we have seen that it is possible for a system constructed by switching between two exponentially stable positive LTI systems to become unstable, even if we only consider initial conditions that lie within the positive orthant. As with general switched linear systems, this observation gives rise to the problem of determining conditions on a family of positive LTI systems that would guarantee the exponential stability of the associated positive switched linear system for arbitrary switching signals. As is the case for general switched linear systems, the existence of a CQLF for a family, $\Sigma_{A_1}, \dots, \Sigma_{A_k}$, of positive LTI systems assures the uniform exponential stability of the associated switched linear system. In this section, we begin our investigation of the stability properties of positive switched linear systems by considering the problem of CQLF existence for sets of exponentially stable positive LTI systems. In particular, we shall present results on CQLF existence for pairs of second order and third order positive LTI systems.

6.3.1 Second order systems

Experimentation with examples and numerical simulation often play key roles in developing insights into a new theoretical problem that is being studied. In the case of the CQLF existence problem for pairs of positive LTI systems, extensive numerical testing, together with a number of preliminary results, had led us to conjecture that a necessary and sufficient condition for a pair of exponentially stable $n-$dimensional positive LTI systems, Σ_{A_1}, Σ_{A_2}, to have a CQLF is that $A_1 A_2^{-1}$ has no negative real eigenvalues. An equivalent condition would then be that the matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ was non-singular, and hence, by Lemma 6.2.1, Hurwitz.

Initially this conjecture arose from considering the marginal situation of two positive

LTI systems Σ_{A_1}, Σ_{A_2} for which there is no CQLF but for which there exists a simultaneous positive semi-definite solution $P = P^T \geq 0$ to the inequalities

$$A_i^T P + PA_i = Q_i \leq 0 \quad \text{for } i = 1, 2.$$

Numerical testing had indicated that in this situation, the matrices Q_i were of rank $n-1$, suggesting that Theorem 4.4.1 could again be applied to obtain necessary and sufficient conditions for CQLF existence in this case. Had this conjecture been true, it would have followed from Theorem 2.3.1 that CQLF existence for a pair of exponentially stable positive LTI systems, Σ_{A_1}, Σ_{A_2}, was equivalent to the exponential stability of the associated positive switched linear system under arbitrary switching. This conjecture, together with some preliminary results that we shall later discuss was presented in [76]. While we shall prove in Theorem 6.3.1 below that the conjecture is true for pairs of second order positive LTI systems, unfortunately it is in general untrue for higher order systems as is shown by the following counterexample.

Example 6.3.1 *Consider the two Hurwitz Metzler matrices $A_1, A_2 \in \mathbb{R}^{3\times 3}$*

$$A_1 = \begin{pmatrix} -0.5302 & 0.0012 & 0.0873 \\ 0.2185 & -0.7494 & 0.5411 \\ 0.7370 & 0.1543 & -0.3606 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -0.5136 & 0.4419 & 0.3689 \\ 0.1840 & -0.3951 & 0.0080 \\ 0.3163 & 0.6099 & -1.0056 \end{pmatrix}.$$

The eigenvalues of the product $A_1 A_2^{-1}$ are given by $\{-0.8217 + 0.0835i, -0.8217 - 0.0835i, 2.2234\}$. However the systems Σ_{A_1}, Σ_{A_2} have no CQLF. In fact, it can be shown using the MATLAB LMI toolbox that there are positive semi-definite matrices P_1, P_2 satisfying $A_1^T P_1 + P_1 A_1 + A_2^T P_2 + P_2 A_2 > 0$. It then follows from (3.11) that *there is no CQLF for Σ_{A_1}, Σ_{A_2} [12]. Thus the condition that $A_1A_2^{-1}$ has no negative eigenvalues is not sufficient for CQLF existence.*

Note that it has recently been shown in [38] that the condition $A_1A_2^{-1}$ having no negative real eigenvalues also fails to be sufficient for the exponential stability of the positive switched linear system

$$\dot{x} = A(t)x, \quad A(t) \in \{A_1, A_2\}.$$

While the above example shows that the conjecture of [76] is not true for general pairs of positive LTI systems, we shall see later in the chapter how the reasoning, based on Theorem 4.4.1, that led us to the conjecture can be successfully applied to the problem of determining necessary and sufficient conditions for CDLF existence for pairs of general positive LTI systems. First of all, we show using Theorem 5.2.1 that the conjecture is indeed true for pairs of second order positive LTI systems.

Theorem 6.3.1 *Let A_1, A_2 in $\mathbb{R}^{2\times 2}$ be Metzler and Hurwitz. Then the following statements are equivalent.*

(i) The matrix product $A_1A_2^{-1}$ has no negative real eigenvalues;

(ii) the systems Σ_{A_1}, Σ_{A_2} have a CQLF;

(iii) the positive switched linear system

$$\dot{x} = A(t)x \quad A(t) \in \{A_1, A_2\}, \tag{6.2}$$

is uniformly exponentially stable for arbitrary switching signals.

Proof: We shall prove the theorem by showing that (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii): The key point here is to show that A_1A_2 cannot have a negative real eigenvalue and then apply Theorem 5.2.1. Now, as A_1, A_2 are Metzler, it follows by direct calculation that the matrix product A_1A_2 has the sign pattern

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}.$$

That is, if we write $B = A_1A_2$, then $b_{ij} \leq 0$ for $i \neq j$ and $b_{ii} > 0$, for $i, j = 1, 2$. Furthermore as A_1 and A_2 are Hurwitz and in $\mathbb{R}^{2\times 2}$, their determinants must both be positive. In particular, it follows that:

(a) $\mathrm{trace}(A_1A_2) > 0$;

(b) $\det(A_1A_2) > 0$.

Now if A_1A_2 has a negative real eigenvalue, it would follow from (b) that both of its eigenvalues were negative. However, this would then imply that the trace of A_1A_2 was negative, contradicting (a). Hence, A_1A_2 cannot have a negative real eigenvalue.

So, assume that $A_1A_2^{-1}$ has no negative real eigenvalues. Then it follows from the above argument that A_1A_2 and $A_1A_2^{-1}$ have no negative real eigenvalues. Theorem 5.2.1 now implies that there is a CQLF for Σ_{A_1}, Σ_{A_2} and that (ii) holds.

(ii)$\Rightarrow$(iii): This is standard and follows from Theorem 2.3.3.

(iii)$\Rightarrow$(i): If the switched linear system (6.2) is uniformly exponentially stable, it follows from Theorem 2.3.1 that the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ must be Hurwitz. Hence, from Lemma 6.2.1 the product $A_1A_2^{-1}$ has no real negative eigenvalues. This completes the proof.

Comments:

It follows from Theorem 6.3.1 that for switched linear systems constructed by switching between a pair of positive second order LTI systems, CQLF existence is not a conservative criterion for uniform exponential stability under arbitrary switching. In fact, the theorem states that for such systems, CQLF existence is equivalent to uniform exponential stability for arbitrary switching signals.

In the proof of Theorem 6.3.1, we have shown that for two Metzler Hurwitz matrices A_1, A_2 in $\mathbb{R}^{2\times 2}$, the matrix product $A_1 A_2$ cannot have any negative real eigenvalues, or, equivalently, the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ must be non-singular. One consequence of this is that any pair of exponentially stable second order positive LTI systems, whose system matrices differ by rank one, will have a CQLF. This is stated in the corollary below. Later, we shall show that a similar result also holds for third order systems.

Corollary 6.3.1 *Let A_1, A_2 be Hurwitz Metzler matrices in $\mathbb{R}^{2\times 2}$ such that* $\mathrm{rank}(A_2 - A_1) = 1$. *Then the LTI systems Σ_{A_1}, Σ_{A_2} have a CQLF.*

Proof: As $\mathrm{rank}(A_2 - A_1) = 1$, it follows from Lemma 5.4.1 that $A_1 A_2^{-1}$ has no negative real eigenvalues. Theorem 6.3.1 now immediately implies that there is a CQLF for Σ_{A_1}, Σ_{A_2}.

Finally, for this subsection, we present a simple numerical example to illustrate the use of Theorem 6.3.1.

Example 6.3.2 *Consider the Metzler, Hurwitz matrices A_1, A_2 in $\mathbb{R}^{2\times 2}$ where*

$$A_1 = \begin{pmatrix} -1.0655 & 0.4398 \\ 0.9943 & -0.8963 \end{pmatrix}, A_2 = \begin{pmatrix} -0.4131 & 0.5915 \\ 0.3932 & -0.6585 \end{pmatrix}.$$

Then, the eigenvalues of the matrix product $A_1 A_2^{-1}$ are given by

$$\sigma(A_1 A_2^{-1}) = \{5.4941, \ \ 2.3907\}.$$

It now follows immediately from Theorem 6.3.1 that the positive LTI systems Σ_{A_1}, Σ_{A_2} have a CQLF.

6.3.2 Third order systems

We have seen above in the proof of Theorem 6.3.1 that for Metzler Hurwitz matrices A_1, A_2 in $\mathbb{R}^{2\times 2}$, the product A_1A_2 cannot have any negative real eigenvalues. This observation led to the simplified condition for CQLF existence for pairs of positive second order LTI systems, given in Theorem 6.3.1, as well as to the fact that any pair of stable positive second order LTI systems, whose system matrices differ by rank one, has a CQLF. We shall now show that this same result about the eigenvalues of the matrix product A_1A_2 holds also when A_1, A_2 are Metzler and Hurwitz matrices in $\mathbb{R}^{3\times 3}$. This fact has a similar consequence for pairs of third order positive LTI systems with system matrices differing by rank one, and for the stability of the associated positive switched linear systems. The main fact that is needed for proving the results of this subsection is the following inequality. In the proof of Theorem 6.3.2, we use the notation $|A|$ to denote the determinant of the matrix A.

Theorem 6.3.2 *Let A_1, $A_2 \in \mathbb{R}^{3\times 3}$ be Metzler and Hurwitz, and let $\gamma > 0$ be any positive real number. Then* $\det(A_1A_2 + \gamma I) > \det(A_1A_2)$.

Proof: If we write $B = A_1A_2$, then the following facts can be easily verified.

(i) $\det(B) > 0$;

(ii) $b_{ii} > 0$ for $1 \leq i \leq 3$;

(iii) $B^{-1} = A_2^{-1}A_1^{-1} \succeq 0$.

From (i) and (iii), it follows that, if we write B_{ii} for the principal sub-matrix of B obtained by removing its i^{th} row and column, then $\det(B_{ii}) \geq 0$ for $1 \leq i \leq 3$.

Now consider

$$\det(B+\gamma I) = \begin{vmatrix} b_{11}+\gamma & b_{12} & b_{13} \\ b_{21} & b_{22}+\gamma & b_{23} \\ b_{31} & b_{32} & b_{33}+\gamma \end{vmatrix}. \tag{6.3}$$

As the determinant is a multi-linear function of the columns of a matrix, we can expand (6.3) using the first column to see that

$$\det(B+\gamma I) = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22}+\gamma & b_{23} \\ b_{31} & b_{32} & b_{33}+\gamma \end{vmatrix} + \gamma \begin{vmatrix} b_{22}+\gamma & b_{23} \\ b_{32} & b_{33}+\gamma \end{vmatrix}. \tag{6.4}$$

Now, considering the first term on the right hand side of (6.4) and repeating the above process using the second column this time, we find that

$$\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22}+\gamma & b_{23} \\ b_{31} & b_{32} & b_{33}+\gamma \end{vmatrix} = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33}+\gamma \end{vmatrix} + \gamma \begin{vmatrix} b_{11} & b_{13} \\ b_{31} & b_{33}+\gamma \end{vmatrix}. \tag{6.5}$$

Finally, if we expand the first term on the right hand side of (6.5) using its third column we can see that

$$\det(B+\gamma I) = \det(B) + \gamma \left(\begin{vmatrix} b_{22}+\gamma & b_{23} \\ b_{32} & b_{33}+\gamma \end{vmatrix} + \begin{vmatrix} b_{11} & b_{13} \\ b_{31} & b_{33}+\gamma \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \right) \tag{6.6}$$

Considering the second order determinants in (6.6) in turn, it follows from points (i), (ii) and (iii) made at the beginning of the proof that

$$\begin{vmatrix} b_{22}+\gamma & b_{23} \\ b_{32} & b_{33}+\gamma \end{vmatrix} > \det(B_{11}) \geq 0,$$

and

$$\begin{vmatrix} b_{11} & b_{13} \\ b_{32} & b_{33}+\gamma \end{vmatrix} > \det(B_{22}) \geq 0.$$

It is now immediate from (6.6) that $\det(A_1A_2 + \gamma I) > \det(A_1A_2)$ as claimed.

Comments:

Theorem 6.3.2 establishes that for real $\gamma > 0$, and Metzler Hurwitz matrices A_1, A_2 in $\mathbb{R}^{3\times 3}$, $\det(A_1A_2+\gamma I)$ is always greater than $\det(A_1A_2)$. In fact, by examining the proof more closely, we can see that the function $\gamma \to \det(A_1A_2 + \gamma I)$ is an increasing function of γ for $\gamma > 0$. Formally, if $\gamma_2 > \gamma_1 > 0$, then $\det(A_1A_2 + \gamma_2 I) > \det(A_1A_2 + \gamma_1 I)$. The next corollary about matrix pencils is an immediate consequence of Theorem 6.3.2, and the fact that for two Hurwitz matrices A_1, A_2, $\det(A_1A_2) > 0$.

Corollary 6.3.2 *Let A_1, A_2 be Hurwitz Metzler matrices in $\mathbb{R}^{3\times 3}$. Then the matrix product A_1A_2 has no real negative eigenvalues. Equivalently, the pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2^{-1}]$ is non-singular.*

It is now a simple matter to combine Corollary 6.3.2 with Theorem 5.3.3 to obtain the following result on CQLF existence for pairs of third order positive LTI systems.

Theorem 6.3.3 *Let A_1, A_2 be Hurwitz Metzler matrices in $\mathbb{R}^{3\times 3}$ with $\operatorname{rank}(A_2 - A_1) = 1$. Then the LTI systems Σ_{A_1}, Σ_{A_2} have a CQLF, and the associated positive switched linear system*

$$\dot{x} = A(t)x \qquad A(t) \in \{A_1, A_2\},$$

is uniformly exponentially stable for arbitrary switching signals.

As a final point for this section, we present a simple numerical example to illustrate the result of Theorem 6.3.3.

Example 6.3.3 *Consider the Metzler, Hurwitz matrices A_1, $A_2 = A_1 + bc^T$ in $\mathbb{R}^{3\times 3}$, where*

$$A_1 = \begin{pmatrix} -1.9545 & 0.2644 & 0.2379 \\ 0.8699 & -2.2528 & 0.6458 \\ 0.9342 & 0.8729 & -1.4462 \end{pmatrix}, b = \begin{pmatrix} -0.2349 \\ -0.1873 \\ 0.1539 \end{pmatrix}, c = \begin{pmatrix} 0.3340 \\ 0.3444 \\ 0.3089 \end{pmatrix}.$$

Now, the rank of $A_2 - A_1$ is 1, so Theorem 6.3.3 guarantees the existence of a CQLF for the positive LTI systems Σ_{A_1}, Σ_{A_2}, and thus the associated positive switched linear system is exponentially stable. Also, Corollary 6.3.2 states that the product $A_1(A_1 + bc^T)$ has no negative real eigenvalues, and a calculation in MATLAB reveals that

$$\sigma(A_1(A_1 + bc^T)) = \{0.5042, \;\; 4.9846, \;\; 7.6010\}.$$

6.4 Common diagonal Lyapunov functions for positive switched linear systems

So far in this thesis, we have mainly been concerned with establishing stability criteria for switched linear systems based on the existence of CQLFs, and with investigating the theoretical aspects of the CQLF existence problem itself. In keeping with this general pattern, in the last section we presented a number of CQLF existence results for pairs of positive LTI systems, and established corresponding stability criteria for positive switched linear systems. Due to the particular properties possessed by positive linear systems, types of Lyapunov function other than CQLFs also arise naturally when considering the stability of positive switched linear systems. For instance, diagonal Lyapunov functions have historically played an important role in the analysis of positive LTI systems [134, 32, 72, 138], and it may be possible to take advantage of the fact that any exponentially stable positive LTI system has a diagonal Lyapunov function in analyzing the stability of positive switched linear systems. Specifically, these considerations give rise to the problem of finding conditions that guarantee the existence of common diagonal Lyapunov functions (CDLFs) for families of exponentially stable positive LTI systems. Of course, such conditions would also be sufficient for the exponential stability of the associated positive switched linear systems. The problem of CDLF existence shall be the main topic under consideration in this and the next section, and we shall see below that it is possible to obtain verifiable sufficient conditions for the stability of positive switched linear systems through investigating the existence of CDLFs for families of positive LTI systems. In the final section of the chapter, we shall turn our attention to another class of Lyapunov functions that are particularly suited to the analysis of positive systems; namely copositive Lyapunov functions.

Diagonal Lyapunov functions for general systems:

While our interest in diagonal Lyapunov functions stems from a desire to derive stability criteria for positive switched linear systems, it should be noted that these functions, and the question of their existence, have been the subject of a considerable amount of work in other contexts as well. In particular, diagonal Lyapunov functions have arisen in areas such as decentralized control and the so-called 'Large Scale Systems' approach [54, 134], in asynchronous computation [55], in the study of Lotka-Volterra predator-prey models [139], and in the robust stability analysis of linear systems subject to certain classes of non-linear perturbations [53]. In many of the above applications, it is important to be able to determine whether or not a given Hurwitz matrix A in $\mathbb{R}^{n\times n}$ admits a diagonal solution to the Lyapunov inequality. More formally, does there exist a positive definite diagonal D such that

$$A^T D + DA < 0. \tag{6.7}$$

Any matrix A for which such a diagonal solution exists is said to be diagonally stable. Of course if A is Metzler, this question is well understood as in this case it is known that there is a diagonal solution to (6.7) if and only if A is Hurwitz [7, 43].

However, the question of identifying diagonally stable matrices in general is far from straightforward, and to date several papers have been written on this very subject [98, 5, 26, 59, 115]. Some sufficient conditions for a matrix to be diagonally stable, as well as necessary and sufficient conditions for restricted classes of matrices (such as so-called combinatorially symmetric matrices) are now known [115]. Furthermore, other authors have obtained theoretical necessary and sufficient conditions for a given Hurwitz matrix to be diagonally stable [5, 59]. Notwithstanding the work that has been done on this problem, there is a marked shortage of algebraic, verifiable conditions that are necessary and sufficient for a general Hurwitz matrix to be diagonally stable. In fact, to the best of the author's knowledge such conditions are only available for 2×2 and 3×3 matrices at the time of writing [98, 26, 59].

While the problem of characterizing diagonally stable matrices is of considerable interest and importance, we shall not be concerned with it here. Instead, we shall be focussing on determining conditions for the existence of a common diagonal Lyapunov function for pairs of stable positive LTI systems, with the aim of obtaining stability criteria for positive switched linear systems. In this context, Theorem 6.2.3 assures us that each of the individual LTI systems has a diagonal Lyapunov function. At this point, we state explicitly what is meant by the term common diagonal Lyapunov function.

Common diagonal Lyapunov functions for positive systems:

Let $A_1, \dots, A_k$ be Metzler Hurwitz matrices in $\mathbb{R}^{n\times n}$. If there exists a single positive definite diagonal matrix D in $\mathbb{R}^{n\times n}$ such that

$$A_i^T D + DA_i < 0 \quad \text{for } 1 \leq i \leq k, \tag{6.8}$$

then $V(x) = x^T Dx$ is said to be a *common diagonal Lyapunov function* (CDLF) for the associated positive LTI systems $\Sigma_{A_1}, \dots, \Sigma_{A_k}$.

In the following subsection we present a number of simple sufficient conditions for pairs of stable positive LTI systems to have a CDLF. The approach that we adopt closely follows that taken in [101] to the general CQLF existence problem, and in [102] to investigate CDLF existence for discrete-time systems.

6.4.1 Sufficient conditions for CDLF existence

In this subsection we give a number of sufficient conditions for a pair of positive LTI systems to have a CDLF. All of the conditions that are presented here are easily checkable and are based on the following lemma, which uses point (iv) of Theorem 6.2.3 to provide a simple means of ensuring the existence of a CDLF for a pair of stable positive LTI systems Σ_{A_1}, Σ_{A_2}.

Lemma 6.4.1 *Let A_1, $A_2, \ldots, A_k$ be Metzler, Hurwitz matrices in $\mathbb{R}^{n\times n}$. Suppose that there exist vectors $v \succ 0$, $w \succ 0$ in $\mathbb{R}^n$ such that*

(i) $-A_i v \succ 0$ for $1 \leq i \leq k$,

(ii) $-A_i^T w \succ 0$, for $1 \leq i \leq k$.

Then the positive definite diagonal matrix D given by

$$D = diag\{w_1/v_1, w_2/v_2, \ldots, w_n/v_n\},$$

satisfies $A_i^T D + DA_i < 0$ for $1 \leq i \leq k$.

Comments:

If we write $\mathbb{R}^n_+$ for the positive orthant of $\mathbb{R}^n$, then the lemma states that if we can find two vectors v, w in $\mathbb{R}^n_+$ such that $A_i v$ and $A_i^T w$ are in

$-\mathbb{R}^n_+$ for $i = 1, 2, \ldots, k$, then a CDLF exists for $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$. Moreover, given the vectors v, w the CDLF can be written down explicitly.

The condition for CDLF existence given in Lemma 6.4.1 naturally gives rise to the following question. Given an arbitrary family of Metzler, Hurwitz matrices, $A_1, \ldots, A_k$ in $\mathbb{R}^{n\times n}$, determine verifiable conditions on the matrices that guarantee the existence of a single vector $v \succ 0$ such that $-A_1 v \succ 0, \ldots -A_k v \succ 0$. If such conditions on $A_1, \ldots, A_k$ were known, then Lemma 6.4.1 could be used to derive corresponding conditions that are sufficient for CDLF existence. This is the approach that underlies the work of the remainder of this subsection. Later in the present chapter, we shall see that this same question also arises when considering the existence of common linear co-positive Lyapunov functions for positive LTI systems.

Theorem 3.6.1, due to Narendra and Balakrishnan, established that a family of stable LTI systems with commuting system matrices has a CQLF. Note that it follows from Lemma 6.4.1 that a corresponding result also holds for CDLF existence for families of stable positive LTI systems. Specifically, if $A_1, \ldots, A_k$ are Hurwitz, Metzler matrices in $\mathbb{R}^{n\times n}$, and $A_iA_j = A_jA_i$ for $1 \leq i, j \leq k$, then there is a CDLF for the positive LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$. The corresponding result for discrete-time systems has already been noted in [102].

