Homework 4

Exercise 1. Let $(N_t,\, t \in \mathbb{R}_+)$ be a Poisson process. Given that there is exactly one arrival during the time interval $[0,t]$, what is the probability that this arrival takes place before time s, where $s \in [0,t]$?

Exercise 2. Let a be a constant such that $0 < a < 2$, and let T be a non-negative random variable such that $\mathbb{E}(T) = 1$.

a) Compute $\mathbb{E}(T \,|\, T \geq a)$ when T is exponentially distributed.

b) Compute $\mathbb{E}(T \,|\, T \geq a)$ when T is uniformly distributed in the interval $[0,2]$.

c) Which of the two expectations is the largest? How can you interpret this result?

Hint: Use the formula $\mathbb{E}(T|T \geq a) = \displaystyle\int_0^\infty dt\, \mathbb{P}(T \geq t|T \geq a)$.

Exercise 3. Let T_1, T_2 be two independent exponential random variables with parameters λ_1 and λ_2, respectively. What is the distribution of the random variable $T = \min\{T_1, T_2\}$?

Exercise 4. A professor experiencing some difficulties in mastering his own agenda has an appointment at his office with two students at the same time. One of the two students is on time, the other one arrives with 5 minutes delay. The duration of each meeting is distributed exponentially, with an average of 30 minutes each. What is the expected amount of time between the arrival of the first student and the departure of the second from the office?

Exercise 5. Let $(N_t,\, t \in \mathbb{R}_+)$ be a Poisson process of intensity $\lambda > 0$.

a) Describe the discrete Markov chain embedded in this continuous-time Markov chain.

b) Derive the Kolmogorov equations for the probabilities $\pi_n(t) = \mathbb{P}(N_t = n)$ and solve them by induction.

c) Does this process admit a stationary distribution?

Exercise 6. A Windows machine breaks down once a day on average, after what it needs to be rebooted, and this typically takes 1 hour (let us exaggerate a bit here...). A Linux machine is on the other hand much more reliable, as it breaks down only once a week on average, but when it breaks down, fixing the problem typically takes 6 hours, as it requires the help of a system administrator, not always easy to find.

a) For each machine, what is the probability that it works after 24 hours (given that it is working at time 0)?

b) For each machine, what is the probability that it works *without interruption* during 24 hours?

Exercise 7. A student spends a whole day (24 hours) working for his final exams of information theory, wireless communications and probability. After two hours on average, he usually gets bored revising his probability course and switches to one of his favourite topics, information theory and wireless communications, with equal probability. But the information theory course is tough also, so he does not work on this topic for more than 4 hours on average and switches to wireless communications after that. Wireless communications is definitely fun and usually keeps him busy for 6 hours in a row. He then goes back to his probability course when he feels bad (2/3 of the time), or again to his information theory course (1/3 of the time).

a) A priori, can you say whether the student will spend more time on probability or information theory?

b) What is the average time spent on each course after 24 hours?

NB: We assume here that the student starts in stationary state.