Question 1

Consider the Lyapunov Function:

$$V(q) = \frac{1}{2}\sum_{ij} q_{i,j}^2$$

The Lyapunov drift is:

$$\Delta V_k = E[V(q(k+1)) - V(q(k))|q[k] = q]$$

$$\Delta V_k = \frac{1}{2}E[\sum_{ij}((q_{ij}(k) + a_{ij}(k) - I_{ij}(k))^+)^2 - q_{ij}(k)^2|q[k] = q]$$

For simplicity, let's drop index k. Therefore, we can bound the Lyapunov drift:

$$\Delta V_k \leq \frac{1}{2}E[\sum_{ij}((q_{ij} + a_{ij} - I_{ij})^2 - q_{ij}^2)|q[k] = q]$$

$$\Delta V_k = \frac{1}{2}E[\sum_{ij}((a_{ij} - I_{ij})^2 + 2q_{ij}(a_{ij} - I_{ij}))|q[k] = q]$$

$$\Delta V_k \leq \frac{1}{2}\sum_{ij} E[((a_{ij} - I_{ij})^2|q[k] = q] + \sum_{ij} q_{ij}E[a_{ij} - I_{ij}|q[k] = q]$$

Since the arrivals into the queue (i, j) are independent Bernoulli with mean λ_{ij}, $E[a_{ij}(k)] = \lambda_{ij}$ and $E[a_{ij}^2(k)] = \lambda_{ij}$ Moreover, given any $q(k)$,

$$\sum_i I_{im}(k) \leq 1$$

$$\sum_j I_{nj}(k) \leq 1$$

Note that $I_{im}^2 = I_{im}$.Thus, we have the following bound for the first term:

$$\frac{1}{2}\sum_{ij} E[((a_{ij} - I_{ij})^2|q[k] = q] \leq \frac{1}{2}\sum_{ij} \lambda_{ij} + E[I_{ij}^2|q[k] = q] \leq N$$

Since λ_{ij} lies in the interior of capacity region and the max-weight scheduling algorithm is used, there exists an ϵ such that the second term is upper bounded:

$$\sum_{ij} q_{ij} E[a_{ij} - I_{ij} | q[k] = q] \leq -\epsilon \sum_{ij} q_{ij}$$

Therefore,

$$\Delta V_k = E[V(q(k+1)) - V(q(k)) | q[k] = q] \leq N - \epsilon \sum_{ij} q_{ij}$$

Taking expectation on both sides:

$$E[V(q(k+1)) - V(q(k)) | q[k] = q] \leq N - \epsilon E[\sum_{ij} q_{ij}]$$

At the steady-state, the left hand-side is equal to zero, and hence,

$$E[\sum_{ij} q_{ij}] \leq \frac{N}{\epsilon}$$

Question 2

Let M be the set of all possible schedules in the switch; and let C denote the capacity region, which is the convex hull of M. The scheduling algorithm chooses the schedule I^*_{ij} at time slot k such that:

$$I^*_{i,j} = \arg\max_{I \in M} \sum_{ij} I_{i,j} q^2_{i,j}(k)$$

Note that $I^*_{i,j}$ is also the optimal solution of the following linear program:

$$I^*_{i,j} = \arg\max_{I \in C} \sum_{ij} I_{i,j} q^2_{i,j}(k) \tag{1}$$

In order to prove the 100% throughput of this algorithm, we need to show that for any vector of arrival rates λ_{ij} strictly within capacity region C, the system is stable. Let us consider the Lyapunov Function:

$$V(q) = \frac{1}{3} \sum_{ij} q^3_{i,j}$$

The Lyapunov drift is:

$$\Delta V_k = E[V(q(k+1)) - V(q(k)) | q[k] = q]$$

$$\Delta V_k = \frac{1}{3}E[\sum_{ij}(q_{ij}(k) + a_{ij}(k) - I_{ij}(k) + u_{ij}(k))^3 - q_{ij}^3(k)|q[k] = q]$$