At this point, it is possible to use Lemma 6.4.1 to immediately write down one simple condition on Metzler Hurwitz matrices $A_1, \ldots, A_k$ in $\mathbb{R}^{n\times n}$ that is sufficient for the existence of a CDLF for $\Sigma_{A_1}, \ldots, , \Sigma_{A_k}$. To do so, we need the notions of row-wise and column-wise diagonal dominance [43]. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be row-wise diagonally dominant if for $1 \leq i \leq n$, $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$. For a Metzler matrix, this means that all of the row sums of A must be negative. Column-wise diagonal dominance is defined in a similar manner. Formally, $A \in \mathbb{R}^{n\times n}$ is said to be column-wise diagonally dominant if for $1 \leq j \leq n$, $|a_{jj}| > \sum_{i=1,i\neq j}^{n} |a_{ij}|$. As before, for A Metzler, this means that all of the column sums of A (and hence the row sums of A^T) are negative.

The following sufficient condition for CDLF existence is now an immediate consequence of Lemma 6.4.1.

Corollary 6.4.1 *Let $A_1, \ldots, A_k$ be Metzler, Hurwitz matrices in $\mathbb{R}^{n\times n}$. Suppose that A_i is row-wise and column-wise diagonally dominant for $i = 1, \ldots, k$. Then there is a CDLF for the associated LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$. In fact, the square of the usual Euclidean norm, $V(x) = x^T x$, defines a CDLF for $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$.*

In Section 3.8, we noted that the existence of a CQLF for a pair of LTI systems Σ_{A_1}, Σ_{A_2} was equivalent to a condition on the Lyapunov operators $\mathcal{L}_{A_1}, \mathcal{L}_{A_2}$. Specifically, $\Sigma_{A_1}, \Sigma_{A_2}$ have a CQLF if and only if there exists some positive definite $P \in \mathbf{P}_n$ such that $\mathcal{L}_{A_1}\mathcal{L}_{A_2}^{-1}(P) \in \mathbf{P}_n$. The next lemma records a similar fact about the existence of a common $v \succ 0$ such that $-A_i v \succ 0$, $i = 1, 2$, for two Metzler Hurwitz matrices A_1, A_2.

Lemma 6.4.2 *Let A_1, A_2 be Metzler, Hurwitz matrices in $\mathbb{R}^{n\times n}$. Then there exists a vector $v \succ 0$ in $\mathbb{R}^n$ such that $-A_i v \succ 0$ for $i = 1, 2$ if and only if there is some $w \succ 0$ such that $A_1 A_2^{-1} w \succ 0$.*

Proof: Firstly suppose that there exists $w \succ 0$ such that $A_1 A_2^{-1} w \succ 0$. Let $v = -A_2^{-1} w$. Then because $-A_2^{-1} \succeq 0$, it follows that $v \succ 0$. Moreover, $-A_1 v = A_1 A_2^{-1} w \succ 0$ and $-A_2 v = w \succ 0$.

Conversely, assume that there exists some $v \succ 0$ such that $-A_1 v \succ 0$ and $-A_2 v \succ 0$, and put $w = -A_2 v \succ 0$. Then $A_1 A_2^{-1} w = -A_1 v \succ 0$. This completes the proof.

In Section 3.8 we discussed a number of sufficient conditions for CQLF existence for pairs of LTI systems due to Ooba and Funahashi [101, 99]. These conditions were expressed in terms of the Lyapunov operators $\mathcal{L}_A$, and were based on the observation that the existence of a CQLF for Σ_{A_1}, Σ_{A_2} is equivalent to the existence of a positive definite $P \in \mathbf{P}_n$ such that $\mathcal{L}_{A_1}\mathcal{L}_{A_2}^{-1}(P) \in \mathbf{P}_n$. A similar approach, based on Lemma 6.4.2 and Lemma 6.4.1, is taken in Theorem 6.4.1 below to derive a number of simple sufficient conditions for a pair of stable positive LTI systems Σ_{A_1}, Σ_{A_2} to have a CDLF. First of all, we state the following technical lemma which will be needed in the proof of Theorem 6.4.1.

Lemma 6.4.3 *Let $A \in \mathbb{R}^{n\times n}$ be given. Suppose that $A + A^T > 0$. Then there exists some $v \succ 0$ in $\mathbb{R}^n$ such that $Av \succ 0$*

Proof: It follows from a standard so-called *Theorem of the Alternative* for convex cones, [7], that exactly one of the following two possibilities must be true.

(i) There is some $v \succ 0$ such that $Av \succ 0$;

(ii) There is some $w \succeq 0$, $w \neq 0$, such that $-A^T w \succeq 0$.

Thus, if there is no $v \succ 0$ with $Av \succ 0$, then there must be some non-zero $w \succeq 0$ such that $(A^T w)^T v \leq 0$ for all $v \succeq 0$. In particular, it follows that $w^T A w \leq 0$ which contradicts $A + A^T > 0$. Thus if $A + A^T > 0$, then there exists some $v \succ 0$ with $Av \succ 0$.

Theorem 6.4.1 *Let A_1, A_2 be Metzler, Hurwitz matrices in $\mathbb{R}^{n\times n}$. Then the following conditions are all sufficient for the existence of a CDLF for the LTI systems Σ_{A_1}, Σ_{A_2}.*

(i) $-A_1A_2^{-1}$ and $-A_2^{-1}A_1$ are both Metzler and Hurwitz;

(ii) $A_1A_2^{-1}$ *and* $A_2^{-1}A_1$ *are both non-negative matrices;*

(iii) there is some finite collection of columns $c_{i_1}, c_{i_2}, \ldots, c_{i_p}$ *taken from* $A_1A_2^{-1}$*, and some finite collection of rows* $r_{j_1}^T, r_{j_2}^T, \ldots, r_{j_l}^T$ *taken from* $A_2^{-1}A_1$ *such that*

$$c_{i_1} + c_{i_2} + \cdots + c_{i_p} \succ 0,$$

$$r_{j_1} + r_{j_2} + \cdots + r_{j_l} \succ 0;$$

(iv) $A_1^TA_2 + A_2^TA_1 > 0$ *and* $A_1A_2^T + A_2A_1^T > 0$*.*

Proof: It follow from Lemma 6.4.1 and Lemma 6.4.2, that it is enough to show that if any of (i)-(iv) are satisfied, then there exist vectors $v \succ 0$, $w \succ 0$ such that $A_1A_2^{-1}v \succ 0$ and $A_1^TA_2^{-T}w \succ 0$.

(i) If $-A_1A_2^{-1}$ and $-A_2^{-1}A_1$ are Metzler and Hurwitz then so are $-A_1A_2^{-1}$ and $-A_1^TA_2^{-T}$. It then follows from Theorem 6.2.3 that there are vectors $v \succ 0$, $w \succ 0$ such that $A_1A_2^{-1}v \succ 0$ and $A_1^TA_2^{-T}w \succ 0$.

(ii) For any invertible non-negative matrix $A \in \mathbb{R}^{n\times n}$, there exists a vector $v \succ 0$ such that $Av \succ 0$. The result now follows immediately.

(iii) If we consider the vector $v = (v_1, v_2, \ldots, v_n)^T$ defined by $v_i = 1$ if $i \in \{i_1, \ldots, i_p\}$ and $v_i = 0$ otherwise, then it is easy to see that $v \succeq 0$ and $A_1A_2^{-1}v \succ 0$. It now follows from continuity that by a suitably small perturbation of the zero components of v, we can obtain a vector $v' \succ 0$ such that $A_1A_2^{-1}v' \succ 0$.

Similarly, defining $w = (w_1, \ldots w_n)^T$ by $w_j = 1$ if $j \in \{j_1, \ldots, j_l\}$ and $w_i = 0$ otherwise, we find that $w \succeq 0$ and $A_1^TA_2^{-T}w \succ 0$. Again a continuity argument can now be applied to show that there is some $w' \succ 0$ such that $A_1^TA_2^{-T}w' \succ 0$.

(iv) By congruence, it follows that

$$A_2^{-T}(A_1^TA_2 + A_2^TA_1)A_2^{-1} = (A_1A_2^{-1})^T + (A_1A_2^{-1}) > 0.$$

Hence, from Lemma 6.4.3 there is some $v \succ 0$ such that $A_1 A_2^{-1} v \succ 0$. A similar argument shows that there is some $w \succ 0$ such that $A_1^T A_2^{-T} w \succ 0$. This completes the proof.

Comments:

> While the conditions given in Theorem 6.4.1 are only sufficient for CDLF existence for pairs of positive LTI systems, they are readily verifiable and can be applied to positive systems of any dimension. Furthermore, if any of the conditions in the theorem are satisfied, then we can of course conclude that the positive switched linear system
>
> $$\dot{x} = A(t)x \quad\quad A(t) \in \{A_1, A_2\}$$
>
> is uniformly exponentially stable for arbitrary switching signals.
>
> Conditions (i) and (ii) of Theorem 6.4.1 were previously reported in [76]. It should also be noted that, following a similar approach, Ooba and Funahashi have derived conditions for CDLF existence for pairs of discrete-time positive LTI systems in [102] that are closely related to condition (iv) above.

We now present a number of simple numerical examples to illustrate the conditions for CDLF existence given in Theorem 6.4.1.

Example 6.4.1 *Consider the Hurwitz, Metzler matrices A_1, A_2 in $\mathbb{R}^{3\times 3}$, given by*

$$A_1 = \begin{pmatrix} -1.8907 & 0.7537 & 0.8447 \\ 0.0099 & -1.4727 & 0.3678 \\ 0.4199 & 0.9200 & -1.6458 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1.8406 & 0.6501 & 0.4001 \\ 0.0492 & -1.1551 & 0.1988 \\ 0.6932 & 0.5527 & -1.5129 \end{pmatrix}.$$

Then, by straightforward calculation we have that;

$$A_1A_2^{-1} = \begin{pmatrix} 0.8789 & -0.3348 & -0.3699 \\ -0.0032 & 1.2340 & -0.0818 \\ 0.1885 & -0.1558 & 1.1172 \end{pmatrix},$$

$$A_2^{-1}A_1 = \begin{pmatrix} 1.1143 & -0.0016 & -0.3352 \\ 0.0843 & 1.2486 & -0.1834 \\ 0.2638 & -0.1527 & 0.8673 \end{pmatrix}.$$

Now the sum of the first two columns of $A_1A_2^{-1}$ is entry-wise positive, as is the sum of row two and three of $A_2^{-1}A_1$. Thus, it follows from condition (iii) of Theorem 6.4.1 that the positive LTI systems Σ_{A_1}, Σ_{A_2} have a CDLF.

Example 6.4.2 *Now consider the Metzler, Hurwitz matrices A_1, A_2 in $\mathbb{R}^{3\times 3}$, where*

$$A_1 = \begin{pmatrix} -1.8800 & 0.6318 & 0.9316 \\ 0.9048 & -1.8395 & 0.3352 \\ 0.5692 & 0.5488 & -1.4184 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1.6010 & 0.4136 & 0.3716 \\ 0.6991 & -1.5731 & 0.4253 \\ 0.3972 & 0.8376 & -1.6336 \end{pmatrix}.$$

The eigenvalues of the symmetric matrix $A_1^T A_2 + A_2^T A_1$ *are* $\{0.4054, \ \ 8.8439, \ \ 11.3290\}$*, and those of* $A_1 A_2^T + A_2 A_1^T$ *are given by* $\{0.3609, \ \ 8.9317, \ \ 11.2856\}$*. Hence, condition (iv) of Theorem 6.4.1 implies that the positive LTI systems* Σ_{A_1}*,* Σ_{A_2} *have a CDLF.*

Discrete-time sufficient conditions:

Finally for this subsection, we note that it is possible to adapt the above techniques to obtain corresponding sufficient conditions for CDLF existence for discrete-time positive systems. In fact, identical arguments can be applied based on the following discrete-time analogue of Lemma 6.4.1, which follows from the arguments presented in [102].

Lemma 6.4.4 *Let* $A_1, \ A_2, \ldots, A_k$ *be non-negative, Schur matrices in* $\mathbb{R}^{n \times n}$*. Suppose that there exist vectors* $v \succ 0$*,* $w \succ 0$ *in* $\mathbb{R}^n$ *such that*

(i) $(I - A_i)v \succ 0$ *for* $1 \leq i \leq k$*,*

(ii) $(I - A_i)^T w \succ 0$*, for* $1 \leq i \leq k$*.*

Then the positive definite diagonal matrix D *given by*

$$D = diag\{w_1/v_1, w_2/v_2, \ldots, w_n/v_n\},$$

satisfies $A_i^T D A_i - D < 0$ *for* $1 \leq i \leq k$*.*

Comments:

Using Lemma 6.4.4 it is possible to obtain a discrete-time condition corresponding to each of the sufficient conditions for CDLF existence for continuous-time systems in Theorem 6.4.1. In fact by replacing A_i with $(A_i - I)$ in conditions (i)-(iv) of Theorem 6.4.1, we immediately obtain sufficient conditions for CDLF existence for pairs of discrete-time positive LTI systems $\Sigma^d_{A_1}$, $\Sigma^d_{A_2}$. Note however, that a stronger version of the discrete-time analogue of condition (iv) in Theorem 6.4.1 has already been reported in [102].

6.5 Necessary and sufficient conditions for CDLFs

In the previous section, we introduced the common diagonal Lyapunov function (CDLF) existence problem for positive LTI systems and presented some preliminary results giving sufficient conditions for a pair of positive LTI systems to have a CDLF. In this section, we continue our investigation of this question and, following similar techniques to those applied to the general CQLF existence problem in Chapter 4 and Chapter 5, derive conditions that are necessary as well as sufficient for CDLF existence for pairs of positive LTI systems.

The convex cones $\mathcal{D}_A$:

The approach to the CQLF existence problem that was described in Chapter 4 and Chapter 5 was based on studying the sets $\mathcal{P}_A$ introduced in Section 3.2. Our main goal in the current section is to apply similar ideas to determine conditions for a pair of positive LTI systems to have a CDLF. In doing so, we shall see that the techniques developed in Chapters 4 and 5 can also provide insights in the context of this problem, and that the same basic ideas can be used to derive necessary and sufficient conditions for CDLF existence for generic pairs of positive LTI systems. Interestingly, the "rank $n-1$" condition of Theorem 4.4.1 shall play a key role once more. In fact, this rank condition is generically satisfied when considering the CDLF existence problem for pairs of positive LTI systems.

Now, for any Metzler Hurwitz matrix A in $\mathbb{R}^{n\times n}$, define $\mathcal{D}_A$ to be the cone of diagonal solutions of the corresponding Lyapunov inequality. Formally,

$$\mathcal{D}_A = \{D \in \mathbb{R}^{n\times n} : D \text{ is diagonal and } A^T D + DA < 0\}. \tag{6.9}$$

Then, in terms of these cones, the stable positive LTI systems Σ_{A_1}, Σ_{A_2} have a CDLF if and only if $\mathcal{D}_{A_1} \cap \mathcal{D}_{A_2}$ is non-empty. This is illustrated in Figure 6.2 below.

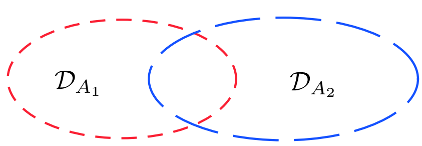

Figure 6.2: CDLF exists for Σ_{A_1}, Σ_{A_2} - $\mathcal{D}_{A_1} \cap \mathcal{D}_{A_2} \neq \emptyset$

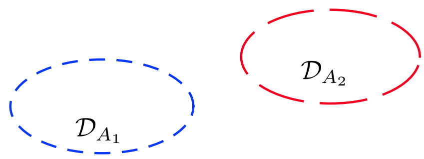

Figure 6.3: CDLF does not exist for Σ_{A_1}, Σ_{A_2} - $\mathcal{D}_{A_1} \cap \mathcal{D}_{A_2} = \emptyset$

As when we considered the general CQLF existence problem, the key aspect of our approach to the problem of CDLF existence is to focus on the marginal situation depicted in Figure 6.4 that divides the scenarios of Figures 6.2 and 6.3.

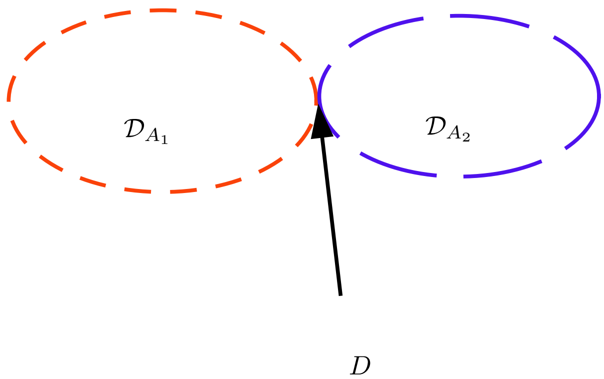

Figure 6.4: Marginal situation for CDLF existence

Formally, in this situation there is no CDLF for Σ_{A_1}, Σ_{A_2}, but there does exist some positive semi-definite diagonal D such that $A_i^T D + DA_i \leq 0$ for $i = 1, 2$. We shall show in the next two subsections that, under the mild assumption that the matrices A_i are irreducible, the rank of $A_i^T D + DA_i$ must be $n - 1$ for $i = 1, 2$ under these circumstances. Furthermore, this fact can then be used to derive necessary and sufficient conditions for Σ_{A_1}, Σ_{A_2} to have a CDLF.

6.5.1 Preliminary results on Metzler matrices and diagonal Lyapunov functions

Before deriving a necessary and sufficient condition for a pair of stable positive LTI systems to have a CDLF, in this subsection we present a number of technical preliminaries related to diagonal Lyapunov functions and Metzler Hurwitz matrices. To begin with, we state the following simple lemma that can easily be verified by direct computation.

Lemma 6.5.1 *Let $A \in \mathbb{R}^{n \times n}$ be a Metzler matrix. Then for any diagonal matrix D in $\mathbb{R}^{n \times n}$ with non-negative entries, $A^T D + DA$ is also Metzler.*

Proof: The (i, j) entry of $A^T D + DA$ is given by $a_{ij} d_i + d_j a_{ji}$. If $i \neq j$, then $a_{ij} \geq 0$, $a_{ji} \geq 0$ and hence $a_{ij} d_i + d_j a_{ji} \geq 0$. Thus $A^T D + DA$ has non-negative off-diagonal entries and is Metzler as claimed.

The next result is concerned with diagonal matrices D on the boundary of the set $\mathcal{D}_A$, for irreducible Metzler Hurwitz matrices A. It establishes that, for such A, any non-zero diagonal $D \geq 0$ such that $A^T D + DA \leq 0$ must in fact be positive definite.

Lemma 6.5.2 *Let A in $\mathbb{R}^{n \times n}$ be Metzler, Hurwitz and irreducible. Suppose that $A^T D + DA \leq 0$ for some non-zero diagonal D in $\mathbb{R}^{n \times n}$. Then $D > 0$.*

Proof: The key fact in the proof of this result is that if $Q \in \mathbb{R}^{n \times n}$ is positive semi-definite, and for some $i = 1, \ldots, n$, $q_{ii} = 0$, then $q_{ij} = 0$ for $1 \leq j \leq n$. Basically, if any element along the diagonal of a positive semi-definite matrix is zero, then every entry in the corresponding row and column must also be zero [42].

We argue by contradiction. Suppose that D is not positive definite. Then we may select a permutation matrix P in $\mathbb{R}^{n \times n}$ such that

$$D' = PDP^T = diag\{d'_1, \ldots, d'_n\},$$

with $d'_1 = 0, \ldots, d'_r = 0$ and $d'_{r+1} > 0, \ldots, d'_n > 0$, for some r with $1 \leq r < n$. It follows by congruence that writing $A' = PAP^T$, we have

$$A'^T D' + D'A' \leq 0.$$

The (i, j) entry of $A'^T D' + D'A'$ is given by $a'_{ij} d'_i + d'_j a'_{ji}$. Now for $i = 1, \ldots, r$, $d'_i = 0$ and hence the corresponding diagonal entry, $2d'_i a'_{ii}$, of $A'^T D' + D'A'$ is zero. From the remarks at the start of the proof, it now follows that for $1 \leq j \leq n$, $a'_{ij} d'_i + d'_j a'_{ji} = 0$ also, and in particular that for $j = r + 1, \ldots, n$, $a'_{ji} = 0$.

To summarize, we have shown that if D is not positive definite, then there is some permutation matrix P, and some r with $1 \leq r < n$ such that for $i = 1, \ldots r$ and $j = r + 1, \ldots, n$, $a'_{ji} = 0$ where $A' = PAP^T$. But this then means that A' is in the form of (6.1) and hence that A is reducible which is a contradiction. Thus, D must be positive definite as claimed.

The following result is crucial for much of what follows and is a relatively straightforward consequence of the previous lemma.

Lemma 6.5.3 *Let $A \in \mathbb{R}^{n \times n}$ be Metzler, Hurwitz and irreducible. Suppose that for some non-zero diagonal D in $\mathbb{R}^{n \times n}$, $A^T D + DA = Q \leq 0$. Then Q is also irreducible.*

Proof: Once again, we shall argue by contradiction. Suppose that Q is reducible. Then there is some permutation matrix P in $\mathbb{R}^{n \times n}$ such that, if we write $A' = PAP^T$,

$D' = PDP^T$, $Q' = PQP^T$, then

(i) $A'^T D' + D'A' = Q' \leq 0$;

(ii) there is some r, with $1 \leq r < n$, such that for $i = r+1, \ldots, n$, $j = 1, \ldots, r$, $q'_{ij} = 0$.

It follows from (ii) that $a'_{ij} d_i + a'_{ji} d_j = 0$ for $i = r+1, \ldots, n$, $j = 1, \ldots, r$. But from Lemma 6.5.2, $d'_i > 0$ for $1 \leq i \leq n$, and hence (as A is Metzler) $a'_{ij} = 0$ for $i = r+1, \ldots, n$, $j = 1, \ldots, r$. This would mean that A' was in the form of (6.1) and hence that A was reducible which is a contradiction. Thus Q must be irreducible as claimed.

