Problem Set 1

ECE 534 Fall 2009
R. Srikant Due: Sept. 8

1. Let x_r be the rate allocated to user r in a network where each user's route is fixed. A link l is called a bottleneck link for a user r if $l \in r$, and

$$y_l = c_l \text{ and } x_s \leq x_r \quad \forall s \text{ such that } \ l \in s.$$

Show that $\{x_r\}$ is a max-min fair allocation if and only if every source has at least one bottleneck link.

2. (a) Show that the allocation $\{x_r\}$ obtained from the algorithm below a max-min fair allocation:

1. Let $S(0)$ be the set of all users in the network and $c_l(0)$ be the capacity of link l. Set $i = 0$.
2. Let $S_l(i) \subseteq S(i)$ be the set of users whose routes include link l, and let $|S_l(i)|$ be the cardinality of this set, i.e., $|S_l(i)|$ is the number of users which use link l. Set $f_l(i) = c_l(i)/|S_l(i)|$. (The parameter $f_l(i)$ is called the fair share on link l at iteration i.)
3. Set $z_r(i) = \min_{l:l\in r} f_l(i)$, $\forall r \in S(i)$. (Thus, each user in $S(i)$ is temporarily allocated the minimum of the fair shares on its route.)
4. Let $T(i)$ be the set of links such that $z_r(i) = \min_s z_s(i)$ $\forall r \in T(i)$ and set $x_r = z_r(i)$ $\forall r \in T(i)$. (The users which get the smallest rates are permanently allocated these rates.)
5. $S(i+1) = S(i) \setminus T(i)$, $c_l(i+1) = c_l(i) - \sum_{r\in T(i)} z_r(i)$, $\forall i$. (Users whose rate allocations are finalized are removed from from the set of all users under consideration and the capacity of each link is reduced by the total rate allocated to such users.)
6. Set $i = i + 1$ and got to Step 2.

(b) Make up your own three-link network and show how the steps of the above algorithm are executed to achieve a max-min fair rate allocation.

3. (a) Consider the primal congestion controller with $\kappa_r(x) = 1$, $\forall x$, and all r. Using the Lyapunov function

$$\sum_r (x_r - \hat{x}_r)^2,$$

show that the controller is globally, asymptotically stable.

(b) How would you modify the above Lyapunov function if $\kappa_r(x) \neq 1$ to show global asymptotic stability?

4. Assume that the link costs $p_l(y_l) \in [0, 1]$ and let $q_r = 1 - \prod_{l:l\in r}(1 - p_l)$. Thus, if p_l is the probability that a packet is *marked* at link l, then q_r is the probability that a packet is marked on route r. Show that the primal congestion controller is also globally, asymptotically stable under this model (without making the assumption that p_l's are really small). *(The following fact may be useful:* log *is an increasing function.)*

5. In this problem, we expand the scope of the utility maximization problem to include adaptive, multi-path routing. Let s denote a source and let $\mathcal{R}(s)$ denote the set of routes used by user s. Each source s is allowed to split its packets along multiple routes. Let z_s denote the rate at which

source s generates data and x_r denote the rate on route r. Thus, the penalty function formulation of the utility maximization problem becomes

$$\max_x \quad \sum_s U_s(z_s) - \sum_l \int_0^{y_l} f_l(y)dy + \epsilon \sum_r \log x_r,$$

where

$$z_s = \sum_{r \in \mathcal{R}(s)} x_r, \qquad y_l = \sum_{r : l \in r} x_r$$

and $\epsilon > 0$ is a small number.

(a) Even when U_s is a strictly concave function, argue that the above objective need not be strictly concave if $\epsilon = 0$. (Thus, we have introduced the ϵ term only to ensure strict concavity. But the impact of this term on the optimal solution will be small if ϵ is chosen to be small.)

(b) Derive a congestion control (and rate-splitting across routes) algorithm and prove that it asymptotically achieves the optimal rates which solve the above utility maximization problem.

6. (a) Linearize the delay differential equation for TCP-Reno derived in the notes.

(b) Obtain sufficient conditions (try to obtain the best possible conditions) for the linear stability of TCP.