For simplicity, let's drop index k. Let $y_{ij} = q_{ij} + a_{ij} - I_{ij}$, then $y_{ij} \geq -1$ and

$$u_{ij} = \begin{cases} 0 & \text{if } y_{ij} \geq 0 \\ -y_{ij} & \text{if } y_{ij} < 0 \end{cases}$$

Hence $(y_{ij} + u_{ij})^3 = y_{ij}^3 + u_{ij}^3 \leq y_{ij}^3 + 1$ Therefore, we can bound the Lyapunov drift:

$$\Delta V_k \leq \frac{1}{3}E[\sum_{ij}((q_{ij} + a_{ij} - I_{ij})^3 - q_{ij}^3 + 1)|q[k] = q]$$

$$\Delta V_k = \frac{1}{3}E[\sum_{ij}(1 + (a_{ij} - I_{ij})^3 + 3q_{ij}^2(a_{ij} - I_{ij}) + 3q_{ij}(a_{ij} - I_{ij})^2)|q[k] = q]$$

$$\Delta V_k \leq \sum_{ij} C_1 + C_2 q_{ij} + q_{ij}^2 E[a_{ij} - I_{ij}|q[k] = q]$$

$$\Delta V_k \leq \sum_{ij} C_1 + C_2 q_{ij} + q_{ij}^2(\lambda_{ij} - I_{ij}^*)$$

Since the vector of arrival rates λ_{ij} is strictly within the capacity region C, there exists an ϵ such that the vector $\lambda_{ij} + \epsilon \in C$. Thus,

$$\Delta V_k \leq \sum_{ij} C_1 + C_2 q_{ij} - \epsilon q_{ij}^2 + q_{ij}^2(\lambda_{ij} + \epsilon - I_{ij}^*)$$

$$\Delta V_k \leq NC_1 + C_2 \sum_{ij} q_{ij} - \epsilon \sum_{ij} q_{ij}^2 \tag{2}$$

where the last inequality is due to (1) and the fact that $\lambda_{ij} + \epsilon \in C$. We want to verify that the drift is negative for all q outside a finite set. Now (2) can be rewritten as:

$$\triangle V_k \leq \sum_{ij}(C_1 + C_2 q_{ij} - \epsilon q_{ij}^2) \tag{3}$$

Therefore, the drift is negative for all values of q outside a finite set B, where:

$$B = \left\{ q : q_{ij} < -C_2 + \frac{\sqrt{C_2^2 + 4\epsilon C_1}}{2\epsilon}, \forall i, j \right\} \tag{4}$$

Positive recurrence follows from the Foster-Lyapunov stability theorem.

Question 3

Let I_{ij} be a maximal matching for positive queues If $q_{ij} > 1$ then

$$\sum_{j'} I_{ij'}(k) \leq 1$$

$$\sum_{i} I_{i'j}(k) \leq 1$$

Let $\epsilon > 0$ s.t.

$$\sum_{j} \lambda_{ij}^2(k) \leq \frac{1}{2} - \epsilon$$

$$\sum_{i} \lambda_{ij}^2(k) \leq \frac{1}{2} - \epsilon$$

Let:

$$V(q) = \sum_{ij} q_{i,j}(\sum_{i'} q_{i',j} + \sum_{j'} q_{i,j'})$$

The Lyapunov drift is:

$$\Delta V_k = E[V(q(k+1)) - V(q(k))|q[k] = q]$$

$$\Delta V_k = E(\sum_{ij}(q_{i,j}+a_{i,j}-I_{i,j})(\sum_{i'}(q_{i',j}+a_{i',j}-I_{i',j})+\sum_{j'}(q_{i,j'}+a_{i,j'}-I_{i,j'}))-\sum_{ij} q_{i,j}(\sum_{i'} q_{i',j}+\sum_{j'} q_{i,j'}))$$