Comments:

The previous technical results establish a number of facts about diagonal matrices on the boundary of $\mathcal{D}_A$ where A is an irreducible Metzler, Hurwitz matrix in $\mathbb{R}^{n \times n}$. In particular, we have shown that for any non-zero D on the boundary of $\mathcal{D}_A$:

(i) D must be positive definite;

(ii) $A^T D + DA$ is Metzler and irreducible.

These two facts lead naturally to the following corollary which plays a crucial role in the proof of the main result in the next subsection.

Corollary 6.5.1 *Let $A \in \mathbb{R}^{n \times n}$ be Metzler, Hurwitz and irreducible. Suppose that $D \in \mathbb{R}^{n \times n}$ is diagonal and that $A^T D + DA = Q \leq 0$. Then* $\mathrm{rank}(Q) = n - 1$*, and there is some vector $v \succ 0$ such that $Qv = 0$.*

Proof: It follows from Lemma 6.5.1 and Lemma 6.5.3 that Q is an irreducible Metzler matrix. Furthermore, as $Q \leq 0$, $\mu(A) = 0$. The result now follows immediately from Theorem 6.2.2.

Comments:

Keeping in mind the 'rank $n-1$' condition of Theorem 4.4.1, it should be noted that Corollary 6.5.1 establishes that if A is Metzler, Hurwitz and irreducible, then for any diagonal D on the boundary of $\mathcal{D}_A$, the rank of $A^T D + DA$ must be $n-1$.

Necessary conditions:

To finish off this subsection, we present simple necessary conditions for a pair of LTI systems to have a CDLF, and a corresponding necessary condition for a single stable LTI system to have a diagonal Lyapunov function. Note how the condition of the next lemma is related to the necessary conditions previously established for the general CQLF problem in Lemma 4.3.1.

Lemma 6.5.4 *Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{n\times n}$ such that Σ_{A_1}, Σ_{A_2} have a CDLF. Then for all non-singular diagonal matrices D in $\mathbb{R}^{n\times n}$, $A_1 + DA_2D$ and $A_1^{-1} + DA_2D$ are Hurwitz and hence non-singular.*

Proof: Firstly, note that if $V(x) = x^T\bar{D}x$ is a CDLF for Σ_{A_1}, Σ_{A_2}, then it is also a CDLF for $\Sigma_{A_1^{-1}}, \Sigma_{A_2}$. Furthermore for any non-singular diagonal D in $\mathbb{R}^{n\times n}$,

$$(DA_2D)^T\bar{D} + \bar{D}(DA_2D) = D(A_2^T\bar{D} + \bar{D}A_2)D < 0.$$

Thus, by congruence $V(x) = x^T\bar{D}x$ would also be a CDLF for Σ_{A_1}, Σ_{DA_2D}, and a diagonal Lyapunov function for $\Sigma_{A_1+DA_2D}$. This implies that $A_1 + DA_2D$ is Hurwitz and hence non-singular. The identical argument shows that $A_1^{-1} + DA_2D$ is also non-singular.

Finally for this section, as Σ_A has a diagonal Lyapunov function if and only if Σ_A, $\Sigma_{A^{-1}}$ have a CDLF, the preceding lemma can be used to derive the following simple necessary conditions for a single stable LTI system to have a diagonal Lyapunov function.

Corollary 6.5.2 *Let Σ_A be a stable LTI system with $A \in \mathbb{R}^{n \times n}$. Then a necessary condition for Σ_A to have a diagonal Lyapunov function is that $A + DA^{-1}D$ and $A + DAD$ are Hurwitz for all non-singular diagonal matrices $D \in \mathbb{R}^{n \times n}$.*

6.5.2 The main result

We now consider a pair of stable positive LTI systems Σ_{A_1}, Σ_{A_2}, where $A_1, A_2 \in \mathbb{R}^{n \times n}$ are Hurwitz, Metzler and irreducible. We shall derive below a necessary and sufficient condition for Σ_{A_1}, Σ_{A_2} to have a CDLF that is related to the matrix pencil conditions obtained for the general CQLF existence problem in Chapter 5, and in [129]. The key step in the derivation of this condition is Theorem 6.5.1 below where we consider the marginal situation depicted in Figure 6.4, of two systems $\Sigma_{A_1}, \Sigma_{A_2}$ for which there is no CDLF but for which there is a non-zero diagonal $D \geq 0$ satisfying

$$\begin{aligned} A_1^T D + DA_1 &\leq 0 \\ A_2^T D + DA_2 &\leq 0. \end{aligned} \tag{6.10}$$

This closely parallels the approach that we have taken to the general CQLF existence problem in Chapter 4. We shall see that in this situation, the necessary conditions of Lemma 6.5.4 are violated.

Theorem 6.5.1 *Let A_1, A_2 in $\mathbb{R}^{n \times n}$ be Hurwitz, Metzler and irreducible. Assume that there is no CDLF for the associated LTI systems, Σ_{A_1}, Σ_{A_2}. Furthermore, suppose that there is some non-zero diagonal $D \geq 0$ satisfying (6.10). Then there exists a positive definite diagonal matrix $D_0 > 0$ such that $A_1 + D_0 A_2 D_0$ is singular.*

<u>**Comments:**</u>

Theorem 6.5.1 is proven in more or less the same way as Theorem 4.4.1 above. For convenience, the main points in the proof are listed below.

(a) We first show using the results of the previous section that the rank of $A_i^T D + DA_i$ is $n-1$ for $i = 1, 2$.

(b) Then it is shown that there exist vectors $x_1 \succ 0$, $x_2 \succ 0$ such that $x_1^T DA_1 x = -x_2^T DA_2 x_2$ for all diagonal matrices D in $\mathbb{R}^{n \times n}$.

(c) The result then follows after some algebraic manipulation.

Proof: First of all, it follows from Corollary 6.5.1 that $Q_1 = A_1^T D + DA_1$, and $Q_2 = A_2^T D + DA_2$ must both have rank $n-1$, and that we can choose vectors $x_1 \succ 0$, $x_2 \succ 0$ such that $Q_1 x_1 = 0, Q_2 x_2 = 0$.

The next stage in the proof is to show that there can be no diagonal matrix D' with

$$\begin{aligned} x_1^T D' A_1 x_1 &< 0 & (6.11) \\ x_2^T D' A_2 x_2 &< 0. & (6.12) \end{aligned}$$

We shall prove this by contradiction. First of all suppose that there is some D' satisfying (6.11), (6.12). We shall show that by choosing $\delta_1 > 0$ sufficiently small, it is possible to guarantee that $A_1^T (D + \delta_1 D') + (D + \delta_1 D') A_1$ is negative definite. Firstly, consider the set

$$\Omega_1 = \{x \in \mathbb{R}^n : x^T x = 1 \text{ and } x^T D' A_1 x \geq 0\}.$$

Note that if the set Ω_1 was empty, then any positive constant $\delta_1 > 0$ would make $A_1^T (D + \delta_1 D') + (D + \delta_1 D') A_1$ negative definite. Hence, we assume that Ω_1 is non-empty.

The function that takes x to $x^T D' A_1 x$ is continuous. Thus Ω_1 is closed and bounded, hence compact. Furthermore x_1 (or any non-zero multiple of x_1) is not in Ω_1 and thus $x^T DA_1 x$ is strictly negative on Ω_1.

Let M_1 be the maximum value of $x^T D' A_1 x$ on Ω_1, and let M_2 be the maximum value of $x^T D A_1 x$ on Ω_1. Then by the final remark in the previous paragraph, $M_2 < 0$. Choose any constant $\delta_1 > 0$ such that

$$\delta_1 < \frac{|M_2|}{M_1 + 1} = C_1$$

and consider the diagonal matrix

$$D + \delta_1 D'.$$

By separately considering the cases $x \in \Omega_1$ and $x \notin \Omega_1$, $x^T x = 1$, it follows that for all non-zero vectors x in $\mathbb{R}^n$ of Euclidean norm 1

$$x^T (A_1^T (D + \delta_1 D') + (D + \delta_1 D') A_1) x < 0$$

provided $0 < \delta_1 < \frac{|M_2|}{M_1+1}$. Since the above inequality is unchanged if we scale x by any non-zero real number, it follows that $A_1^T (D + \delta_1 D') + (D + \delta_1 D') A_1$ is negative definite. As A_1 is Hurwitz, this implies that the diagonal matrix $D + \delta_1 D'$ is positive definite.

The same argument can be used to show that there is some $C_2 > 0$ such that

$$x^T (A_2^T (D + \delta_1 D') + (D + \delta_1 D') A_2) x < 0$$

for all non-zero x, provided $0 < \delta_1 < C_2$. So, if we choose δ less than the minimum of C_1, C_2, we would have a positive definite diagonal matrix

$$D_1 = D + \delta D'$$

which defined a CDLF for Σ_{A_1} and Σ_{A_2}.

As there is no diagonal solution to (6.11), (6.12) it follows that

$$x_1^T D' A_1 x_1 < 0 \iff x_2^T D' A_2 x_2 > 0 \tag{6.13}$$

for diagonal D'. It follows from this that

$$x_1^T D' A_1 x_1 = 0 \iff x_2^T D' A_2 x_2 = 0.$$

The expressions $x_1^T D' A_1 x_1$, $x_2^T D' A_2 x_2$, viewed as functions of D', define linear functionals on the space of diagonal matrices in $\mathbb{R}^{n \times n}$. Moreover, we have seen that the null sets of these functionals are identical. So they must be scalar multiples of each other. Furthermore, (6.13) implies that they are negative multiples of each other. So there is some constant $k > 0$ such that

$$x_1^T D' A_1 x_1 = -k x_2^T D' A_2 x_2 \tag{6.14}$$

for all diagonal D' in $\mathbb{R}^{n \times n}$. In fact, we can take $k = 1$ as we may replace x_2 with $x_2/\sqrt{k}$ if necessary.

On expanding out equation (6.14) (with $k = 1$) and equating coefficients, it follows that

$$x_1 \circ A_1 x_1 = -x_2 \circ A_2 x_2, \tag{6.15}$$

where $x \circ y$ denotes the element-wise or Hadamard product of the vectors x, y, given by $(x \circ y)_i = x_i y_i$. Now as $x_1 \succ 0$, $x_2 \succ 0$, there is some diagonal matrix $D_0 > 0$ such that $x_2 = D_0 x_1$. But then, it follows from (6.15) that $A_2 x_2 = -D_0^{-1} A_1 x_1$ and hence that $(D_0^{-1} A_1 + A_2 D_0) x_1 = 0$. This means that

$$\det(A_1 + D_0 A_2 D_0) = \det(D_0)\det(D_0^{-1} A_1 + A_2 D_0) = 0$$

as claimed.

We can now apply Lemma 6.5.4 and Theorem 6.5.1 to derive the main result of this section.

Theorem 6.5.2 *Let A_1, A_2 in $\mathbb{R}^{n \times n}$ be Hurwitz, Metzler and irreducible. Then a necessary and sufficient condition for the exponentially stable positive LTI systems, $\Sigma_{A_1}, \Sigma_{A_2}$, to have a CDLF is that $A_1 + D A_2 D$ is non-singular for all diagonal $D > 0$.*

Proof: The necessity was proven in Lemma 6.5.4. Now suppose that there is no CDLF for Σ_{A_1}, Σ_{A_2}. Then:

(i) for $\alpha > 0$ sufficiently large, $\Sigma_{A_1-\alpha I}$, Σ_{A_2} will have a CDLF;

(ii) if we define $\alpha_0 = \inf\{\alpha > 0 : \Sigma_{A_1-\alpha I}, \Sigma_{A_2}$ have a CDLF $\}$, then $\Sigma_{A_1-\alpha_0 I}$, Σ_{A_2} satisfy the conditions of Theorem 6.5.1;

(iii) it follows that there is some diagonal $D > 0$ such that $A_1 - \alpha_0 I + DA_2D$ is singular.

From item (iii), it follows that $A_1 + DA_2D$ is not Hurwitz. However both A_1 and DA_2D are Hurwitz Metzler matrices, and it therefore follows from Lemma 6.2.1 that there is some positive $\gamma > 0$ such that $A_1 + \gamma DA_2D$ is singular. Hence, defining $\bar{D} = \sqrt{\gamma}D$, we have that $A_1 + \bar{D}A_2\bar{D}$ is singular. This completes the proof of the theorem.

Comments:

Theorem 6.5.2 establishes an algebraic condition that is necessary and sufficient for a pair of exponentially stable positive LTI systems of any dimension to have a CDLF, under the assumption that both of the system matrices are irreducible. To the best of the author's knowledge, it is the first such general result available on CDLF existence for a significant class of n-dimensional systems. Note also that the condition for CDLF existence given in the theorem takes the form of a multi-variable matrix pencil condition, and as such it is related to the matrix pencil conditions for general CQLF existence discussed in previous chapters. Corresponding to what we saw for second order systems and systems differing by rank one in Chapter 5, the key factor in the proof of Theorem 6.5.2 is the result of Theorem 6.5.1. In particular, once again the fact that the rank of the matrices $A_i^T D + DA_i$ is $n-1$ for positive LTI systems in the marginal situation depicted in Figure 6.4 is crucial.

While the condition given in Theorem 6.5.2 is both general and compact, it is far from straightforward to check. However, it does provide insight into the question of CDLF existence for pairs of positive LTI systems, and it is hoped that the insights it provides may lead to simpler conditions that can be used in the design of positive switched linear systems. As an initial step in this direction, in the next subsection two straightforward applications of the theorem are described.

6.5.3 Two applications

We now present two simple applications of Theorem 6.5.2. First of all, note the following easily verifiable facts.

(i) If $A_1 \in \mathbb{R}^{n \times n}$ is Metzler and Hurwitz, then it follows from Corollary 6.5.2 that $DA_1D + A_1$ is also Hurwitz and Metzler for all diagonal $D > 0$.

(ii) If B is any Metzler matrix with $A_1 \succeq B$, then B is also Hurwitz [43].

(iii) If $A_1 \succeq B$, then for any diagonal $D > 0$, $DA_1D + A_1 \succeq DA_1D + B$.

Thus if $A_1 \succeq A_2$, it follows from item (iii) that for all positive diagonal D, $DA_1D + A_1 \succeq DA_1D + A_2$. Hence from (i) and (ii) it follows that $DA_1D + A_2$ is Hurwitz for all diagonal $D > 0$. Thus applying Theorem 6.5.2 we have the following known result [76, 89].

Theorem 6.5.3 *Let $A_1, A_2 \in \mathbb{R}^{n \times n}$ be Metzler, Hurwitz and irreducible, and suppose $A_1 \succeq A_2$. Then the positive LTI systems, $\Sigma_{A_1}, \Sigma_{A_2}$, have a CDLF.*

It is in fact possible to slightly strengthen Theorem 6.5.3 by noting that, for a fixed diagonal $D_1 > 0$, as D ranges over all positive diagonal matrices, so too does

$DD_1 = D_1D$. So if we know that $DD_1A_1D_1D + A_2$ is non-singular for all positive diagonal D, then $DA_1D + A_2$ is also non-singular for all positive diagonal D. This gives us the following result.

Corollary 6.5.3 *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Metzler, Hurwitz and irreducible. Suppose that for some diagonal $D_1 > 0$, $D_1A_1D_1 \succeq A_2$. Then the positive LTI systems, $\Sigma_{A_1}, \Sigma_{A_2}$, have a CDLF.*

6.6 Copositive Lyapunov functions

Up to this point, we have been trying to find stability criteria for positive switched linear systems that are based on the existence of common quadratic Lyapunov functions or common diagonal Lyapunov functions. While this approach has yielded a number of results, it is important to appreciate that it may lead to conservative stability criteria for positive switched linear systems, as it fails to take into account the most fundamental property of such systems: namely, that their trajectories are constrained to remain within the non-negative orthant for all time. For example, in order for $V(x) = x^TPx$ to be a CQLF in the usual sense for the n-dimensional positive LTI systems, $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$, the following conditions must be satisfied for every non-zero x in $\mathbb{R}^n$.

(i) $x^TPx > 0$;

(ii) $x^T(A_i^TP + PA_i)x < 0$ for $1 \leq i \leq k$.

Requiring that (i) and (ii) are satisfied globally is unnecessarily restrictive for positive switched linear systems, as the state vector can only lie in the non-negative orthant in this case. In view of this fact, it is enough to require the existence of a function $V(x) = x^TPx$ for which (i) and (ii) are satisfied for all non-zero $x \succeq 0$ [8]. Such a function is then said to be a *common quadratic copositive Lyapunov function* for the positive LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$. More generally, $V(x)$ is a common copositive Lyapunov function for Σ_{A_i}, $1 \leq i \leq k$, if $V(x) > 0$ for all non-zero $x \succeq 0$, and $\dot{V}(x) < 0$ along all trajectories of Σ_{A_i} within the non-negative orthant for $i = 1, \ldots, k$. These functions shall be the focus of the current section and we shall consider the two closely related problems of determining conditions that can be used to test for the existence of common quadratic copositive Lyapunov functions and common linear copositive Lyapunov functions.

6.6.1 Quadratic copositive Lyapunov functions

We begin our discussion of copositive Lyapunov functions by considering quadratic copositive Lyapunov functions. Formally, the function $V(x) = x^T P x$ is a common quadratic copositive Lyapunov function for the positive LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$ if:

(i) $P = P^T$, $x^T P x > 0$ for all $x \succeq 0$, $x \neq 0$;

(ii) $x^T(A_i^T P + P A_i)x < 0$ for all $x \succeq 0$, $x \neq 0$, for $1 \leq i \leq k$.

If such a function exists for the positive LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$, then the associated positive switched linear system will be exponentially stable. Of course, any CQLF will automatically define a common quadratic copositive Lyapunov function also. However, by only requiring that the inequalities of (i) and (ii) are satisfied for x in the non-negative orthant, it may sometimes be possible to find common quadratic copositive Lyapunov functions for systems that do not have CQLFs. In this way, less conservative stability criteria for positive switched linear systems may be obtained.

Incidentally, it should be mentioned that a symmetric matrix P that satisfies (i) is often referred to as a copositive matrix in the mathematics literature [52], and that there has been a considerable amount of work done on finding ways of determining whether or not a given symmetric matrix is copositive. While necessary and sufficient conditions for copositivity have been recently derived in [52], these only provide a practical means of testing for copositivity in low dimensions.

Copositivity and the Lyapunov equation:

Before proceeding, it is worth noting the following straightforward fact about the existence of quadratic copositive Lyapunov functions for stable positive LTI systems.

Lemma 6.6.1 *Let A be a Metzler, Hurwitz matrix in $\mathbb{R}^{n \times n}$. Then for any copositive matrix Q in $\mathbb{R}^{n \times n}$, the unique symmetric solution P to the Lyapunov equation*

$$A^T P + PA = -Q,$$

(guaranteed by Theorem 3.2.1) is also copositive.

Proof: Recall that the solution P is given by

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt.$$

The result now follows immediately from the facts that Q is copositive and $e^{A^T t}$, e^{At} are both non-negative for all $t \geq 0$, as A and A^T are both Metzler.

Necessary and sufficient conditions for second order systems:

In Theorem 6.3.1, we established that a necessary and sufficient condition for a pair of stable second order positive LTI systems Σ_{A_1}, Σ_{A_2} to have a CQLF was that $A_1 A_2^{-1}$ had no negative real eigenvalues. In the following theorem, we show that the same condition is also necessary and sufficient for the existence of a common quadratic copositive Lyapunov function in this case.

Theorem 6.6.1 *Let A_1, A_2 be Metzler, Hurwitz matrices in $\mathbb{R}^{2 \times 2}$. Then there exists a common quadratic copositive Lyapunov function for the systems Σ_{A_1}, Σ_{A_2} if and only if $A_1 A_2^{-1}$ has no negative real eigenvalues.*

Proof: First of all assume that the product $A_1 A_2^{-1}$ has no negative real eigenvalues. Then it follows immediately from Theorem 6.3.1 that Σ_{A_1}, Σ_{A_2} have a CQLF, $V(x) = x^T P x$. But $V(x)$ is then also a common quadratic copositive Lyapunov function for Σ_{A_1}, Σ_{A_2}.

To see that the converse is also true, suppose that $A_1 A_2^{-1}$ has a negative real eigenvalue. Then for some $\gamma > 0$, the Metzler matrix $A_1 + \gamma A_2$ has an eigenvalue in the closed right half plane. Thus, for this γ, the Perron eigenvalue of $A_1 + \gamma A_2$ must be non-negative. That is, $\mu(A_1 + \gamma A_2) \geq 0$, and there is some $x \succeq 0$, $x \neq 0$, with $(A_1 + \gamma A_2)x = \mu(A_1 + \gamma A_2)x$. It now follows that for any copositive matrix P,

$$x^T P(A_1 + \gamma A_2)x = \mu(A_1 + \gamma A_2)x^T P x \geq 0.$$

This implies that Σ_{A_1}, Σ_{A_2} cannot have a quadratic common copositive Lyapunov function as claimed.

Comments:

Theorem 6.6.1 shows that for pairs of second order positive LTI systems, the existence of a CQLF is equivalent to the existence of a common quadratic copositive Lyapunov function. Also, it is important to point out that if the product $A_1 A_2^{-1}$ has a negative real eigenvalue, then it follows from the proof of Theorem 2.3.1 in [130] that the corresponding switched linear system will fail to be exponentially stable, even if we restrict ourselves to only considering initial conditions that lie in the non-negative orthant. Thus CQLF existence, and common quadratic copositive Lyapunov function existence, are not conservative criteria for the stability of second order positive switched linear systems.

6.6.2 Linear copositive Lyapunov function

Another possible approach to establishing the stability of positive switched linear systems is to look for a common copositive Lyapunov function of the form, $V(x) = v^T x$, for its constituent systems, where v is some fixed vector in $\mathbb{R}^n$. We shall refer to these functions as common linear copositive Lyapunov functions from now on. Before discussing conditions for the existence of such functions, we note that the linear function, $V(x) = v^T x$, will be positive for all non-zero $x \succeq 0$ if and only if $v \succ 0$, and that the derivative of $V(x) = v^T x$ along any trajectory of a given positive LTI system, Σ_A, will be negative in the non-negative orthant if and only if $-A^T v \succ 0$.

The above remarks show that $V(x) = v^T x$ will define a linear copositive Lyapunov function for the positive LTI system Σ_A if and only if $v \succ 0$ and $-A^T v \succ 0$. Given that a matrix A in $\mathbb{R}^{n \times n}$ is Metzler and Hurwitz if and only if it transpose A^T is also Metzler and Hurwitz, it follows from Theorem 6.2.3 that, for any Hurwitz Metzler matrix A in $\mathbb{R}^{n \times n}$, there is some vector $v \succ 0$ with $-A^T v \succ 0$. Thus, any exponentially stable positive LTI system has a linear copositive Lyapunov function and the problem of determining when a family of such systems has a common linear copositive Lyapunov function amounts to answering the following question.

