Problem Set 3

ECE567 Fall 2009
Each problem is worth 10 points. Due: Oct. 7

1. Consider a mobile radio that is moving on the integer points of the real line according to a random walk. Let $S(n)$ denote the position of the mobile at time instant n and define $S(n)$ as follows: $S(0) = 0$ and

$$S(n+1) = \begin{cases} S(n)+1, & w.p. \quad p \\ S(n)-1, & w.p. \quad 1-p, \end{cases}$$

where $w.p.$ denotes with probability. Let $Y(n) = |S(n)|$. Show that $Y(n)$ is also a Markov chain and determine its probability transition matrix.

Hint: First compute

$$P(S(n) = i \mid Y(n) = i, Y(n-1), Y(n-2), \ldots, Y(1)),$$

and using this, compute

$$P(Y(n+1) = i+1 \mid Y(n) = i, Y(n-1), Y(n-2), \ldots, Y(1))$$

and

$$P(Y(n+1) = i-1 \mid Y(n) = i, Y(n-1), Y(n-2), \ldots, Y(1)).$$

2. Suppose that there are N i.i.d. sources accessing a bufferless router with capacity Nc. Let X_i denote the rate at which Source i transmits data. Assume that the peak rate of each source is equal to M and the mean rate is less than or equal to ρ. Show that the distribution of X_i that maximizes the Chernoff bound estimate of the overflow probability is given by

$$X_i = \begin{cases} M, & w.p. \quad q \\ 0, & w.p. \quad 1-q, \end{cases}$$

where $q = \rho/M$.

Hint: Maximizing the Chernoff bound estimate of the overflow probability is the same as minimizing the rate function $I_X(c)$ over all distributions such that $X \leq M$ w.p.1 and $E(X) \leq \rho$.

3. Let X be a random variable. Its rate function was defined in class as

$$I(x) := \sup_s sx - \log M(s),$$

where $M(s) := E(e^{sX})$. Compute the rate function of each of the following random variables.

(a) X is Gaussian with mean μ and variance σ^2.
(b) X is exponential with mean $1/\lambda$.
(c) X is Poisson with mean λ.
(d) X is one with probability p and zero with probability $1-p$.

4. In many communication networks, traffic policing is used to reduce the burstiness of traffic that enters the network. Let $a(k)$ be the number of bits generated by a discrete-time source in slot k and assume that $\{a(k)\}$ are i.i.d. Suppose that the network has an agreement with the source that it will allow, on average, μ bits per time slot into the network where $\mu \leq E(a(k))$. The network's goal is to minimize the effective bandwidth of the source while satisfying the agreement. Thus, the network inserts a policing device at the edge, which allows $h(x)$ bits into the network when the

source generates x bits. In other words, if we define $y(k) = h(a(k))$, then $y(k)$ is the number of bits from the source that enters the network in slot k. The network's goal is to minimize the effective bandwidth of $y(k)$. Show that the optimal strategy for the network is to choose $h(x) = \min(x, M)$, where M is chosen such that $E(h(a(k))) = \mu$.

Hint: The following fact may be useful: $e^u \geq 1 + u$.

5. Consider a simple model of a discrete-time wireless channel: the channel is ON w.p. μ and OFF w.p. $1 - \mu$, and when the channel is ON it can serve one packet in the slot and when the channel is OFF, the channel cannot serve any packets. Packets arrive at this wireless channel according to the following arrival process: a maximum of one arrival occurs in each instant. The probability of an arrival in the current slot is 0.8 if there was an arrival in the previous time slot. The probability of an arrival in the current slot is 0.1 if there was no arrival in the previous time slot. (i) For what values of μ would you expect this system to be stable? (ii) For these values of μ, show that an appropriate Markov chain describing the state of this queueing system is stable (i.e., positive recurrent) using the Foster-Lyapunov theorem.

Hint: (i) The arrival process is itself a Markov chain, but it is a simple two-state Markov chain. Find the mean arrival rate of the packet arrival process. The mean service rate must be larger than this mean arrival rate. (ii) The state of the Markov chain describing the queueing system is the number of packets in the queue along with the state of the arrival process.

6. Show that the heavy-traffic tightness of the Kingman bound in Problem 5, Problem Set 2 holds even without the assumption $s(k) \leq S_{max}$. In other words, show that the bound is tight in heavy traffic without assuming that there is an upper bound on the amount of packets per time slot; however, continue to assume that the number of arrivals and service per time slot have finite second moments.