$$\Delta V_k \leq \sum_{ij}[(\lambda_{ij} - I_{ij})(\sum_{i'} q_{i',j} + \sum_{j'} q_{i,j'}) + q_{ij}(\sum_{i'}(\lambda_{ij} - I_{ij}) + \sum_{j'}(\lambda_{ij} - I_{ij}))] + C$$

$$\Delta V_k \leq 2 \times \sum_{ij} q_{i,j}[\sum_{i'} \lambda_{i'j} + \sum_{j'} \lambda_{ij'} - I_{i'j} - I_{ij'}] + C$$

$$\Delta V_k \leq 2 \times \sum_{ij} q_{i,j}[\frac{1}{2} - \epsilon + \frac{1}{2} - \epsilon - 1] + C$$

$$\Delta V_k \leq -4 \times \epsilon \times \sum_{ij} q_{i,j} + C$$

Positive recurrent follows from large enough $\sum_{ij} q_{ij}$

QUESTION 4

Let X_A and X_B be random variables representing the channel state of user A and B respectively. The rate allocated to user A is given below:

$$\begin{aligned}\lambda_A &< P(X_A=1,X_B=0)+3P(X_A=3,X_B=0)+p_1P(X_A=1,X_B=1)\\ &\quad+p_2P(X_A=1,X_B=3)+3p_3P(X_A=3,X_B=1)+3p_4P(X_A=3,X_B=3)\\ &=\frac{4+p_1+p_2+3p_3+3p_4}{9}\end{aligned} \tag{5}$$

Similarly, the rate allocated to user B is:

$$\lambda_B < \frac{4+p_1'+3p_2'+p_3'+3p_4'}{9} \tag{6}$$

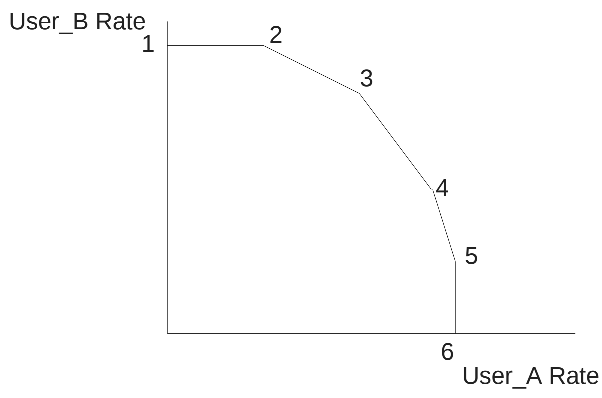
Moreover, we also have for each $i \in \{1,2,3,4\}$:

$$p_i + p_i' \leq 1 \tag{7}$$

The rate pairs defined by (λ_A, λ_B) using (5), (6), and (7) for various values of p_i and p_i' gives the capacity region. The corner points of the capacity region are as follows:

(1) $(0, \frac{12}{9})$ at $p_i = 0$ and $p_i' = 1$
(2) $(\frac{4}{9}, \frac{12}{9})$ at $p_i = 0$ and $p_i' = 1$
(3) $(\frac{7}{9}, \frac{11}{9})$ at $p_1 = p_2 = p_4 = 0$, $p_3 = 1$, and $p_i' = 1 - p_i$
(4) $(\frac{11}{9}, \frac{7}{9})$ at $p_1 = p_2 = p_4 = 1$, $p_3 = 0$, and $p_i' = 1 - p_i$
(5) $(\frac{12}{9}, \frac{4}{9})$ at $p_i = 1$ and $p_i' = 1$
(6) $(\frac{12}{9}, 0)$ at $p_i = 1$ and $p_i' = 1$

A graph of this capacity region is shown below.