> Given Metzler, Hurwitz matrices $A_1, \ldots, A_k$ in $\mathbb{R}^{n \times n}$, when does there exist a single vector $v \succ 0$ such that $-A_i^T v \succ 0$ for $1 \leq i \leq k$?

A similar question arose earlier in Section 6.4 when we were interested in obtaining sufficient conditions for the existence of CDLFs for families of positive LTI systems. In fact, the sufficient conditions for CDLF existence presented in Theorem 6.4.1 were based on related conditions for the existence of vectors $v \succ 0$, $w \succ 0$ in $\mathbb{R}^n$ such that:

(i) $-A_i v \succ 0$ for $1 \leq i \leq k$;

(ii) $-A_i^T w \succ 0$ for $1 \leq i \leq k$.

Hence, the following sufficient conditions for the existence of common linear copositive Lyapunov functions for pairs of n-dimensional positive LTI systems follow immediately from the arguments presented in Theorem 6.4.1.

Theorem 6.6.2 *Let A_1, A_2 be Metzler, Hurwitz matrices in $\mathbb{R}^{n \times n}$. Then the following conditions are all sufficient for the existence of a common linear copositive Lyapunov function for the exponentially stable positive LTI systems Σ_{A_1}, Σ_{A_2}.*

(i) $-A_2^{-1}A_1$ *is Metzler and Hurwitz;*

(ii) $A_2^{-1}A_1$ *is a non-negative matrix;*

(iii) there is some finite collection of rows $r_{j_1}^T, r_{j_2}^T, \ldots, r_{j_p}^T$ *taken from* $A_2^{-1}A_1$ *such that*

$$r_{j_1} + r_{j_2} + \cdots + r_{j_p} \succ 0;$$

(iv) $A_1 A_2^T + A_2 A_1^T > 0$.

Linear copositive Lyapunov functions and LMIs:

At this point, it should be mentioned that it is possible to cast the question of common linear copositive Lyapunov function existence as a linear matrix inequality (LMI) feasibility problem. To see this, recall that $v^T x$ is a linear copositive Lyapunov function for the exponentially stable positive LTI system Σ_A if and only if $v \succ 0$ and all of the entries of the row vector $v^T A$ are negative. This is equivalent to requiring that $v \succ 0$ and that the matrix

$$v_1 \mathrm{diag}\{a_{11}, \ldots, a_{1n}\} + v_2 \mathrm{diag}\{a_{21}, \ldots, a_{2n}\} \ldots + v_n \mathrm{diag}\{a_{n1}, \ldots, a_{nn}\} \quad (6.16)$$

is negative definite. Here for a row vector w, $\mathrm{diag}\{w\}$ denotes the diagonal matrix whose diagonal entries are the components of w.

Thus, if we write $A_i^{(j)}$ for the j^{th} row of the matrix A_i, there is a common linear copositive Lyapunov function for the positive LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_k}$, if and only if the system of LMIs (6.17) in the decision variables $v_1, \ldots, v_n$ is feasible.

$$\begin{aligned} \mathrm{diag}\{v_1, \ldots, v_n\} &> 0 \\ \sum_{j=1}^{n} v_j \mathrm{diag} A_i^{(j)} &< 0 \text{ for } 1 \leq i \leq k \end{aligned} \tag{6.17}$$

Hence, it is possible to use numerical tools such as the MATLAB LMI toolbox to test for the existence of common linear copositive Lyapunov functions.

Necessary and sufficient conditions - general systems:

While LMIs provide a numerical means of testing for the existence of common linear copositive Lyapunov functions, the reservations that were raised about their use in investigating the existence of CQLFs are again pertinent in this context. In particular, the fact that the numerical LMI-based approach provides little real insight into the problem of common linear copositive Lyapunov function existence and its precise relationship with the stability of positive switched linear systems means that a fuller theoretical understanding of the problem is both desirable and important. For this reason, we now turn our attention to the question of determining necessary and sufficient conditions for common linear copositive Lyapunov function existence for pairs of stable positive LTI systems. The approach that we take to this problem is based on the analysis of convex sets, and mirrors that previously taken to the general CQLF existence problem in Chapters 4 and 5, and to the CDLF existence problem for pairs of positive LTI systems earlier in this chapter. As before, we shall see that this approach can give insight into the problem, and that it can lead to verifiable necessary and sufficient conditions for common linear copositive Lyapunov function existence for certain system classes.

In analogy with the approaches taken before to the CDLF and CQLF existence problems, in Theorem 6.6.3 below we consider the marginal situation of two n-

dimensional exponentially stable positive LTI systems Σ_{A_1}, Σ_{A_2} that do not have a common linear copositive Lyapunov function, but for which there is some vector $v \succeq 0$, $v \neq 0$, with $-A_i^T v \succeq 0$ for $i = 1, 2$. Thus, if we define the convex cones

$$S_{A_i} = \{v \succ 0 : -A_i^T v \succ 0\},$$

for $i = 1, 2$, then we are considering the situation where the open cones S_{A_1} and S_{A_2} are disjoint, but there is some non-zero vector v common to the closures of S_{A_1} and S_{A_2}. Notice that, for any Metzler Hurwitz matrix A in $\mathbb{R}^{n \times n}$, the cone S_A is given by

$$S_A = \{y = -A^{-T}x : x \in \mathbb{R}^n,\ x \succ 0\}. \tag{6.18}$$

In the proof of Theorem 6.6.3 we shall make use of the standard fact from convex analysis [116] that two non-intersecting open convex cones, C_1, C_2 in $\mathbb{R}^n$, can be separated by a hyperplane through the origin. Formally, this means that there must exist some vector z in $\mathbb{R}^n$ (parameterizing the hyperplane) such that $z^T x < 0$ for all x in C_1, and $z^T x > 0$ for all x in C_2.

Theorem 6.6.3 *Let A_1, A_2 be Metzler, Hurwitz matrices in $\mathbb{R}^{n \times n}$. Suppose that there is no common linear copositive Lyapunov function for the systems Σ_{A_1}, Σ_{A_2}, but that there is some non-zero vector $v \succeq 0$ with $-A_i^T v \succeq 0$ for $i = 1, 2$. Then there exist non-zero vectors $w_1 \succeq 0$, $w_2 \succeq 0$, such that $A_1 w_1 + A_2 w_2 = 0$. Moreover, both w_1 and w_2 have at least one component equal to zero.*

Outline of Proof:

For convenience, we now list the main steps that are involved in proving Theorem 6.6.3.

(i) First, we show that there must exist a vector z in $\mathbb{R}^n$ such that the hyperplane given by

$$\mathcal{H} = \{x \in \mathbb{R}^n : z^T x = 0\}$$

separates the sets S_{A_1}, S_{A_2}, and that the vector v is contained in $\mathcal{H}$.

(ii) We then make use of the fact that $\mathcal{H}$ is tangential to both S_{A_1} and S_{A_2} to obtain two different parameterizations of $\mathcal{H}$.

(iii) The result then follows from equating these parameterizations.

Proof: As the open convex cones S_{A_1} and S_{A_2} do not intersect, there exists some hyperplane through the origin that separates them. Formally, this means that there is some vector z in $\mathbb{R}^n$ such that $z^T x < 0$ for all x in S_{A_1}, and $z^T x > 0$ for all x in S_{A_2}.

We now show that because the vector v is common to the closures of S_{A_1} and S_{A_2}, it must satisfy $z^T v = 0$. To see this, first note that as v lies in the closure of S_{A_1} there exists a sequence of vectors v_n in S_{A_1} that converges to v as n tends to infinity. But then, for each n, $z^T v_n < 0$ and, by continuity, in the limit we must have that $z^T v \leq 0$. Similarly, as v is in the closure of S_{A_2}, there exists some sequence w_n in S_{A_2}, with $z^T w_n > 0$ for all n, that converges to v as n tends to infinity. This then implies that $z^T v \geq 0$. Hence, $z^T v \leq 0$ and $z^T v \geq 0$ and we must have that $z^T v = 0$ as claimed.

We next obtain two different representations of the vector z. First of all, as $z^T x < 0$ for all $x \in S_{A_1}$, it follows that $z^T A_1^{-T} y > 0$ for all $y \succ 0$. This implies that $A_1^{-1} z \succeq 0$ or, equivalently, that $z = A_1 w_1$ for some $w_1 \succeq 0$, $w_1 \neq 0$.

Similarly, it follows from $z^T x > 0$ for all $x \in S_{A_2}$, that $z^T A_2^{-T} y < 0$ for all $y \succ 0$. Thus, we also have that $z = -A_2 w_2$ for some non-zero $w_2 \succeq 0$. Equating the two expressions for z, we have that $A_1 w_1 + A_2 w_2 = 0$.

Finally, from $z^T v = 0$ it follows that $w_1^T A_1^T v = 0$. But $-A_1^T v \succeq 0$ by assumption, so if $w_1 \succ 0$, then $w_1^T A_1^T v < 0$. Thus, at least one component of w_1 must be zero. Similarly, it also follows that at least one component of w_2 is zero. This completes the proof of Theorem 6.6.3.

Comments:

Theorem 6.6.3 provides a characterization of pairs of exponentially stable positive LTI systems that are on the 'boundary' of having a common linear copositive Lyapunov functions, and in this sense it is intimately related to the results presented above in Theorem 4.4.1 and Theorem 6.5.1. Furthermore, we shall now use Theorem 6.6.3 to derive necessary and sufficient conditions for a pair of exponentially stable positive LTI systems to have a common linear copositive Lyapunov function in much the same way as the earlier results were used to obtain necessary and sufficient conditions for CQLF and CDLF existence. First of all, we note the following simple necessary condition for the existence of a common linear copositive Lyapunov function.

Lemma 6.6.2 *Let A_1, A_2 be Metzler, Hurwitz matrices in $\mathbb{R}^{n\times n}$. Suppose that the systems Σ_{A_1}, Σ_{A_2} have a common linear copositive Lyapunov function. Then there cannot exist non-zero vectors $w_1 \succeq 0$, $w_2 \succeq 0$, with $A_1w_1 + A_2w_2 = 0$.*

Proof: Let $v^T x$ be a common linear copositive Lyapunov function for Σ_{A_1}, Σ_{A_2}. Then $-A_i^T v \succ 0$ for $i = 1, 2$. Now suppose that there are two vectors $w_1 \succeq 0$, $w_2 \succeq 0$ such that $A_1w_1 + A_2w_2 = 0$. Then

$$-(v^T A_1 w_1 + v^T A_2 w_2) = 0 \tag{6.19}$$

also. However, the expression (6.19) will be strictly positive unless $w_1 = 0$ and $w_2 = 0$, and thus there can be no non-zero vectors $w_1 \succeq 0$, $w_2 \succeq 0$, with $A_1w_1 + A_2w_2 = 0$ as claimed.

We can now combine the above lemma with Theorem 6.6.3 to obtain the following general result.

Theorem 6.6.4 *Let A_1, A_2 be Metzler, Hurwitz matrices in $\mathbb{R}^{n\times n}$. Then a necessary and sufficient condition for the systems Σ_{A_1}, Σ_{A_2}, to have a common linear copositive Lyapunov function is that there are no non-zero vectors $w_1 \succeq 0$, $w_2 \succeq 0$ such that $A_1w_1 + A_2w_2 = 0$.*

Proof: The necessity has already been shown in Lemma 6.6.2. To see the converse, suppose that Σ_{A_1}, Σ_{A_2} have no common linear copositive Lyapunov function. Then, for large enough values of $\alpha > 0$, Σ_{A_1}, $\Sigma_{A_2-\alpha I}$ will have a common linear copositive Lyapunov function (LCLF). If we now define

$$\alpha_0 = \inf\{\alpha : \Sigma_{A_1}, \Sigma_{A_2-\alpha I} \text{ have a common LCLF } \},$$

then $\alpha_0 \geq 0$, and by continuity, the systems Σ_{A_1}, $\Sigma_{A_2-\alpha_0 I}$ will satisfy Theorem 6.6.3. Thus, there exist non-zero vectors $w_1 \succeq 0$, $w_2 \succeq 0$ with

$$A_1w_1 + (A_2 - \alpha_0 I)w_2 = 0.$$

But then,

$$A_1w_1 + A_2\bar{w}_2 = 0,$$

where $\bar{w}_2 = w_2 - \alpha_0 A_2^{-1}w_2 \succeq 0$, and $\bar{w}_2 \neq 0$. Hence, if there are no non-zero vectors $w_1 \succeq 0$, $w_2 \succeq 0$ such that $A_1w_1 + A_2w_2 = 0$, then Σ_{A_1}, Σ_{A_2} must have a common linear copositive Lyapunov function. This completes the proof.

Comments:

> The existence of a common linear copositive Lyapunov function for the positive LTI systems Σ_{A_1}, Σ_{A_2} is equivalent to the existence of a vector $v \succ 0$ in $\mathbb{R}^n$ with $A_1^T A_2^{-T} v \succ 0$. It is possible to combine this fact with

a Theorem of the Alternative for convex cones [7] to obtain a different derivation of the conditions for common linear copositive Lyapunov function existence given in Theorem 6.6.4. However, the proof that we have presented here further illustrates how the methods introduced in Chapter 4 for the CQLF existence problem can also be used to derive results on the existence of other types of common Lyapunov functions. Furthermore, in Theorem 6.6.5 below we shall see how to use the result of Theorem 6.6.3 to derive simple conditions for a pair of second order stable positive LTI systems to have a common linear copositive Lyapunov function.

Note that it is also possible to derive a slightly different version of Theorem 6.6.4. Specifically, if A_1 and A_2 are Metzler, Hurwitz matrices in $\mathbb{R}^{n\times n}$ and there exist non-zero vectors $w_1 \succeq 0$ and $w_2 \succeq 0$ in $\mathbb{R}^n$ such that $A_1w_1 + A_2w_2 = v \succeq 0$, then defining $\bar{w}_2 = w_2 - A_2^{-1}v$ we have that

$$\bar{w}_2 \succeq 0, \bar{w}_2 \neq 0 \text{ and } A_1w_1 + A_2\bar{w}_2 = 0.$$

This simple observation leads to the following alternative version of Theorem 6.6.4.

Corollary 6.6.1 *Let A_1, A_2 be Metzler, Hurwitz matrices in $\mathbb{R}^{n\times n}$. Then a necessary and sufficient condition for the systems Σ_{A_1}, Σ_{A_2}, to have a common linear copositive Lyapunov function is that there are no non-zero vectors $w_1 \succeq 0$, $w_2 \succeq 0$ such that $A_1w_1 + A_2w_2 \succeq 0$.*

Necessary and sufficient conditions - Second order systems:

Finally for this chapter, we present the following application of Theorem 6.6.3 giving verifiable conditions that are necessary and sufficient for the existence of a common linear copositive Lyapunov function for a pair of second order exponentially stable positive LTI systems.

Theorem 6.6.5 *Let A_1, A_2 be Metzler, Hurwitz matrices in $\mathbb{R}^{2\times 2}$. Then, writing $a_{ij}^{(k)}$ for the (i,j) entry of A_k, a necessary and sufficient condition for the positive LTI systems Σ_{A_1}, Σ_{A_2} to have a common linear copositive Lyapunov function is that both of the determinants*

$$\begin{vmatrix} a_{11}^{(1)} & a_{12}^{(2)} \\ a_{21}^{(1)} & a_{22}^{(2)} \end{vmatrix}, \quad \begin{vmatrix} a_{11}^{(2)} & a_{12}^{(1)} \\ a_{21}^{(2)} & a_{22}^{(1)} \end{vmatrix},$$

are positive.

<u>**Comments:**</u>

> Theorem 6.6.5 provides the following simple test for the existence of a common linear copositive Lyapunov function for second order positive LTI systems. Firstly construct the matrices T_1, T_2, where the first and second columns of T_1 are the first column of A_1 and the second column of A_2 respectively, and the first and second columns of T_2 are the first column of A_2 and the second column of A_1 respectively. Then the systems Σ_{A_1}, Σ_{A_2} have a common linear copositive Lyapunov function if and only if the determinants of T_1 and T_2 are both positive.

Proof: To begin with, we shall demonstrate the necessity of the above conditions by showing that if either one is violated, then Σ_{A_1}, Σ_{A_2} cannot have a common linear copositive Lyapunov function. Firstly, suppose that

$$a_{11}^{(1)}a_{22}^{(2)} - a_{21}^{(1)}a_{12}^{(2)} \leq 0.$$

Because A_1 and A_2 are both Metzler and Hurwitz, their diagonal entries must be negative and thus $a_{12}^{(2)}$ cannot be zero. In fact, $a_{12}^{(2)} > 0$. Dividing across by $-a_{12}^{(2)}$, it follows that

$$a_{21}^{(1)} - (a_{11}^{(1)}/a_{12}^{(2)})a_{22}^{(2)} = \alpha \geq 0. \tag{6.20}$$

Now, if we define $w_1 = (1 \;\; 0)^T$ and $w_2 = (0 \quad -(a_{11}^{(1)}/a_{12}^{(2)}))^T$, then $w_1 \succeq 0$, $w_2 \succeq 0$, and (6.20) implies that

$$A_1 w_1 + A_2 w_2 = (0 \;\; \alpha)^T \succeq 0.$$

Thus, by Corollary 6.6.1, the systems Σ_{A_1}, Σ_{A_2} cannot have a common linear copositive Lyapunov function. The necessity of the second condition follows by an identical argument, or by symmetry.

Conversely, suppose that Σ_{A_1}, Σ_{A_2} do not have a common linear copositive Lyapunov function. Then, by the same arguments used in Theorem 6.6.4, there exists some constant $\alpha \geq 0$ such that Σ_{A_1} and $\Sigma_{A_2 - \alpha I}$ satisfy the hypotheses of Theorem 6.6.3. Applying Theorem 6.6.3, we then have that, for this α, there exist two vectors $w_1 \succeq 0$, $w_2 \succeq 0$, such that w_1 and w_2 both have one positive component and one zero component, and

$$A_1 w_1 + (A_2 - \alpha I) w_2 = 0. \tag{6.21}$$

Without loss of generality, we can assume that the non-zero component of w_1 is equal to one.

Given that A_1 and $A_2 - \alpha I$ are both Metzler and Hurwitz, it follows from (6.21) that one of the two following possibilities must be true.

(i) $w_1 = (1 \;\; 0)^T$, $w_2 = (0 \;\; \lambda)^T$ for some $\lambda > 0$,

(ii) $w_1 = (0 \;\; 1)^T$, $w_2 = (\lambda \;\; 0)^T$ for some $\lambda > 0$.

In the first case, (6.21) implies that

$$\begin{pmatrix} a_{11}^{(1)} \\ a_{21}^{(1)} \end{pmatrix} + \lambda \begin{pmatrix} a_{12}^{(2)} \\ a_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda\alpha \end{pmatrix}, \tag{6.22}$$

and hence,

$$\lambda = -(a_{11}^{(1)}/a_{12}^{(2)}),$$

and

$$a_{21}^{(1)} - (a_{11}^{(1)}/a_{12}^{(2)})a_{22}^{(2)} = \lambda\alpha \geq 0.$$

It now follows easily that

$$a_{11}^{(1)}a_{22}^{(2)} - a_{21}^{(1)}a_{12}^{(2)} \leq 0. \tag{6.23}$$

An identical argument will show that in the second case (ii) above, we have that

$$a_{11}^{(2)}a_{22}^{(1)} - a_{21}^{(2)}a_{12}^{(1)} \leq 0. \tag{6.24}$$

This completes the proof.

Comments:

> From the point of view of the exponential stability of positive switched linear systems, it should be noted that it has already been established in Theorem 6.6.1 that the existence of a common quadratic copositive Lyapunov function for a pair of exponentially stable positive LTI systems is necessary and sufficient for the exponential stability of the associated positive switched linear system. However, the conditions for common linear copositive Lyapunov function existence given in Theorem 6.6.5 are extremely simple to check, and moreover it may be possible to extend the above analysis to obtain corresponding conditions for higher order systems.

Finally for this chapter, we give a numerical example to illustrate the result of Theorem 6.6.5.

Example 6.6.1 *Consider the Metzler, Hurwitz matrices A_1, A_2 in $\mathbb{R}^{2\times 2}$ given by*

$$A_1 = \begin{pmatrix} -0.7125 & 0.7764 \\ 0.5113 & -0.9397 \end{pmatrix}, A_2 = \begin{pmatrix} -1.3768 & 0.8066 \\ 0.9827 & -1.3738 \end{pmatrix}.$$

Then

$$\begin{vmatrix} -0.7125 & 0.8066 \\ 0.5113 & -1.3738 \end{vmatrix} = 0.5664, \quad \begin{vmatrix} -1.3768 & 0.7764 \\ 0.9827 & -0.9397 \end{vmatrix} = 0.5308.$$

Thus, Theorem 6.6.5 implies that the systems Σ_{A_1}, Σ_{A_2} have a common linear copositive Lyapunov function. In fact, it can easily be verified that $-A_i^T v \succ 0$ for $i = 1, 2$, where $v = (1.1499, \ \ 1.1636)^T$.

6.7 Concluding remarks

In this chapter, we have considered a number of problems in the stability analysis of positive switched linear systems. Specifically, we have concentrated on three problems: namely, the CQLF existence problem for families of positive LTI systems, the common diagonal Lyapunov function (CDLF) existence problem for families of positive LTI systems, and the problem of common copositive Lyapunov function existence for families of positive LTI systems. The major contributions made in the chapter are listed below.

- We provided a brief review of the most relevant aspects of the theory of positive LTI systems and non-negative matrices.
- We proved that, for two second order exponentially stable positive LTI systems, Σ_{A_1}, Σ_{A_2}, the matrix product $A_1 A_2$ cannot have any negative real eigenvalues. We also pointed out the following consequences of this fact.
  - (a) For a pair of exponentially stable second order positive LTI systems Σ_{A_1}, Σ_{A_2}, a necessary and sufficient condition for the existence of a CQLF is that $A_1 A_2^{-1}$ has no negative real eigenvalues;
  - (b) CQLF existence is not a conservative criterion for the exponential stability of switched systems constructed by switching between a pair of

exponentially stable second order positive LTI systems;

(c) any two exponentially stable second order positive LTI systems, whose system matrices differ by rank one have a CQLF.

- We showed that the previous result on the eigenvalues of the product $A_1 A_2$ is also true for third order positive LTI systems. Consequently, any two exponentially stable third order positive LTI systems, whose system matrices differ by rank one have a CQLF, and the associated switched linear system is guaranteed to be exponentially stable under arbitrary switching.

- We presented a number of verifiable sufficient conditions for CDLF existence for families of positive LTI systems.

- Using a similar approach to that taken to the CQLF existence problem in Chapter 4, we derived an algebraic condition that is necessary and sufficient for a generic pair of exponentially stable positive LTI systems, of any order, to have a CDLF.

- The problem of common copositive Lyapunov function existence was considered and the following results were derived.

  (a) Verifiable necessary and sufficient conditions for a pair of exponentially stable second order positive LTI systems to have a common quadratic copositive Lyapunov function.

  (b) Verifiable necessary and sufficient conditions for a pair of exponentially stable second order positive LTI systems to have a common linear copositive Lyapunov function.

  (c) Necessary and sufficient conditions for a general pair of exponentially stable positive LTI systems (of any order) to have a common linear copositive Lyapunov function.

- We also presented some simple sufficient conditions for a pair of exponentially stable positive LTI systems to have a common linear copositive Lyapunov function, and described how the problem of common linear copositive Lyapunov function existence can be cast as a feasibility problem in LMIs.

Chapter 7

The geometry of the sets $\mathcal{P}_A$ and some open questions

In this chapter, we first describe some technical facts about the geometry of the boundary of the convex cones $\mathcal{P}_A$ for a given Hurwitz A, and show how these facts give a novel geometrical perspective on some of our earlier results. We also discuss ways of extending the analysis of Chapters 4 and 5, and explain why we have concentrated on finding applications of Theorem 4.4.1 rather than on trying to generalize this result. A number of preliminary results on the possibility of extending the work of earlier chapters to non-linear systems are also presented, and several open problems that arise out of the work of the thesis are described.

7.1 Introductory remarks

Throughout our discussions on the CQLF existence problem and the CDLF existence problem for positive systems, certain convex cones of matrices have played a key role.

In particular, the approach to the CQLF existence problem described in Chapters 4 and 5 was based on a study of the open convex cones $\mathcal{P}_A$ defined by

$$\mathcal{P}_A = \{P = P^T > 0 : A^T P + PA < 0\}.$$

In terms of these cones, the problem of CQLF existence for a pair of exponentially stable LTI systems Σ_{A_1}, Σ_{A_2}, amounts to determining whether or not the intersection $\mathcal{P}_{A_1} \cap \mathcal{P}_{A_2}$ is empty. It is important to appreciate that in the specific approach that we adopted to this question, the *boundary* structure of the cones $\mathcal{P}_A$ has played a pivotal role. To see this, recall that a key aspect of our approach was to focus on the marginal situation of a pair of exponentially stable LTI systems Σ_{A_1}, Σ_{A_2}, for which the *open* cones $\mathcal{P}_{A_1}$, $\mathcal{P}_{A_2}$, are disjoint, while their *boundaries* have a non-trivial intersection. Through the consideration of this situation, we were led to the key result of Theorem 4.4.1 which we were later able to use to obtain insights into the CQLF existence problem, and to derive verifiable necessary and sufficient conditions for CQLF existence for certain system classes.

The importance of Theorem 4.4.1, together with the fact that it is essentially a result about a type of intersection between the boundaries of $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$ where A_1, A_2 are Hurwitz, suggests that a thorough understanding of the boundary structure of the cones $\mathcal{P}_A$ may well lead to further insights into the general CQLF existence problem, and to other results similar to Theorem 4.4.1. In view of this, in this chapter we begin to study the properties of the boundary of the cones $\mathcal{P}_A$ where A is a given Hurwitz matrix. In particular, we shall describe a number of technical facts that provide a more geometric perspective on some of our earlier results; showing that Theorem 4.4.1 is actually a natural consequence of a geometrical property of the boundary of the cones $\mathcal{P}_A$. We shall also devote some time to discussing how the work of earlier chapters may be extended to more general cases, highlighting some of the complications that arise, and explaining why we have focussed on finding applications of Theorem 4.4.1 rather than seeking to obtain more general versions of that result.

As we are nearing the end of the thesis at this point, we shall also discuss a number of open problems suggested by the work of earlier chapters that may form the basis of future research. In particular, we shall discuss:

(i) the possibility of extending some of the results that we have obtained for switched linear systems to systems obtained by switching between non-linear constituent systems;

(ii) some open questions relating to the CQLF existence problem for families of LTI systems;

(iii) some problems in the stability of positive switched linear systems that arise out of the work discussed in Chapter 6.

Thus in the present chapter, we have two objectives. Firstly, to provide an alternative perspective on some of the earlier work of the thesis, and secondly, to highlight a number of questions raised by that work that may be used to direct future research efforts.

7.2 A geometric perspective on some earlier results

In this section, we shall present a number of technical results about the boundary structure of the cones $\mathcal{P}_A$, where A is a Hurwitz matrix in $\mathbb{R}^{n \times n}$, and indicate the relevance of these to some of the work of earlier chapters. In particular, we shall show how the facts described here provide a more geometric perspective on the key result of Theorem 4.4.1. We shall also briefly discuss some ways of extending our earlier analysis, in the hope of obtaining further results similar to Theorem 4.4.1.

To begin with, recall that, for a Hurwitz A in $\mathbb{R}^{n\times n}$ the set $\mathcal{P}_A$ is defined to be the open convex cone

$$\mathcal{P}_A = \{P = P^T > 0 : A^T P + PA < 0\},$$

with its closure being given by

$$\overline{\mathcal{P}_A} = \{P = P^T \geq 0 : A^T P + PA \leq 0\}.$$

Thus the boundary of $\mathcal{P}_A$ consists of positive semi-definite matrices $P = P^T \geq 0$ such that $A^T P + PA \leq 0$, and $(A^T P + PA)x = 0$ for at least one non-zero $x \in \mathbb{R}^n$.

The next lemma notes that those matrices P on the boundary of $\mathcal{P}_A$ for which the rank of $A^T P + PA$ is $n-1$ are *dense* in the boundary. This means that arbitrarily close to any P on the boundary of $\mathcal{P}_A$ for which the rank of $A^T P + PA$ is less than $n-1$, there is some P_0, also on the boundary, such that $\text{rank}(A^T P_0 + P_0 A) = n-1$. The proof of Lemma 7.2.1 is given in the appendices. In the statement of the lemma, $\|.\|$ denotes the matrix norm on $\mathbb{R}^{n\times n}$ induced from the usual Euclidean norm on $\mathbb{R}^n$ [42].

Lemma 7.2.1 *Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz, and suppose that $P = P^T \geq 0$ is such that $A^T P + PA \leq 0$ and $\text{rank}(A^T P + PA) = n-k$ for some k with $1 < k \leq n$. Then for any $\epsilon > 0$, there exists some $P_0 = P_0^T \geq 0$ such that:*

(i) $\|P - P_0\| < \epsilon$;

(ii) $A^T P_0 + P_0 A = Q_0 \leq 0$;

(iii) $\text{rank}(Q_0) = n-1$.

Comments:

It should be noted that Lemma 7.2.1 indicates that the 'rank $n-1$' assumption of Theorem 4.4.1 is not overly restrictive. This lemma also partially explains why the conditions of Theorem 4.4.1 have been so often satisfied in numerically generated examples.

Tangent hyperplanes to $\mathcal{P}_A$:

We next turn our attention to hyperplanes in $Sym(n,\mathbb{R})$ that are *tangential* to the cone $\mathcal{P}_A$ at various points on its boundary. We shall see below that the result of Theorem 4.4.1 follows in a natural way from certain properties of such hyperplanes. First of all, note that any hyperplane through the origin in $Sym(n,\mathbb{R})$ can be written as

$$\mathcal{H}_f = \{H \in Sym(n,\mathbb{R}) : f(H) = 0\},$$

for some fixed linear functional $f : Sym(n,\mathbb{R}) \to \mathbb{R}$. We say that the hyperplane $\mathcal{H}_f$ is *tangential* to the set $\mathcal{P}_A$ at a point P on its boundary if:

(i) $f(P) = 0$;

(ii) $f(H) \neq 0$ for all $H \in \mathcal{P}_A$.

Now consider a point P_0 on the boundary of $\mathcal{P}_A$ for which $A^T P_0 + P_0 A$ has rank $n-1$. In this case, there is a *unique* hyperplane tangential to $\mathcal{P}_A$ that passes through P_0, and moreover this hyperplane can be parameterized in a natural way. Formally, we have the following result.

Theorem 7.2.1 *Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz. Suppose that P_0 lies on the boundary of $\mathcal{P}_A$, and that the rank of $A^T P_0 + P_0 A$ is $n-1$, with $(A^T P_0 + P_0 A)x_0 = 0$, $x_0 \neq 0$. Then:*

(i) there is a unique hyperplane tangential to $\mathcal{P}_A$ at P_0;

(ii) this plane is given by

$$\{H \in Sym(n,\mathbb{R}) : x_0^T H A x_0 = 0\}.$$

A geometric perspective on Theorem 4.4.1:

We now describe how Theorem 7.2.1 can be used to give a more geometric proof of Theorem 4.4.1, highlighting the critical part played by the geometry of the cones $\mathcal{P}_A$ in establishing that result. So, as in Theorem 4.4.1, suppose that we have two n-dimensional exponentially stable LTI systems Σ_{A_1}, Σ_{A_2}, which do not have a CQLF, but for which there does exist a positive semi-definite $P = P^T \geq 0$ such that $A_i^T P + PA_i = Q_i \leq 0$, with $\text{rank}(Q_i) = n - 1$ for $i = 1, 2$. Then:

(i) there exists a hyperplane, $\mathcal{H}$, through the origin in $Sym(n, \mathbb{R})$ that separates the disjoint open convex cones $\mathcal{P}_{A_1}$, $\mathcal{P}_{A_2}$ [116];

(ii) any hyperplane separating $\mathcal{P}_{A_1}$, $\mathcal{P}_{A_2}$ must contain the matrix P, and be tangential to both $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$ at P;

(iii) there exist non-zero vectors x_1, x_2 in $\mathbb{R}^n$ such that $Q_i x_i = 0$ for $i = 1, 2$.

Now on combining (i) and (ii) with Theorem 7.2.1, we can see that in fact there is a unique hyperplane $\mathcal{H}$ separating $\mathcal{P}_{A_1}$, $\mathcal{P}_{A_2}$. Furthermore, we can use (iii) and Theorem 7.2.1 to parameterize $\mathcal{H}$ in two different ways. Namely:

$$\begin{aligned} \mathcal{H} &= \{H \in Sym(n, \mathbb{R}) : x_1^T H A_1 x_1 = 0\} \\ &= \{H \in Sym(n, \mathbb{R}) : x_2^T H A_2 x_2 = 0\}. \end{aligned} \tag{7.1}$$

It now follows that there must be some constant $k > 0$ such that

$$x_1^T H A_1 x_1 = -k x_2^T H A_2 x_2,$$

for all H in $Sym(n, \mathbb{R})$. The result of Theorem 4.4.1 now follows on applying Lemma 4.3.6.

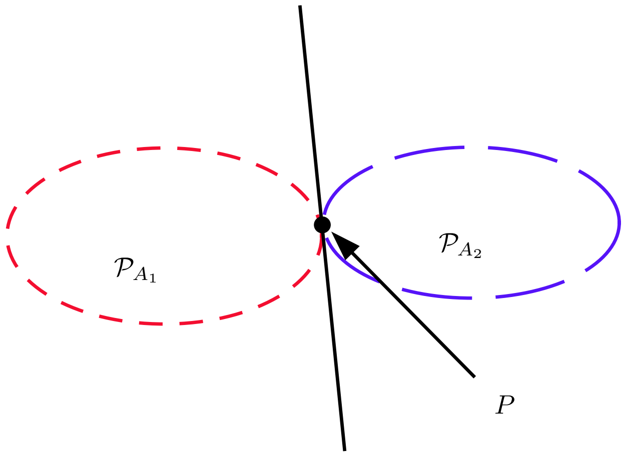

Figure 7.1: Under the hypotheses of Theorem 4.4.1, there is a unique separating hyperplane through P

Comments:

The above argument highlights those properties of the boundary of $\mathcal{P}_A$ that play a role in determining the result of Theorem 4.4.1. A key idea in the argument is that under the hypotheses of the theorem, any hyperplane that separates $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$ must be tangential to both cones. The fact that it is tangential to both cones at a point P for which $\mathrm{rank}(A_i^T P + PA_i) = n - 1$, $i = 1, 2$ means that we can use Theorem 7.2.1 to parameterize this hyperplane in two different ways. The result then follows by equating these two parameterizations.

Possible extensions of earlier results:

One natural way of extending the work of Chapters 4 and 5 is to attempt to derive results similar to Theorem 4.4.1 for cases where the ranks of the matrices Q_i appearing in the statement of that theorem take values other than $n - 1$. For instance, we could consider a pair of exponentially stable LTI systems Σ_{A_1}, Σ_{A_2} that do not have a CQLF, but for which there is some $P = P^T \geq 0$ with

$$A_i^T P + PA_i = Q_i \leq 0, \quad \text{rank}(Q_i) \in \{n-1, n-2\} \text{ for } i = 1, 2. \tag{7.2}$$

If it were possible to prove a result similar to Theorem 4.4.1 for this situation, then it could be used to derive necessary and sufficient conditions for CQLF existence for pairs of third order systems following similar arguments to those that led to the result of Theorem 5.2.1 for pairs of second order LTI systems.

Geometric arguments such as those employed above to re-derive Theorem 4.4.1 could again be used to obtain a corresponding result for the situation (7.2) if a parametrization of the hyperplanes that are tangential to the cone $\mathcal{P}_A$ at a point P on its boundary for which $\text{rank}(A^T P + PA) = n - 2$ was known. Unfortunately, parameterizing these hyperplanes is not nearly as straightforward as it was in the 'rank $n-1$' case. For instance, consider a point P on the boundary of $\mathcal{P}_A$ for which the rank of $A^T P + PA$ is $n-2$. Then, for any linearly independent vectors, x, y in the kernel of $Q = A^T P + PA$, and any non-negative constants α, β (not both zero), the hyperplane

$$\mathcal{H} = \{H \in Sym(n, \mathbb{R}) : \alpha x^T HAx + \beta y^T HAy = 0\} \tag{7.3}$$

is tangential to the set $\mathcal{P}_A$ at P. Hence, there are many more possible parameterizations for hyperplanes tangential to $\mathcal{P}_A$ at such points P than was the case for the 'rank $n-1$' case. Furthermore, the more complicated nature of the parameterizations (7.3) makes the task of extending Theorem 4.4.1 difficult and suggests that the conditions that would result from an analysis of situations where Theorem 4.4.1 cannot be applied would be too complicated to be of practical use. In fact, recently published work [58] indicates that necessary and sufficient conditions for CQLF existence for general systems, while interesting, will be overly complex and impossible to check. It is for this reason that we have mainly focussed on identifying system classes to which Theorem 4.4.1 applies, as we have seen that for such systems, it is possible to obtain conditions for CQLF existence that are easily verifiable and that relate the existence of CQLFs to the dynamics of the associated switched linear systems.

To emphasize some of the difficulties involved in extending Theorem 4.4.1 that were outlined in the last paragraph, consider two LTI systems Σ_{A_1}, Σ_{A_2} which do not have a CQLF but for which there is a positive semi-definite P satisfying (7.2). Then, as before, there is a hyperplane that separates the two open convex cones $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$, and this hyperplane must be tangential to both cones at P. Now, in the scenario considered in Theorem 4.4.1, we were able to conclude that there was some pair of vectors x_1, x_2 in $\mathbb{R}^n$, and a positive constant $k > 0$ such that

$$x_1^T H A_1 x_1 = -k x_2^T H A_2 x_2$$

for all $H \in Sym(n, \mathbb{R})$. However, in the case (7.2) the more complex boundary structure of $\mathcal{P}_A$ means that several other possibilities could also arise. For instance, one possibility that would have to be considered is where there exist vectors x_1, x_2, y_2 in $\mathbb{R}^n$ and constants $k > 0$, $\beta > 0$ such that

$$x_1^T H A_1 x_1 = -k(x_2^T H A_2 x_2 + \beta y_2^T H A_2 y_2)$$

for all $H \in Sym(n, \mathbb{R})$. Another case that would have to be dealt with is where there are vectors x_1, y_1, x_2, y_2 in $\mathbb{R}^n$ and constants $\beta_1 > 0$, $\beta_2 > 0$, $k > 0$ such that

$$(x_1^T H A_1 x_1 + \beta_1 y_1^T H A_1 y_1) = -k(x_2^T H A_2 x_2 + \beta_2 y_2^T H A_2 y_2)$$

for all $H \in Sym(n, \mathbb{R})$.

The above observations show that extending Theorem 4.4.1 to situations such as that described by (7.2) is far from straightforward and, more significantly, that an analysis of such situations would result in conditions on the matrices A_1, A_2 which are too complex to be verifiable or useful in any real sense. This provides further motivation for the problem of finding system classes to which Theorem 4.4.1 can be applied as we have seen before that it is possible to obtain conditions for CQLF existence for such systems that are practically useful and dynamically meaningful.

The final two results considered in this section concern pairs of Hurwitz matrices A_1, A_2 with $\mathrm{rank}(A_2 - A_1) = 1$. First of all, we note the following fact about the types of simultaneous solutions that can exist to a pair of Lyapunov inequalities corresponding to two Hurwitz matrices A_1, A_2 with $\mathrm{rank}(A_2 - A_1) = 1$.

Lemma 7.2.2 *Let $A_1 \in \mathbb{R}^{n\times n}$ be Hurwitz and suppose that P lies on the boundary of $\mathcal{P}_{A_1}$ with*

$$A_1^T P + PA_1 = Q_1 \leq 0.$$

Then if $P \in \mathcal{P}_{A_2}$ for some Hurwitz matrix $A_2 \in \mathbb{R}^{n\times n}$ with $\mathrm{rank}\,(A_2 - A_1) = 1$, the rank of Q_1 must be $n-1$.

<u>**Comments:**</u>

> An immediate consequence of Lemma 7.2.2 is that for two Hurwitz matrices A_1, A_2 in $\mathbb{R}^{n\times n}$ with $\mathrm{rank}(A_2 - A_1) = 1$, there can exist no positive definite matrix P such that
>
> $$\begin{aligned} A_1^T P + PA_1 = Q_1 &\leq 0 \\ A_2^T P + PA_2 = Q_2 &< 0 \end{aligned}$$
>
> with $\mathrm{rank}\,(Q_1) \leq n-2$.

The result of Lemma 7.2.2 is illustrated in Figure 7.2 below.

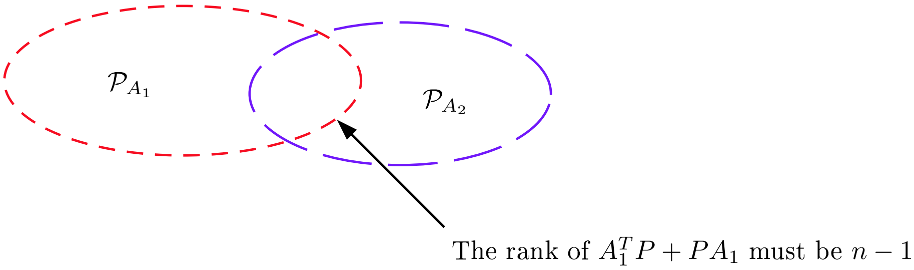

Figure 7.2: P on the boundary of $\mathcal{P}_{A_1}$, and in the interior of $\mathcal{P}_{A_2}$ with $\text{rank}(A_2 - A_1) = 1$ implies $\text{rank}(A_1^T P + PA_1) = n-1$

Finally for this section, we note a curious technical fact about the left and right eigenvectors of singular matrix pencils, which is a consequence of our earlier work on the CQLF existence problem for systems whose system matrices differ by rank one.

Theorem 7.2.2 *Let A_1, A_2 be Hurwitz matrices in $\mathbb{R}^{n\times n}$ with $\text{rank}(A_2 - A_1) = 1$. Suppose that there is exactly one value of $\gamma_0 > 0$ for which $A_1^{-1} + \gamma_0 A_2$ is singular. Then for this γ_0:*

(i) Up to scalar multiples, there exist unique vectors $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^n$ such that

$$(A_1^{-1} + \gamma_0 A_2)x_0 = 0, \;\; y_0^T(A_1^{-1} + \gamma_0 A_2) = 0;$$

(the left and right eigenspaces are one dimensional)

(ii) for this x_0 and y_0, it follows that

$$y_0^T A_1^{-1} x_0 = 0, \quad y_0^T A_2 x_0 = 0.$$

7.3 Open questions

In this section, we shall discuss a number of open questions related to the work of earlier chapters. First of all, we shall consider the problem of extending the analysis of the CQLF existence problem for linear systems presented in Chapters 4 and 5 to switched non-linear systems, and point out some aspects of the earlier analysis that can be carried over to the non-linear case. Subsequently, we shall also describe several open problems on CQLF existence for families of linear time-invariant systems.

7.3.1 CQLFs and switched non-linear systems

While thus far we have only dealt with linear systems, the problem of CQLF existence can also arise in the stability analysis of switched non-linear systems of the form

$$\dot{x} = f(x,t), \qquad f(x,t) \in \{f_1(x), \ldots, f_k(x)\}, \tag{7.4}$$

which are constructed by switching between the associated family of non-linear systems

$$\Sigma_{f_i} : \dot{x} = f_i(x) \quad 1 \leq i \leq k, \tag{7.5}$$

where $f_1, \ldots, f_k$ are globally Lipschitz [141] non-linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$, with $f_i(0) = 0$ for $1 \leq i \leq k$. [1] Specifically, if each individual system Σ_{f_i} has a quadratic Lyapunov function, $V(x) = x^T P x$, then we may ask under what conditions the family of systems $\Sigma_{f_1}, \ldots, \Sigma_{f_k}$ has a CQLF. Formally, this amounts to asking if there exists some positive definite matrix $P = P^T > 0$ such that

$$x^T P f_i(x) < 0 \text{ for all non-zero } x \in \mathbb{R}^n, \tag{7.6}$$

[1] For the remainder of this section, all vector fields being considered shall be assumed to be globally Lipschitz and to satisfy $f(0) = 0$. This is to ensure the existence of solutions, and that the origin is an equilibrium point of each of the systems.

for $1 \leq i \leq k$.

Unsurprisingly, there are a number of complications that arise when we attempt to apply our earlier methods to non-linear systems, and several simplifying assumptions are necessary in order to make any progress. For instance, for the CQLF existence problem to make sense, it is necessary to assume that each of the constituent non-linear systems, Σ_{f_i}, has a quadratic Lyapunov function. While for linear systems this merely amounts to assuming the exponential stability of the constituent systems, this is not necessarily the case for non-linear systems and it may be a restrictive condition to impose on the systems Σ_{f_i}. Moreover in the linear case, as we assume that the constituent systems are exponentially stable, it follows that the system matrices are all invertible (Hurwitz in fact). Once again, the situation for non-linear systems may be more complicated, and we shall need to assume that the mappings f_i being considered are invertible. In this subsection, we shall consider the CQLF existence problem for a pair of non-linear systems, Σ_{f_1}, Σ_{f_2}, and look at ways of applying the techniques previously described for linear systems to the non-linear case. Throughout, we shall make the following simplifying assumptions:

(i) each individual system Σ_{f_i}, for $1 \leq i \leq k$ has a quadratic Lyapunov function;

(ii) each f_i is invertible.

We shall now show that under these assumptions, it is possible to extend some of the techniques developed earlier for linear systems to the CQLF existence problem for a pair of non-linear systems Σ_{f_1}, Σ_{f_2}.

The convex cones $\mathcal{P}_f$:

First of all, recall that our analysis of the CQLF existence problem for families of LTI systems was based on studying the convex cones $\mathcal{P}_A$, for A Hurwitz. Analogously, for a globally Lipschitz, non-linear mapping, $f : \mathbb{R}^n \to \mathbb{R}^n$, we can define the set

$$\mathcal{P}_f = \{P = P^T > 0 : x^T P f(x) < 0 \text{ for all non-zero } x \in \mathbb{R}^n\}. \tag{7.7}$$

Then, as with LTI systems, $\mathcal{P}_f$ is an open convex cone in $Sym(n, \mathbb{R})$.

Furthermore, the fundamental result of Theorem 3.2.2 has the following analogue for the non-linear case.

Lemma 7.3.1 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be globally Lipschitz and invertible, with inverse f^{-1}. Then for any positive definite matrix $P = P^T > 0$,*

$$x^T P f(x) < 0 \textit{ for all non-zero } x \in \mathbb{R}^n$$

if and only if

$$x^T P f^{-1}(x) < 0 \textit{ for all non-zero } x \in \mathbb{R}^n.$$

Lemma 7.3.1 means that the cones $\mathcal{P}_f$ and $\mathcal{P}_{f^{-1}}$ coincide, as was the case for LTI systems.

Necessary conditions for CQLF existence:

Lemma 4.3.1 described two simple matrix pencil conditions that were necessary for a pair of exponentially stable LTI systems Σ_{A_1}, Σ_{A_2} to have a CQLF. We shall now derive corresponding necessary conditions for CQLF existence for a pair of non-linear systems Σ_{f_1}, Σ_{f_2}.

Lemma 7.3.2 *Let $f_1 : \mathbb{R}^n \to \mathbb{R}^n$, $f_2 : \mathbb{R}^n \to \mathbb{R}^n$ be globally Lipschitz and invertible, and suppose that there exists a CQLF for the associated systems Σ_{f_1}, Σ_{f_2}. Then for all $\gamma > 0$, and all non-zero $x \in \mathbb{R}^n$:*

$$\begin{aligned} f_1(x) + \gamma f_2(x) &\neq 0; \\ f_1^{-1}(x) + \gamma f_2(x) &\neq 0. \end{aligned}$$

Comments:

The above result states that if there exists a CQLF for the systems Σ_{f_1}, Σ_{f_2}, then there can be no non-zero vector x in $\mathbb{R}^n$ for which $f_1^{-1}(f_2(x)) = -\lambda x$, or $f_1(f_2(x)) = -\lambda x$ with $\lambda > 0$. Thus Lemma 7.3.2 is a non-linear extension of the fact that a necessary condition for the exponentially stable LTI systems, Σ_{A_1}, Σ_{A_2}, to have a CQLF is that the matrix products $A_1^{-1}A_2$ and A_1A_2 have no negative real eigenvalues.

Thus far, we have assumed the non-linear vector fields f_i being considered are globally Lipschitz and invertible, and that the associated non-linear systems Σ_{f_i} have quadratic Lyapunov functions. In the following results, we make the additional assumption that the individual vector fields are *homogeneous* of degree one, where a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be homogeneous of degree one if $f(\lambda x) = \lambda f(x)$ for all $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$. Under these conditions, we then have the following non-linear version of Lemma 4.3.2.

Lemma 7.3.3 *Let $f_1 : \mathbb{R}^n \to \mathbb{R}^n$, $f_2 : \mathbb{R}^n \to \mathbb{R}^n$ be globally Lipschitz, invertible, and homogeneous of degree one. Suppose that the systems Σ_{f_1}, Σ_{f_2} have no CQLF, and write $g = f_2 - f_1$. Then:*

(i) for sufficiently large values of $\alpha > 0$, the systems

$$\begin{aligned} \dot{x} &= f_1(x) \\ \dot{x} &= f_2(x) - \alpha x \end{aligned}$$

have a CQLF;

(ii) for sufficiently small values of $k > 0$, the systems

$$\begin{aligned} \dot{x} &= f_1(x) \\ \dot{x} &= f_1(x) + kg(x) \end{aligned}$$

have a CQLF.

Non-linear version of Theorem 4.4.1:

The final point that we wish to make in this brief discussion of possible non-linear extensions of our earlier work, is that it is possible to adapt the arguments used in Theorem 4.4.1 to derive the following corresponding result for non-linear systems.

Theorem 7.3.1 *Let $f_1 : \mathbb{R}^n \to \mathbb{R}^n$, $f_2 : \mathbb{R}^n \to \mathbb{R}^n$ be globally Lipschitz, invertible, and homogeneous of degree one, and suppose that the systems Σ_{f_1}, Σ_{f_2} have no CQLF. Furthermore, suppose that there is some positive definite $P = P^T > 0$ such that, for $i = 1, 2$:*

(i) $x^T P f_i(x) \leq 0$ *for all* x *in* $\mathbb{R}^n$*;*

(ii) *there is a non-zero vector* $x_i \in \mathbb{R}^n$ *such that*

$$\{x \in \mathbb{R}^n : x^T P f_i(x) = 0\} = \{x \in \mathbb{R}^n : x = \lambda x_i \text{ for } \lambda \in \mathbb{R}\}.$$

Then either

(i) there exists some non-zero x in $\mathbb{R}^n$ and some $\gamma > 0$ such that $f_1(x) + \gamma f_2(x) = 0$

or

(ii) there exists some non-zero x in $\mathbb{R}^n$ and some $\gamma > 0$ such that $f_1^{-1}(x) + \gamma f_2(x) = 0$.

Comments:

In Theorem 7.3.1, we assume the existence of a positive definite P such that for $i = 1, 2$, $x^T P f_i(x)$ is negative for all values of x except for scalar multiples of some non-zero vector x_i. This mirrors the 'rank $n - 1$' assumption of Theorem 4.4.1. Note that under the hypotheses of Theorem 7.3.1, the necessary conditions for CQLF existence given in Lemma 7.3.2 are violated.

In this subsection we have seen that it is possible, under appropriate assumptions, to extend some of our earlier methods to non-linear systems. While the results presented here only represent an initial step along the road to deriving practically useful conditions for CQLF existence for non-linear systems, they raise the hope that it may be possible to derive results similar to Theorem 5.2.1 or Theorem 5.3.3 for nonlinear systems, following arguments like those developed in Chapter 5, and using some of the preliminary ideas discussed here.

7.3.2 Some open problems on linear systems

In the previous subsection, we discussed the substantial issue of extending the work of earlier chapters to the case of non-linear systems. We shall now describe several open problems concerning linear systems suggested by the work of earlier chapters. First of all, we shall consider a number of questions related to the results presented in Chapters 4 and 5 on the CQLF existence problem. We shall then turn our attention to the stability question for positive switched linear systems dealt with in Chapter 6, highlighting several open problems in this area also.

CQLF existence for systems whose system matrices differ by rank two:

Throughout the work on the CQLF existence problem presented in Chapters 4 and 5, our primary interest was in obtaining necessary and sufficient conditions for CQLF existence that were dynamically meaningful, and at the same time simple enough to be practically useful. For this reason, we have focussed on specific system classes for which it is possible to obtain such conditions, such as pairs of second order systems, and pairs of systems whose system matrices differ by rank one. Furthermore, it should be noted that recent work on the CQLF existence problem for general LTI systems

[58] has illustrated that any necessary and sufficient conditions for CQLF existence derived for general pairs of LTI systems would be too complex to be verifiable in practice.

Initial numerical investigations have indicated that it may also be possible to derive simple conditions that are necessary and sufficient for CQLF existence for pairs of LTI systems whose system matrices differ by rank two. In view of this, we suggest the following open problem.

> *Determine necessary and sufficient conditions for a pair of exponentially stable LTI systems Σ_{A_1}, Σ_{A_2}, to have a CQLF, where* $\mathrm{rank}(A_2 - A_1) = 2$.

Identification of system classes to which Theorem 4.4.1 applies:

We have seen in Chapter 5 that for system classes to which the result of Theorem 4.4.1 can be applied, it is possible to obtain necessary and sufficient conditions for CQLF existence that are both easily verifiable and dynamically meaningful. Moreover, in view of the discussion about possible extensions of Theorem 4.4.1 in Section 7.2, it seems likely that in situations where this theorem cannot be applied, it will not be possible to obtain such simple conditions for CQLF existence. These considerations, naturally motivate the following very important problem.

> *Identify further classes of LTI systems (in addition to second order systems and systems whose system matrices differ by rank one) to which Theorem 4.4.1 can be successfully applied.*

CQLFs and conservatism:

The necessary and sufficient conditions for CQLF existence for pairs of second order LTI systems, given in Theorem 5.2.1, provided insights into the conservatism of CQLF existence as a criterion for the exponential stability for second order switched linear systems. Further, in Corollary 5.3.1 on pairs of exponentially stable LTI systems Σ_{A_1}, Σ_{A_2} with $\mathrm{rank}(A_2 - A_1^{-1}) = 1$, and Theorem 6.3.1 on pairs of exponentially stable second order positive LTI systems, we have identified classes of switched linear systems for which CQLF existence is equivalent to exponential stability under arbitrary switching.

In this regard, consider a switched linear system that becomes unstable through a 'chattering' like motion, such as is depicted in Figure 7.3 below.

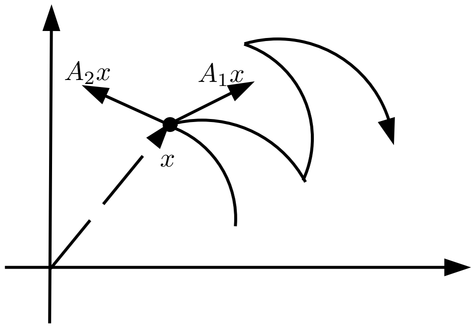

Figure 7.3: Instability and 'chattering'

Loosely speaking, this indicates that in the marginal situation between the existence and non-existence of such a chattering instability, the vector fields of the two systems Σ_{A_1}, Σ_{A_2} would be similar to those shown in Figure 7.4.

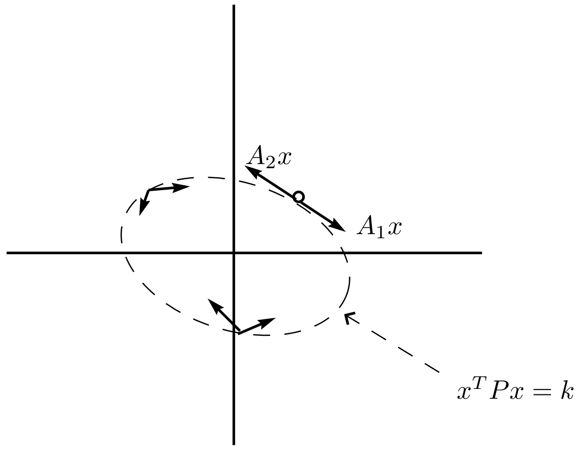

Figure 7.4: CQLF existence and 'chattering'

This simple observation suggests that for a switched linear system that can only become unstable through chattering, the limit of CQLF existence may well coincide with the limit of exponential stability under arbitrary switching and hence that CQLF existence may not be a conservative way of establishing exponential stability under arbitrary switching for such systems.

In the light of the above points and the need for a greater understanding of the precise relationship between CQLF existence and the exponential stability of switched linear systems, we suggest the following general problem.

> *Identify classes of switched linear systems for which the existence of a CQLF for their constituent systems is equivalent to exponential stability under arbitrary switching.*

CQLF existence for positive LTI systems:

While the conjecture of [76], that $A_1A_2^{-1}$ having no negative real eigenvalues was necessary and sufficient for the stable positive LTI systems Σ_{A_1}, Σ_{A_2} to have a CQLF has been shown to be untrue, numerical simulations indicate that counterexamples to the conjecture are quite rare. This observation raises the hope that it may be possible to derive related necessary and sufficient conditions for a pair of exponentially stable positive LTI systems to have a CQLF. Thus, we have the following problem.

> *Determine necessary and sufficient conditions for a pair of exponentially stable positive LTI systems Σ_{A_1}, Σ_{A_2} to have a CQLF.*

Positive systems whose system matrices differ by rank one:

In Theorem 6.3.1 and Theorem 6.3.2, we have seen that for Metzler, Hurwitz matrices, A_1, A_2 in $\mathbb{R}^{2\times2}$ or $\mathbb{R}^{3\times3}$ the product A_1A_2 cannot have any negative real eigenvalues. These facts meant that there exists a CQLF for any pair of exponentially stable positive LTI systems Σ_{A_1}, Σ_{A_2}, where $A_1, A_2 \in \mathbb{R}^{2\times2}$ or $A_1, A_2 \in \mathbb{R}^{3\times3}$, and $\text{rank}(A_2 - A_1) = 1$. These results lead to the problem of either proving or refuting the following conjecture.

> *Given any Metzler, Hurwitz matrices A_1, A_2 in $\mathbb{R}^{n\times n}$, the product A_1A_2 cannot have any negative real eigenvalues.*

Conditions for CDLF existence for positive systems:

Theorem 6.5.2 described a compact algebraic condition that was necessary and sufficient for a generic pair of exponentially stable positive LTI systems of any order to have a CDLF. However, the condition given in the theorem is not readily verifiable in its current form. In view of this, it is natural to ask whether it is possible to derive alternative conditions that are simpler to check, or to use the result of Theorem 6.5.2 to obtain verifiable conditions for CDLF existence for significant classes of positive LTI systems. Hence, we have the following problem.

> *Derive necessary and sufficient conditions for CDLF existence for pairs*

of positive LTI systems that are simpler than the condition given in Theorem 6.5.2. In particular, determine necessary and sufficient conditions for CDLF existence for pairs of third order positive LTI systems.

Necessary and sufficient conditions for common copositive Lyapunov function existence:

In Chapter 6, we presented verifiable necessary and sufficient conditions for the existence of common quadratic and linear copositive Lyapunov functions for pairs of second order positive LTI systems. While the general conditions for common linear copositive Lyapunov function existence given in Theorem 6.6.4 apply to systems of any order, they are not in a form that may be readily checked. Hence, it is of interest to determine simple conditions, similar to those obtained for second order systems, for common copositive Lyapunov function existence for higher order systems. This leads us to the following problem which is the final one that we shall list here.

Determine verifiable necessary and sufficient conditions for the existence of a common quadratic (or linear) copositive Lyapunov function for a pair of exponentially stable positive LTI systems, Σ_{A_1}, Σ_{A_2}, where $A_1 \in \mathbb{R}^{n\times n}$, $A_2 \in \mathbb{R}^{n\times n}$ with $n > 2$.

7.4 Concluding remarks

In this chapter, we presented some preliminary results on the boundary structure of the convex cones $\mathcal{P}_A$ for a given Hurwitz matrix A, and described a variety of open questions related to the work of this thesis. The most important points in the chapter are now listed.

- We derived some technical results that provide a more geometric perspective on the key result of Theorem 4.4.1, and have highlighted the role played by the geometry of the cones $\mathcal{P}_A$ in the proof of that result.

- We have explained why we concentrated on finding systems to which Theorem 4.4.1 can be applied rather than on attempting to generalize that result. In particular, we have shown why necessary and sufficient conditions for CQLF existence for general systems would be too complicated to be of practical use.

- We have also presented two facts about simultaneous solutions of pairs of Lyapunov inequalities, and singular matrix pencils.

- We discussed the possibility of extending our methods and results to non-linear switched systems, and have shown that some of our earlier results can be thus extended.

- Several open problems arising from the work of earlier chapters have been described.

Chapter 8

Conclusions and some final remarks

In this final chapter, we present some brief concluding remarks, summarizing the work of earlier chapters, and highlighting the main contributions made in the course of the thesis.

8.1 Conclusions and discussion

The recent increase in the number of switched systems occurring in engineering applications, and the fact that the critical issue of stability is still unresolved for such systems, have been the major motivations for the work described in this thesis. With the aim of adding to the current understanding of the stability issues associated with switched linear systems, we have considered the question of common Lyapunov function existence for families of exponentially stable LTI systems; concentrating on deriving simple, *verifiable* conditions for common Lyapunov function existence that can be used to establish the exponential stability of switched linear systems, and on gaining insights into the relationship between common Lyapunov function existence and the dynamics of switched linear systems. We shall now review the contents of the thesis, discussing the major points made throughout the work.

The motivation and background for the work of the thesis were presented in the opening two chapters. In particular, in the first chapter we discussed the practical importance of switched systems, pointed out some of the problems that can arise as a result of switching, and highlighted the need for conditions that can be used to establish stability for switched systems. In the second chapter we then formally defined the class of switched linear systems and illustrated the difficulties that can arise in analyzing the stability of such systems; thus motivating the work of the remainder of the thesis. We also described some of the general work done on the stability theory of switched linear systems in the recent past.

Most of the original results presented in the thesis are concerned with the CQLF existence problem for families of exponentially stable LTI systems, and in Chapter 3, we formally introduced this problem and provided a survey of the literature available on it, presenting results from both the mathematics and engineering literatures. In the same chapter, we pointed out the need to understand the precise connection between CQLF existence and the exponential stability of switched linear systems. In particular, the issue of identifying classes of switched linear systems for which the existence of a CQLF for their constituent systems is not a conservative way of establishing stability was raised. It is my view that this is a highly important problem, as the knowledge that CQLF existence is equivalent to exponential stability for a class of switched linear systems would considerably simplify the stability analysis of this class of systems.

One of the main contributions of the thesis appeared in Chapter 4, where we presented a novel framework within which to tackle the CQLF existence problem for a pair of stable LTI systems, and derived the key result of Theorem 4.4.1. Several points about this result should be emphasized. Firstly, it provides insight into the relationship between CQLF existence and the exponential stability of switched linear systems, and has implications for the conservatism of CQLF existence as a criterion for the exponential stability of switched linear systems. In fact, suppose that we have a pair of exponentially stable LTI systems, Σ_{A_1}, Σ_{A_2}, satisfying the hypotheses of Theorem 4.4.1. Then on the one hand, Σ_{A_1}, Σ_{A_2} are right on the 'boundary' of those pairs of systems that have a CQLF, while at the same time, it follows from Theorem 2.3.1 that at least one of the associated switched linear systems

$$\begin{aligned} \dot{x} &= A(t)x, \qquad A(t) \in \{A_1, A_2\} \\ \dot{x} &= A(t)x, \qquad A(t) \in \{A_1, A_2^{-1}\} \end{aligned}$$

is *not* exponentially stable. Thus, loosely, Theorem 4.4.1 describes situations in which the boundary of CQLF existence and the boundary of exponential stability coincide.

As well as giving insights into the nature of the conservatism of CQLF existence as a criterion for the exponential stability of switched linear systems, Theorem 4.4.1 provides a scheme for obtaining simple, meaningful conditions for CQLF existence for classes of switched linear systems. A key aspect of the result is that its conclusion describes conditions on the system matrices A_1, A_2 that are not only simple to check, but, in the light of Theorem 2.3.1, also have a clear meaning in terms of the dynamics of switched linear systems. In fact, the simple form of these conditions suggests that if we can identify classes of systems to which the theorem applies, then it will be possible to derive conditions for CQLF existence for these system classes that are both easily verifiable and have direct implications for the exponential stability of the associated switched linear systems. Two examples illustrating this point were provided in Chapter 5, where we saw how Theorem 4.4.1 unifies two of the most powerful results giving necessary and sufficient conditions for CQLF existence that have previously appeared in the literature. The work of Chapter 5 demonstrated the crucial role played by Theorem 4.4.1 in determining the simple conditions for CQLF existence for pairs of second order systems, and pairs of systems of arbitrary dimension whose system matrices differ by rank one. In discussing Theorem 4.4.1, it should also be noted that the same underlying ideas that were used to derive this result for continuous-time LTI systems could also be successfully applied to obtain a corresponding result for pairs of discrete-time LTI systems in Theorem 4.4.2.

In Chapter 5, in addition to showing how to use the results of Chapter 4 to obtain verifiable necessary and sufficient conditions for CQLF existence, and demonstrating the role that these results play in determining simple conditions for CQLF existence, we also extended the result of Theorem 3.5.4 (derived in [128]) on CQLF existence for pairs of LTI systems with system matrices in companion form to the case of a pair of exponentially stable LTI systems whose system matrices differ by a general rank one matrix. Further extensions of this result were also given in Corollary 5.3.1 (to systems Σ_{A_1}, Σ_{A_2} with $\text{rank}(A_2 - A_1^{-1}) = 1$) and Corollary 5.3.2 (to systems Σ_{A_1}, Σ_{A_2} for which there is some $c > 0$ with $\text{rank}(A_2 - cA_1) = 1$). It should also be noted that in Corollary 5.3.1 we described an entire class of switched linear systems for which CQLF existence was equivalent to uniform exponential stability under arbitrary switching.

The work of Chapter 6 on the stability of positive switched linear systems arose out of an attempt to identify further system classes to which Theorem 4.4.1 could be applied. In fact, numerical investigations had initially led us to conjecture that exponentially stable positive LTI systems would provide another example of a system class which could be treated within the framework of that theorem. Based on these numerical observations and some preliminary results, we formed the conjecture (reported in [76]) that the matrix product $A_1 A_2^{-1}$ having no negative real eigenvalues was equivalent to the existence of a CQLF for a pair of exponentially stable positive LTI systems Σ_{A_1}, Σ_{A_2}. While this conjecture is false in general, we have shown it to be true for second order systems in Theorem 6.3.1. This result then furnished us with another class of switched linear systems for which CQLF existence is equivalent to exponential stability. Furthermore, our investigation of the CQLF existence problem for positive LTI systems has led to the facts, expressed in Corollary 6.3.1 and Theorem 6.3.3, that for second and third order systems, any pair of exponentially stable positive LTI systems whose system matrices differ by rank one must have a CQLF, and that the associated switched linear systems must be uniformly exponentially stable under arbitrary switching.

While the conjecture of [76] turned out to be false, it has led to the results discussed in the previous paragraph. Moreover, the rationale behind this conjecture was successfully used to derive the necessary and sufficient condition for CDLF existence presented in Theorem 6.5.2. In this context, the result of Corollary 6.5.1 is particularly noteworthy as it shows that the 'rank $n-1$' conditions required in Theorem 4.4.1, are guaranteed to be satisfied when considering the CDLF existence problem for pairs of exponentially stable positive LTI systems with irreducible system matrices. It is also of interest that the form of the condition for CDLF existence given in Theorem 6.5.2 is related to that of the matrix pencil conditions for CQLF existence derived for second order systems and systems with system matrices differing by rank one, and that the techniques developed in Chapter 4 for the CQLF existence problem can be fruitfully applied to the CDLF existence problem as well.

When investigating the stability of positive switched linear systems, copositive Lyapunov functions arise naturally. In the final sections of Chapter 6, we considered the problems of common quadratic, and common linear, copositive Lyapunov function existence for pairs of exponentially stable positive LTI systems, and derived simple necessary and sufficient conditions for the existence of such functions (in both the linear and quadratic cases) for pairs of positive second order LTI systems. Furthermore, in Theorem 6.6.4, the ideas that were developed in Chapter 4 in the context of the CQLF existence problem were used to derive a necessary and sufficient condition for the existence of a common linear copositive Lyapunov function for a general pair of exponentially stable positive LTI systems. Once again, this further illustrates the adaptability of these techniques. Other minor contributions made in Chapter 6 include the simple sufficient conditions for CDLF existence and common linear copositive Lyapunov function existence for pairs of positive LTI systems given in Theorem 6.4.1 and Theorem 6.6.2 respectively, and the description of how to cast the common linear copositive Lyapunov function existence problem as a feasibility problem in LMIs.

The convex cones,

$$\mathcal{P}_A = \{P = P^T > 0 : A^T P + PA < 0\},$$

for a given Hurwitz A, and their boundary structure in particular, played a crucial role in much of the analysis of the CQLF existence problem presented in Chapters 4 and 5. In Chapter 7, we described a number of technical results on the boundary structure of these cones that provided new geometrical insights into the work and results of earlier chapters. In particular, we showed that those matrices P on the boundary of $\mathcal{P}_A$ for which the rank of $A^T P + PA$ is $n - 1$ are dense within the boundary. This explains why the conditions of Theorem 4.4.1 are so often satisfied in numerical simulations, and indicates that it may well be possible to find further system classes to which Theorem 4.4.1 may be applied to obtain simple necessary and sufficient conditions for CQLF existence. Another fact established in Chapter 7 was that through any matrix P on the boundary of $\mathcal{P}_A$ for which the rank of $A^T P + PA$ is $n - 1$, there exists one and only one hyperplane that is tangential to the cone at that point. The extremely simple structure of the boundary of $\mathcal{P}_A$ at such points provides a straightforward geometrical explanation of why Theorem 4.4.1 is true.

In Chapter 7, we also explained our reasons for concentrating on identifying system classes to which Theorem 4.4.1 could be applied rather than attempting to derive more general versions of that theorem. Essentially, in situations where the theorem does not apply, necessary and sufficient conditions for CQLF existence are likely to be too complicated to be of genuine practical use. This is also indicated by the results recently published in [58]. In the light of this observation, the question of identifying system classes to which Theorem 4.4.1 can be applied takes on added importance, as we have seen that it is likely that simple, dynamically meaningful conditions for CQLF existence can be derived for such system classes.

In the final section of Chapter 7, we listed a number of problems suggested by the work of this thesis, and presented some preliminary results indicating that it may be possible to extend the techniques that we have developed for the CQLF existence problem for linear systems to non-linear systems. Two other results of Chapter 7 that should be noted are Lemma 7.2.2 and Theorem 7.2.2 on simultaneous solutions of the Lyapunov inequality and the right and left eigenvectors of singular matrix pencils respectively.

8.2 Future work

As we have already described a number of open problems relating to the work of this thesis in Chapter 7, here we merely present a brief discussion of some of the major topics for future work suggested by the results of earlier chapters.

In the context of the CQLF existence problem and its relation to the exponential stability of switched linear systems, arguably the most important question that needs to be addressed is the problem of identifying classes of systems to which Theorem 4.4.1 can be applied. The point has been made earlier that for such system classes it should be possible to derive conditions for CQLF existence, using techniques similar to those developed in Chapters 4 and 5, that are both easy to verify and interpretable in terms of the dynamics of switched linear systems. On the other hand, the work of Chapter 7 indicates that for more general system classes where Theorem 4.4.1 cannot be applied, necessary and sufficient conditions for CQLF existence will be overly complicated, and difficult both to verify and interpret. Hence, the problem of finding further system classes to which the theorem can be applied is of paramount importance and great practical interest.

Of the other problems relating to CQLF existence that were listed in Chapter 7, perhaps the next most natural to ask is whether or not conditions for CQLF existence can be derived for pairs of LTI systems with system matrices differing by a rank two matrix. The fact that the conditions for the case of a pair of systems whose system matrices differ by a rank one matrix take so simple a form, together with the results of some initial numerical investigations indicate that it may be possible to obtain useful conditions for CQLF existence in this case also. Of course, a natural approach to the problem would be to investigate whether or not Theorem 4.4.1 can be applied in this situation.

There are also a number of open questions related to the work on positive switched linear systems described in Chapter 6. We have remarked above that while the conjecture on CQLF existence for pairs of positive LTI systems made in [76] is false, the rationale behind it has led to a number of novel results. In addition to this, numerical testing indicates that counterexamples to the conjecture are quite rare. This suggest that it may well be possible to derive some related conditions for CQLF existence for pairs of positive LTI systems, or to identify some large subclass of positive LTI systems for which the original conjecture is in fact true. Another issue that needs to be addressed is the question of whether or not the results of Corollary 6.3.1 and Theorem 6.3.3 on CQLF existence for pairs of second and third order positive LTI systems with system matrices differing by rank one, extend to higher dimensions. If so, this would mean that any positive switched linear system (of arbitrary dimension) constructed by switching between a pair of exponentially stable positive LTI systems whose system matrices differ by rank one must be uniformly exponentially stable under arbitrary switching.

A drawback of the general condition for CDLF existence for pairs of positive LTI systems given in Theorem 6.5.2 is that it is not verifiable in its current form. While some simple applications of the theorem were described in the text, it is natural to ask if this result can be used to obtain simplified conditions for CDLF existence for certain system classes. In this context, the problem of deriving verifiable necessary and sufficient conditions for a pair of third order positive LTI systems to have a CDLF was raised in the last chapter. Finally, it would also be of interest to investigate further the linear and quadratic copositive Lyapunov functions discussed in Chapter 6, and to determine whether simple, verifiable conditions, such as those given in Theorem 6.6.5 for second order systems, can be obtained for higher order systems also.

8.3 Final remarks

In the opening pages of this thesis, we presented a broad mission statement for the work as a whole; that being to add to the current understanding of the stability issues associated with switched linear systems, and, where possible, to derive verifiable conditions for the stability of such systems based on the existence of common Lyapunov functions for families of LTI systems. Looking back at the thesis at this point in the light of these goals, the following brief points should be noted.

(a) We have derived a number of results that give simple, useful conditions for CQLF existence for pairs of LTI systems. For instance, in Theorem 5.3.3 a simple condition for CQLF existence for pairs of LTI systems with system matrices differing by rank one was described, and this result was then extended in Corollary 5.3.2 to the case of a pair of LTI systems Σ_{A_1}, Σ_{A_2} for which there is some $c > 0$ with $\mathrm{rank}(A_2 - cA_1) = 1$. Similarly, the results of Theorem 6.3.1 and Theorem 6.3.3 on pairs of second and third order positive LTI systems are in a form that renders them readily usable.

(b) We have identified classes of switched linear systems for which CQLF existence is not a conservative criterion for exponential stability under arbitrary switching, and thereby gained insights into the relationship between CQLF existence and the dynamics of switched linear systems.

(c) The key result of Theorem 4.4.1 provides a general scheme for deriving simple, dynamically meaningful conditions for CQLF existence for certain system classes, and unifies two of the most important results on CQLF existence previously obtained in the literature.

(d) We have derived simple, verifiable sufficient conditions for CDLF existence and common linear copositive Lyapunov function existence for pairs of positive LTI systems.

(e) The same techniques used to tackle the CQLF existence problem led us to an algebraic condition that is necessary and sufficient for CDLF existence for a generic pair of n-dimensional positive LTI systems.

(f) Similar techniques were again used to derive necessary and sufficient conditions for a common linear copositive Lyapunov function to exist for a pair of positive LTI systems.

Finally, it is clear that there are many challenging problems relating to the stability of switched systems and the question of common Lyapunov function existence that are of considerable practical importance. While we have addressed a number of these in the course of this thesis, the remarks made in the previous section and the open problems listed in Chapter 7 demonstrate that the well is far from dry, and that there is still ample scope for further research within this interesting and important area.

Appendix A

Technical proofs for Chapter 4

Proof of Lemma 4.3.5:

Consider the norm $\|A\|_\infty = sup\{|a_{ij}| : 1 \le i, j \le n\}$ on $\mathbb{R}^{n\times n}$, and let z be any non-zero vector in $\mathbb{R}^n$. Then it is easy to see that the set $\{T \in \mathbb{R}^{n\times n} : \det(T) \neq 0, (Tz)_i \neq 0, 1 \le i \le n\}$ is open. On the other hand, if $T \in \mathbb{R}^{n\times n}$ is such that $(Tz)_i = 0$ for some i, an arbitrarily small change in an appropriate element of the i^{th} row of T will result in a matrix T' such that $(T'z)_i \neq 0$. From this it follows that arbitrarily close to the original matrix T, there is some $T_1 \in \mathbb{R}^{n\times n}$ such that $T_1 z$ is non-zero component-wise.

Now to prove the lemma, simply select a non-singular T_0 such that $T_0 x$ is non-zero component-wise. Suppose that some component of $T_0 y$ is zero. By the arguments in the previous paragraph, it is clear that we can select a non-singular $T_1 \in \mathbb{R}^{n\times n}$ such that each component of $T_1 x$ and $T_1 y$ is non-zero. Now it is simply a matter of repeating this step for the remaining vectors u and v to complete the proof of the lemma.

Proof of Lemma 4.3.6:

We can assume that all components of x, y, u, v are non-zero. To see why this is so, suppose that the result was proven for this case and that we are given four arbitrary non-zero vectors x, y, u, v. We could transform them via a single non-singular transformation T such that each component of Tx, Ty, Tu, Tv was non-zero (Lemma 4.3.5). Then for all symmetric matrices P we would have $(Tx)^T P(Ty) = x^T(T^T PT)y$, and hence, that $(Tx)^T P(Ty) = -k(Tu)^T P(Tv)$. Then $Tx = \alpha Tu$ and thus $x = \alpha u$ or $Tx = \beta Tv$ and $x = \beta v$. So we shall assume that all components of x, y, u, v are non-zero. Suppose that x is not a scalar multiple of u to begin with. Then for any index i with $1 \leq i \leq n$, there is some other index j and two non-zero real numbers c_i, c_j such that

$$x_i = c_i u_i,\ x_j = c_j u_j,\ c_i \neq c_j \tag{A.1}$$

Choose one such pair of indices i, j. Equating the coefficients of p_{ii}, p_{jj} and p_{ij} respectively in the identity $x^T Py = -ku^T Pv$ yields the following equations.

$$\begin{aligned} x_i y_i &= -k u_i v_i && \text{(A.2)} \\ x_j y_j &= -k u_j v_j && \text{(A.3)} \\ (x_i y_j + x_j y_i) &= -k(u_i v_j + u_j v_i) && \text{(A.4)} \end{aligned}$$

If we combine (A.1) with (A.2) and (A.3), we find

$$\begin{aligned} y_i &= -\frac{k}{c_i} v_i && \text{(A.5)} \\ y_j &= -\frac{k}{c_j} v_j && \text{(A.6)} \end{aligned}$$

Using (A.2)-(A.6) we find that $c_i u_i y_j + c_j u_j y_i = -k(u_i v_j + u_j v_i)$. Hence, $u_i v_j (\frac{c_j - c_i}{c_j}) = u_j v_i (\frac{c_j - c_i}{c_i})$. Recall that $c_i \neq c_j$ so we can divide by $c_j - c_i$ and rearrange terms to get

$$\frac{c_i}{c_j} = (\frac{v_i}{v_j})(\frac{u_j}{u_i}) \tag{A.7}$$

But using (A.1) we find

$$\frac{c_i}{c_j} = (\frac{x_i}{x_j})(\frac{u_j}{u_i}) \tag{A.8}$$

Combining (A.7) and (A.8) yields

$$\frac{v_i}{v_j} = \frac{x_i}{x_j} \tag{A.9}$$

Thus $x_i = cv_i$, $x_j = cv_j$ for some constant c. Now if we select any other index k with $1 \leq k \leq n$, and write $x_k = c_k u_k$ then c_k must be different to at least one of c_i, c_j. Without loss of generality, we may take it that $c_k \neq c_i$. Then the above argument can be repeated with the indices i and k in place of i and j to yield

$$x_i = cv_i,\, x_k = cv_k. \tag{A.10}$$

But this can be done for any index k so we conclude that $x = cv$ for a scalar c. So we have shown that if x is not a scalar multiple of u, then it is a scalar multiple of v.

To complete the proof, note that if $x = \beta v$ for a scalar β then by (A.2), $\beta v_i y_i = -k u_i v_i$ for all i. Thus $y = -(\frac{k}{\beta})u$ as claimed. The same argument will show that if $x = \alpha u$ for a scalar α, then $y = -(\frac{k}{\alpha})v$.

Appendix B

Technical proofs for Chapter 5

Proof of Lemma 5.4.2:

Without loss of generality, we may assume that the rank one matrix bc^T is in one the Jordan canonical forms given by:

$$bc^T = \begin{pmatrix} \mu & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & & & \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

or

$$bc^T = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 1 & \dots & \dots & 0 \\ \vdots & & & \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

, where $\mu \in \mathbb{R}$.

Suppose that $\lambda_0 > 0$ is such that $A(A - \lambda_0 bc^T)$ has a real negative eigenvalue. It follows that for this λ_0 there is some $\gamma_0 > 0$ such that $A^{-1} + \gamma_0(A - \lambda_0 bc^T)$ is singular.

Thus

$$\det(A^{-1} + \gamma_0(A - \lambda_0 bc^T)) = \det((A^{-1} + \gamma_0 A) - \lambda_0\gamma_0 bc^T) = 0.$$

For any $\gamma > 0$, it follows from considering the form of the matrix $(A^{-1}+\gamma A)-\lambda_0\gamma bc^T$, that we may write

$$\det((A^{-1} + \gamma A) - \lambda_0\gamma bc^T) = M(\gamma) + \lambda_0 N(\gamma)$$

where M and N are polynomials in γ.

We now note the following facts about the polynomials M and N.

(i) $M(\gamma) = \det(A^{-1} + \gamma A)$ is non-zero and of the same sign for all $\gamma > 0$ ($(A^{-1} + \gamma A)$ is always Hurwitz).

(ii) $M(0) + \lambda N(0) = M(0)$ for any $\lambda > 0$.

For convenience, assume that $M(\gamma) > 0$ for all $\gamma > 0$. Now, $M(\gamma_0) + \lambda_0 N(\gamma_0) = 0$ and as $M(\gamma_0) > 0$, we must have $N(\gamma_0) < 0$. Then for any $\lambda > \lambda_0$

$$M(\gamma_0) + \lambda N(\gamma_0) < M(\gamma_0) + \lambda_0 N(\gamma_0) = 0.$$

But $M(0)+\lambda N(0) = M(0) > 0$, so by the intermediate value theorem, there is some γ_1 with $0 < \gamma_1 < \gamma_0$ such that

$$\det(A^{-1} + \lambda(A - \gamma_1 bc^T)) = M(\gamma_1) + \lambda N(\gamma_1) = 0$$

and hence the matrix product $A(A-\lambda bc^T)$ has a real negative eigenvalue as claimed.

Proof of Theorem 5.4.1:

Under the hypotheses of the theorem, we know that

$$\Gamma(\omega) = 1 + \mathrm{Re}\{c^T(j\omega I - A)^{-1}b\} \geq 0$$

for all $\omega \in \mathbb{R}$ and that there is definitely some value of ω for which this expression is zero. The proof is broken into a number of steps.

Step 1:

First of all, recall from Section 5.4.2 that we can write

$$\Gamma(\omega) = \frac{p(\omega)}{\det(\omega^2 I + A^2)},$$

where p is a monic even polynomial of degree $2n$. We shall first show that there exists a vector c' arbitrarily close to c such that if we define

$$\frac{p'(\omega)}{\det(\omega^2 I + A^2)} = 1 + \mathrm{Re}\{c'^T(j\omega I - A)^{-1}b\} \text{ for } \omega \in \mathbb{R}, \tag{B.1}$$

then the monic even polynomial p' satisfies

$$p'(\omega) \geq 0 \text{ for all } \omega \in \mathbb{R},$$

and has a unique positive real zero of multiplicity 2 at some $\omega_c > 0$ in $\mathbb{R}$.

By assumption, the mapping T defined in (5.13) is invertible with continuous inverse T^{-1}. Throughout the proof we shall identify a monic even polynomial in ω of degree $2n$ with the vector in $\mathbb{R}^n$ given by the coefficients of $\omega^0, \omega^2, \ldots, \omega^{2n-2}$. Write $p = T(c)$. Then the continuity of T^{-1} means that for any $\epsilon > 0$ there is some $\delta > 0$ such that $\|p - p'\| < \delta$ implies $\|c - T^{-1}(p')\| < \epsilon$.

Now $p(\omega) \geq 0$ for all $\omega \in \mathbb{R}$, so the zeroes of p occur as complex conjugate pairs or as real zeroes of even multiplicity. If p has more than a single positive real zero of multiplicity 2, then by replacing terms like $(\omega - \omega_1)(\omega - \omega_1)$ in the linear factorization of p by the terms $(\omega - (\omega_1 + j\delta_1))(\omega - (\omega_1 - j\delta_1))$, we can construct a monic even polynomial p' of degree $2n$, whose coefficients are arbitrarily close to those of the original p such that:

(i) $p'(\omega) \geq 0$ for all $\omega \in \mathbb{R}$;

(ii) p' has a single positive real zero of multiplicity two at some $\omega_c > 0$ in $\mathbb{R}$.

In particular, we can choose such a p' with $\|p'-p\| < \delta$ and thus defining $c' = T^{-1}(p')$, we have $\|c' - c\| < \epsilon$ and

$$1 + \mathrm{Re}\{c'^T(j\omega I - A)^{-1}b\} \geq 0$$

with a single positive real zero of multiplicity 2 at $\omega_c > 0$ as required. We can actually say slightly more than this. In fact, it is possible to combine the above argument with the fact that the real and imaginary parts of the vector $(j\omega I - A)^{-1}b$ cannot be co-linear to show that we can select a vector c' arbitrarily close to the original c with the above properties such that the imaginary part of $c'^T(j\omega I - A)^{-1}b$ is non-zero. We shall show that this c' is the vector required in the theorem.

Step 2:

For the rest of the proof, we shall write $S(\omega)$ for the expression $j\omega I - A$. The next stage is to show that for the matrices A and $A - bc'^T$, it is possible to find a positive semi-definite matrix $R \geq 0$ of rank $n-2$ such that

$$1 + \mathrm{Re}\{(c')^T S(\omega)^{-1}b\} \geq b^T(S(\omega)^*)^{-1}RS(\omega)^{-1}b \tag{B.2}$$

To begin with, choose any positive semi-definite matrix R' in $\mathbb{R}^{n\times n}$ of rank $n-2$ such that $R'S(\omega_c)^{-1}b = 0$ where ω_c is the only positive real zero of $1+\mathrm{Re}\{(c')^T S(\omega)^{-1}b\}$. We shall show that it is possible to scale this R' so that it satisfies the condition (B.2). First of all note that

$$b^T(S(\omega)^*)^{-1}R'S(\omega)^{-1}b = \frac{p_2(\omega)}{\det(\omega^2 I + A^2)}$$

where p_2 is an even polynomial of degree $2n-2$ in ω. Similarly, write

$$1 + \mathrm{Re}\{(c')^T S(\omega)^{-1}b\} = \frac{p_1(\omega)}{\det(\omega^2 I + A^2)}$$

where p_1 is a monic even polynomial of degree $2n$ in ω. As both p_1 and p_2 are even, it is enough to consider values of ω in $[0, \infty)$. Now as ω tends to infinity, the expression

$$1 + \mathrm{Re}\{(c')^T S(\omega)^{-1}b\} - b^T(S(\omega)^*)^{-1}R'S(\omega)^{-1}b$$

tends to 1. Thus, by continuity there is some constant $K > 0$ such that $1 + \mathrm{Re}\{(c')^T S(\omega)^{-1} b\} - b^T (S(\omega)^*)^{-1} R' S(\omega)^{-1} b > 0$ for all ω with $\omega > K$. Furthermore

$$\begin{aligned} 1 + \mathrm{Re}\{(c')^T S(\omega)^{-1} b\} - b^T (S(\omega)^*)^{-1} R' S(\omega)^{-1} b &= \frac{p_1(\omega) - p_2(\omega)}{\det(\omega^2 I + A^2)} \\ &= (\omega - \omega_c)^2 \frac{(p_1'(\omega) - p_2'(\omega))}{\det(\omega^2 I + A^2)} \end{aligned}$$

with $p_1'(\omega) > 0$ for all positive real ω. Thus there is some constant $M_1 > 0$ such that $p_1'(\omega) \geq M_1$ for ω in the compact interval $[0, K]$. Furthermore there is some $M_2 > 0$ such that $p_2'(\omega) \leq M_2$ for all $\omega \in [0, K]$. If we now choose some constant $C > 0$ such that $C < min\{M_1/M_2, 1\}$, then by separately considering the cases of $\omega \in [0, K]$ and $\omega > K$ we see that

$$1 + \mathrm{Re}\{c'^T S(\omega)^{-1} b\} \geq b^T (S(\omega)^*)^{-1} (CR') S(\omega)^{-1} b$$

for all real $\omega \geq 0$. Hence, because the expressions on either side of the above inequality are even functions of ω, it follows that $R = CR'$ is a positive semi-definite matrix of rank $n - 2$ satisfying condition (B.2) as claimed.

Now the numerator of the rational function

$$1 + \mathrm{Re}\{c'^T S(\omega)^{-1} b\} - b^T (S(\omega)^*)^{-1} R S(\omega)^{-1} b$$

is a monic even polynomial of degree $2n$ with real coefficients, and is non-negative for all ω in $\mathbb{R}$. By arguments identical to those presented in [51], it follows that there is some monic polynomial θ of degree n with real coefficients such that

$$1 + \mathrm{Re}\{c'^T S(\omega)^{-1} b\} - b^T (S(\omega)^*)^{-1} R S(\omega)^{-1} b = \frac{|\theta(j\omega)|^2}{\det(\omega^2 I + A^2)}.$$

As the leading coefficient of θ is one, the polynomial $-\theta(s) + \det(sI - A)$ has degree $n - 1$, and thus by (5.11), there is some real vector q (the vector formed with the coefficients of $-\theta(s) + \det(sI - A)$) such that

$$\frac{-\theta(j\omega)}{\det(S(\omega))} + 1 = q^T S(\omega)^{-1} b$$

and hence

$$\begin{aligned}\frac{|\theta(j\omega)|^2}{\det(\omega^2 I + A^2)} &= 1 + \mathrm{Re}\{c'^T S(\omega)^{-1} b\} - b^T (S(\omega)^*)^{-1} R S(\omega)^{-1} b \\ &= |q^T S(\omega)^{-1} b - 1|^2 \end{aligned} \tag{B.3}$$

<u>*Step 3:*</u>

We now show that (B.3) means that there is a positive semi-definite matrix P such that

$$\begin{aligned} A^T P + PA &= -qq^T - R \\ Pb &= q + c'/2 \end{aligned} \tag{B.4}$$

As A is Hurwitz, we can certainly find a positive semi-definite matrix P such that $A^T P + PA = -qq^T - R$. This P then satisfies

$$S(\omega)^* P + PS(\omega) = qq^T + R. \tag{B.5}$$

Next expand the identity (B.3) to get

$$\begin{aligned} 1 \quad &+ \quad \mathrm{Re}\{c'^T S(\omega)^{-1} b\} - b^T (S(\omega)^*)^{-1} (R) S(\omega)^{-1} b \\ &= \quad (b^T (S(\omega)^*)^{-1} q - 1)(q^T S(\omega)^{-1} b - 1) \\ &= \quad 1 - 2\mathrm{Re}\{q^T S(\omega)^{-1} b\} + b^T (S(\omega)^*)^{-1} qq^T S(\omega)^{-1} b \end{aligned} \tag{B.6}$$

Collecting the terms in (B.6) and using (B.5) to substitute for $qq^T + R$ we find that

$$\mathrm{Re}\{(c'/2 + q - Pb)^T S(\omega)^{-1} b\} = 0 \text{ for all } \omega \in \mathbb{R}. \tag{B.7}$$

But this implies that $c'/2 + q - Pb = 0$ or $Pb - c'/2 = q$. (This follows from the fact that the matrix $L(A)$ in (5.14) is assumed to be invertible.)

<u>*Step 4:*</u>

The final step in this proof is to show that the matrix P in (B.4) satisfies

$$\begin{aligned} A^T P + PA = Q_1 \quad &\leq \quad 0, \\ (A - bc'^T)^T P + P(A - bc'^T) = Q_2 \quad &\leq \quad 0 \end{aligned} \tag{B.8}$$

where Q_1 and Q_2 are both of rank $n-1$.

Obviously, $A^TP + PA = -qq^T - R = Q_1 \leq 0$. We shall show later that the rank of Q_1 is $n-1$. First of all consider

$$\begin{aligned}(A - bc'^T)^TP + P(A - bc'^T) &= -qq^T - R - c'b^TP - Pbc'^T \\ &= -qq^T - R - c'(c'/2+q)^T - (c'/2+q)c'^T \\ &= -R - (c'+q)(c'+q)^T \leq 0 \qquad \text{(B.9)}\end{aligned}$$

Next as R is of rank $n-2$, we can write $R = \Sigma_{i=1}^{n-2} v_i v_i^T$ for $n-2$ linearly independent vectors $v_1, \ldots v_{n-2}$. Also recall that $S(\omega_c)^{-1}b$ is a zero eigenvector of R. If the rank of $qq^T + R$ was less than $n-1$, then q would lie in the span of $v_1, \ldots, v_{n-2}$, and thus $q^TS(\omega_c)^{-1}b = 0$. But (B.3) implies that $q^TS(\omega_c)^{-1}b = 1$. This contradiction shows that the rank of $qq^T + R$ must be $n-1$. A similar argument also shows that $R + (c'+q)(c'+q)^T$ is of rank $n-1$. This completes the proof of Theorem 5.4.1.

Appendix C

Technical proofs for Chapter 7

Proof of Lemma 7.2.1:

Let $Q = A^T P + PA$, and note that as the inverse, $\mathcal{L}_A^{-1}$, of the Lyapunov operator $\mathcal{L}_A$ is continuous, there is some $\delta > 0$ such that if $\|Q' - Q\| < \delta$, then $\|\mathcal{L}_A^{-1}(Q') - \mathcal{L}_A^{-1}(Q)\| < \epsilon$. Now, as Q is symmetric and has rank $n-k$, there exists some orthogonal matrix T in $\mathbb{R}^{n\times n}$ such that

$$\tilde{Q} = T^T Q T = \mathrm{diag}\{\lambda_1, \ldots, \lambda_{n-k}, 0, \ldots, 0\},$$

where $\lambda_1 < 0, \ldots, \lambda_{n-k} < 0$. Now, define

$$\tilde{Q_0} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_{n-k}, -\delta/2, \ldots, -\delta/2, 0\},$$

and let $Q_0 = T\tilde{Q_0}T^T$. Then we have that:

(i) $Q_0 \leq 0$, and $\mathrm{rank}(Q_0) = n-1$;

(ii) $\|Q - Q_0\| < \delta$.

It now follows that the matrix $P_0 = \mathcal{L}_A^{-1}(Q_0)$ lies on the boundary of $\mathcal{P}_A$ and satisfies:

(i) $\|P_0 - P\| < \epsilon$;

(ii) $\operatorname{rank}(A^T P_0 + P_0 A) = n - 1$,

as required

Proof of Theorem 7.2.1:

First of all, note that any hyperplane that is tangential to the cone $\mathcal{P}_A$ must pass through the origin. Now, let

$$\mathcal{H}_f = \{H \in Sym(n, \mathbb{R}) : f(H) = 0\}$$

be a hyperplane that is tangential to $\mathcal{P}_A$ at P_0, where f is a linear functional defined on $Sym(n, \mathbb{R})$. We shall show that $\mathcal{H}_f$ must coincide with the hyperplane

$$\mathcal{H} = \{H \in Sym(n, \mathbb{R}) : x_0^T H A x_0 = 0\}.$$

We shall argue by contradiction so suppose that this was not true. This would mean that there was some $\overline{P}$ in $Sym(n, \mathbb{R})$ such that $f(\overline{P}) = 0$ but $x_0^T \overline{P} A x_0 < 0$.

Now, consider the set

$$\Omega = \{x \in \mathbb{R}^n : x^T x = 1 \text{ and } x^T \overline{P} A x \geq 0\},$$

and note that if Ω was empty, this would mean that $\overline{P}$ was in $\mathcal{P}_A$, contradicting the fact that $\mathcal{H}_f$ is tangential to $\mathcal{P}_A$. Thus, we can assume that Ω is non-empty.

Note that the set Ω is closed and bounded, hence compact. Furthermore x_0 is not in Ω and thus $x^T P_0 A x < 0$ for all x in Ω.

Let M_1 be the maximum value of $x^T \overline{P} A x$ on Ω, and let M_2 be the maximum value of $x^T P_0 A x$ on Ω. Then by the final remark in the previous paragraph, $M_2 < 0$. Choose any constant $\delta > 0$ such that

$$\delta < \frac{|M_2|}{M_1 + 1} = C_1$$

and consider the symmetric matrix

$$P_0 + \delta_1 \overline{P}.$$

By separately considering the cases $x \in \Omega$ and $x \notin \Omega$, $x^T x = 1$, it follows that for all non-zero vectors x of Euclidean norm 1

$$x^T(A^T(P_0 + \delta\overline{P}) + (P_0 + \delta\overline{P})A)x < 0$$

provided $0 < \delta < \frac{|M_2|}{M_1+1}$. Since the above inequality is unchanged if we scale x by any non-zero real number, it follows that $A^T(P_0 + \delta\overline{P}) + (P_0 + \delta\overline{P})A$ is negative definite. Thus, $P_0 + \delta\overline{P}$ is in $\mathcal{P}_A$. However,

$$f(P_0 + \delta\overline{P}) = f(P_0) + \delta f(\overline{P}) = 0,$$

which implies that $\mathcal{H}_f$ intersects the interior of the cone $\mathcal{P}_A$ which is a contradiction. Thus, there can be only one hyperplane tangential to $\mathcal{P}_A$ at P_0, and this is given by

$$\{H \in Sym(n, \mathbb{R}) : x_0^T H A x_0 = 0\},$$

as claimed.

Proof of Lemma 7.2.2:

Let $B = A_2 - A_1$. To begin with, we assume that B is in Jordan canonical form so that (as B is of rank 1) either

$$B = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ \vdots & & & \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

or

$$B = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 1 & \dots & \dots & 0 \\ \vdots & & & \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

Now partition $Q_1 = A_1^T P + PA_1$ as

$$Q_1 = \begin{pmatrix} c_1 & q_1^T \\ q_1 & Q \end{pmatrix}$$

where $c_1 \in \mathbb{R}$, $q_1 \in \mathbb{R}^{n-1}$ and Q is a symmetric matrix in $\mathbb{R}^{(n-1)\times(n-1)}$. Then it is a simple calculation to verify that $Q_2 = A_2^T P + PA_2$ takes the form

$$Q_2 = \begin{pmatrix} c_2 & q_2^T \\ q_2 & Q \end{pmatrix}$$

with the same Q as before.

From the interlacing theorem for bordered matrices [42], it follows that the eigenvalues of Q_i for $i = 1, 2$ must interlace with the eigenvalues of Q. However as $P \in \mathcal{P}_{A_2}$, $Q_2 < 0$ and thus Q must be non-singular in $\mathbb{R}^{(n-1)\times(n-1)}$. Therefore, as the eigenvalues of Q_1 must also interlace with the eigenvalues of Q, it follows that Q_1 cannot have rank less than $n-1$.

Now suppose that B is not in Jordan canonical form and write $\Lambda = T^{-1}BT$ where Λ is the Jordan form for B (one of the two possible forms given above). Consider $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{P}$ given by

$$\tilde{A}_1 = T^{-1}A_1T,\ \tilde{A}_2 = T^{-1}A_2T,\ \tilde{P} = T^T PT.$$

Then it is a straightforward exercise in congruences to verify that

$$\tilde{A}_2{}^T \tilde{P} + \tilde{P}\tilde{A}_2 = T^T Q_2 T < 0$$

and that

$$\tilde{A}_1{}^T \tilde{P} + \tilde{P}\tilde{A}_1 = T^T Q_1 T \leq 0.$$

Furthermore $\mathrm{rank}(\tilde{A}_1 - \tilde{A}_2) = 1$ and $\tilde{A}_1$, $\tilde{A}_2$ are both Hurwitz. Hence by the previous argument, $T^T Q_1 T$ must have rank $n-1$, and thus by congruence the rank of Q_1 must also be $n-1$.

Proof of Theorem 7.2.2:

Write $B = A_2 - A_1$. First note that the hypotheses of the theorem mean that $\det(A_1^{-1} + \gamma A_2) = \det(A_1^{-1} + \gamma A_1 + \gamma B)$ never changes sign for $\gamma > 0$, as A_1^{-1} and A_2 are both Hurwitz. So we may assume that $\det(A_1^{-1} + \gamma A_1 + \gamma B) \geq 0$ for all $\gamma > 0$.

By examining the proof of Lemma 5.4.2, it can be seen that this implies that for all k, with $0 < k < 1$, $\det(A_1^{-1} + \gamma(A_1 + kB)) > 0$ for all $\gamma > 0$. It now follows from Theorem 5.3.3, and the results of Meyer in [84], that there must exist some $P = P^T > 0$ such that

$$\begin{aligned} A_1^{-T}P + PA_1^{-1} &\leq 0 \\ A_2^T P + PA_2 &\leq 0. \end{aligned}$$

Furthermore, we must then have that

$$(A_1^{-T} + \gamma_0 A_2^T)Px_0 + P(A_1^{-1} + \gamma_0 A_2)x_0 = 0. \tag{C.1}$$

Thus, as $(A_1^{-1} + \gamma_0 A_2)x_0 = 0$,

$$(A_1^{-T} + \gamma_0 A_2^T)Px_0 = 0,$$

and $Px_0 = \lambda y_0$ for some real $\lambda \neq 0$. Now,

$$y_0^T A_1^{-1} x_0 + \gamma_0 y_0^T A_2 x_0 = 0$$

implies that

$$x_0^T P A_1^{-1} x_0 + \gamma_0 x_0^T P A_2 x_0 = 0,$$

from which we can conclude the result of the theorem.

Proof of Lemma 7.3.1:

Let P be a positive definite matrix in $\mathbb{R}^n$ such that

$$x^T P f(x) < 0 \quad \text{for all non-zero } x \in \mathbb{R}^n. \tag{C.2}$$

Now let y be any non-zero vector in $\mathbb{R}^n$. Then, $f^{-1}(y)$ will also be non-zero and it follows from (C.2) that $(f^{-1}(y))^T Py < 0$, and thus that $y^T P f^{-1}(y) < 0$. Hence, we have shown that

$$x^T P f(x) < 0 \quad \text{for all non-zero } x \in \mathbb{R}^n$$

implies that

$$x^T P f^{-1}(x) < 0 \quad \text{for all non-zero } x \in \mathbb{R}^n.$$

The converse follows by interchanging the roles of f and f^{-1} in the previous argument.

<u>Proof of Lemma 7.3.2:</u>

Let $V(x) = x^T P x$ be a CQLF for Σ_{f_1}, Σ_{f_2}. Then it follows using Lemma 7.3.1 that for all non-zero $x \in \mathbb{R}^n$,

$$x^T P f_1(x) < 0, \; x^T P f_2(x) < 0, \; x^T P f_1^{-1}(x) < 0.$$

Thus, for any non-zero x in $\mathbb{R}^n$ and any $\gamma > 0$,

$$\begin{aligned} x^T P(f_1(x) + \gamma f_2(x)) &< 0 \\ x^T P(f_1^{-1}(x) + \gamma f_2(x)) &< 0. \end{aligned}$$

The result now follows immediately.

<u>Proof of Lemma 7.3.3:</u>

(i) Take any positive definite P such that $x^T P f_1(x) < 0$ for all non-zero x in $\mathbb{R}^n$. Then, as the functions $x^T P f_2(x)$ and $x^T P x$ are continuous, and $P > 0$, there must be some constants $M_1 \geq 0, M_2 > 0$ such that

$$\begin{aligned} x^T P f_2(x) &\leq M_1 \quad \text{for all } x \text{ with } x^T x = 1, \\ x^T P x &\geq M_2 \quad \text{for all } x \text{ with } x^T x = 1. \end{aligned}$$

Now, if we choose any $\alpha > M1/M2$, then for all vectors x of Euclidean norm one, we have that

$$x^T P(f_2(x) - \alpha x) < 0.$$

The homogeneity of f_2 now implies that this inequality is valid for all non-zero x in $\mathbb{R}^n$ and that $V(x)$ is a CQLF for the systems

$$\begin{aligned} \dot{x} &= f_1(x) \\ \dot{x} &= f_2(x) - \alpha x. \end{aligned}$$

(ii) As in (i), take any positive definite P such that $x^T P f_1(x) < 0$ for all non-zero x in $\mathbb{R}^n$. Once again, as the functions $x^T P f_1(x)$ and $x^T P g(x)$ are continuous, there must be some constants $M_1 > 0, M_2 \geq 0$ such that

$$\begin{aligned} x^T P f_1(x) &\leq -M_1 \quad \text{for all } x \text{ with } x^T x = 1, \\ x^T P g(x) &\leq M_2 \quad \text{for all } x \text{ with } x^T x = 1. \end{aligned}$$

If we now choose any k with $0 < k < M_1/(M_2+1)$, then for all vectors x of Euclidean norm one, we have that

$$x^T P(f_1(x) + kg(x)) < 0.$$

As above, the homogeneity of f_1 and g imply that this inequality is valid for all non-zero x in $\mathbb{R}^n$ and that $V(x)$ is a CQLF for the systems

$$\begin{aligned} \dot{x} &= f_1(x) \\ \dot{x} &= f_1(x) + kg(x). \end{aligned}$$

Proof of Theorem 7.3.1:

The arguments used are practically identical to those used to derive Theorem 4.4.1. As before the proof is split into two main stages.

Stage 1:

First of all, we shall show that if there exists some $\overline{P}$ in $Sym(n, \mathbb{R})$ satisfying

$$x_1^T \overline{P} f_1(x_1) < 0, \;\; x_2^T \overline{P} f_2(x_2) < 0 \tag{C.3}$$

then the systems Σ_{f_1} and Σ_{f_2} would have a CQLF.

So, suppose that there is some $\overline{P}$ satisfying (C.3)and consider the set

$$\Omega_1 = \{x \in \mathbb{R}^n : x^T x = 1 \text{ and } x^T \overline{P} f_1(x) \geq 0\}.$$

Note that if Ω_1 was empty, then any positive constant $\delta_1 > 0$ would make $x^T(P + \delta_1 \overline{P}) f_1(x) < 0$ for all $x \in \mathbb{R}^n$. Now assume that Ω_1 is non-empty.

The function that takes x to $x^T \overline{P} f_1(x)$ is continuous. Thus Ω_1 is closed and bounded, hence compact. Furthermore x_1 (or any non-zero multiple of x_1) is not in Ω_1 and thus $x^T P f_1(x) < 0$ for all x in Ω_1.

Let M_1 be the maximum value of $x^T \overline{P} f_1(x)$ on Ω_1, and let M_2 be the maximum value of $x^T P f_1(x)$ on Ω_1. Then by the final remark in the previous paragraph, $M_2 < 0$. Choose any constant $\delta_1 > 0$ such that

$$\delta_1 < \frac{|M_2|}{M_1 + 1} = C_1$$

and consider the symmetric matrix

$$P + \delta_1 \overline{P}.$$

By separately considering the cases $x \in \Omega_1$ and $x \notin \Omega_1$, $x^T x = 1$, it follows that for all non-zero vectors x of Euclidean norm 1

$$x^T (P + \delta_1 \overline{P}) f_1(x) < 0$$

provided $0 < \delta_1 < \frac{|M_2|}{M_1+1}$. In fact, as f_1 is homogeneous of degree one, it follows that

$$x^T (P + \delta_1 \overline{P}) f_1(x) < 0$$

for all $x \in \mathbb{R}^n$, provided $0 < \delta_1 < \frac{|M_2|}{M_1+1}$.

The identical argument can now be used to show that there is some positive constant $C_2 > 0$ such that

$$x^T(P + \delta_1 \overline{P})f_2(x) < 0$$

for all non-zero x in $\mathbb{R}^n$, provided $0 < \delta_1 < C_2$. Furthermore, as P is positive definite, there exists some $\epsilon > 0$ such that $P + \delta_1 \overline{P}$ will be positive definite provided $0 < \delta_1 < \epsilon$. Now choose any $\delta > 0$ such that $\delta < \min\{C_1, C_2, \epsilon\}$ and consider the positive definite matrix

$$P_1 = P + \delta \overline{P}.$$

Then $V(x) = x^T P_1 x$ will be a CQLF for Σ_{f_1} and Σ_{f_2}.

Stage 2:

Thus, there is no $\overline{P}$ in $Sym(n, \mathbb{R})$ satisfying the conditions (C.3). We next show that this implies the conclusion of the theorem.

As there is no $\overline{P}$ satisfying (C.3), any $\overline{P}$ in $Sym(n, \mathbb{R})$ that makes the expression $x_1^T \overline{P} f_1(x_1)$ negative will make the expression $x_2^T \overline{P} f_2(x_2)$ positive. More formally,

$$x_1^T \overline{P} f_1(x_1) < 0 \iff x_2^T \overline{P} f_2(x_2) > 0 \qquad \text{(C.4)}$$

for $\overline{P} \in Sym(n, \mathbb{R})$. This implies that

$$x_1^T \overline{P} f_1(x_1) = 0 \iff x_2^T \overline{P} f_2(x_2) = 0.$$

The mappings $\overline{P} \to x_1^T \overline{P} f_1(x_1)$ and $\overline{P} \to x_2^T \overline{P} f_2(x_2)$ define linear functionals on the space $Sym(n, \mathbb{R})$. Moreover, we have seen that the null sets of these functionals are identical. Thus, they must be scalar multiples of each other. Furthermore, (C.4) implies that they are *negative* multiples of each other. Therefore there is some constant $k > 0$ such that

$$x_1^T \overline{P} f_1(x_1) = -k x_2^T \overline{P} f_2(x_2) \qquad \text{(C.5)}$$

for all $\overline{P} \in Sym(n, \mathbb{R})$.

Now Lemma 4.3.6 implies that either $x_1 = \alpha x_2$ with $f_1(x_1) = -(\frac{k}{\alpha})f_2(x_2)$ for some real α, or $x_1 = \beta f_2(x_2)$ and $f_1(x_1) = -(\frac{k}{\beta})x_2$ for some real β. The result now follows easily using the facts that both of the functions f_1, f_2 are invertible and homogeneous of degree one.

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