We now prove analytically that no points outside of the region in the graph can be obtained through (5) and (6) using values of p_i and p'_i chosen according to (7). In order to verify this claim, we show that there are no admissible values of p_i and p'_i that can give a point beyond the five lines that define the region in the graph. First consider the line that runs through the points 5 and 6. Clearly, no point with $\lambda_A > \frac{12}{9}$ is achievable under (5), since the maximum value of (5) is $\frac{12}{9}$ (when $p_i = 0$ and $p'_i = 1$). By a similar argument, no point with $\lambda_B > \frac{12}{9}$ is admissible under (6) either. Now consider the line that runs through the points 3 and 4. The equation of this line is given by:

$$\lambda_A + \lambda_B = 2 \tag{8}$$

From (5) and (6) we know that:

$$\begin{aligned} \lambda_A + \lambda_B &\leq \frac{8 + p_1 + p'_1 + p_2 + 3p'_2 + 3p_3 + p'_3 + 3p_4 + 3p'_4}{9} \\ &\leq \frac{8 + 1 + 1 + 2p'_2 + 1 + 2p_3 + 3}{9} \\ &\leq \frac{14 + 2 + 2}{9} \\ &= 2 \end{aligned} \tag{9}$$

where the second inequality follows by substituting $p'_i = 1 - p_i$, and the third inequality follows by choosing $p'_2 = 1$ and $p_3 = 1$. By comparing (8) and (9), it follows that no point beyond the line given by (8) is admissible under (5) and (6). Now consider the line running through points 2 and 3. The equation of this line is given by:

$$9\lambda_A + 27\lambda_B = 40 \tag{10}$$

From (5) and (6) we know that:

$$\begin{aligned} 9\lambda_A + 27\lambda_B &\leq 4 + p_1 + p_2 + 3p_3 + 3p_4 + 12 + 3p'_1 + 9p'_2 + 3p'_3 + 9p'_4 \\ &\leq 16 + 1 + 2p'_1 + 1 + 8p'_2 + 3 + 3 + 6p'_4 \\ &\leq 24 + 2 + 8 + 6 \\ &= 40 \end{aligned} \tag{11}$$

where the second inequality follows by substituting $p'_i = 1 - p_i$, and the third inequality follows by choosing $p'_1 = p'_2 = p'_4 = 1$. By comparing (10) and (11), it follows that no point beyond the line given by (11) is admissible under (5) and (6). By a similar argument, no point beyond the line running through points 4 and 5 is admissible since the equation of this line is $27\lambda_A + 9\lambda_B = 40$. Therefore, no point beyond the curve of the polygon in the graph is admissible under the inequalities (5), (6), and (7).

Question 5

From the course notes, we have the following relationship:

$$
\begin{aligned}
E(\triangle V_k \,|\, q[k] = q) &= E(V(q[k+1]) - V(q[k]) \,|\, q[k] = q) \\
&\leq K - \epsilon \sum_{i=0}^{2} q_i \lambda_i
\end{aligned} \tag{12}
$$

where

$$
K = E(\sum_{i=1}^{2} (a_i - d_i)^2 \,|\, q[k] = q), \tag{13}
$$

and where $\{\lambda_i(1+\epsilon)\} \in \mathcal{C}$ (the capacity region). In steady state $E(\triangle V_k) = 0$. Taking expectations of both sides of equation (12) gives:

$$
\begin{aligned}
\sum_{i=1}^{2} \lambda_i E(q_i) &\leq \frac{E(\sum_{i=1}^{2}(a_i - d_i)^2)}{2\epsilon} \\
&= \frac{E(\sum_{i=1}^{2}(a_i^2 - 2a_i d_i + d_i^2)}{2\epsilon} \\
&\leq \frac{\sum_{i=1}^{2} E(a_i^2) + \sum_{i=1}^{2} E(d_i^2)}{2\epsilon}
\end{aligned} \tag{14}
$$

since the arrival process a_i and the service process d_i are independent. Moreover we also know that:

$$
E(a_i)^2 = E(a_i) = \lambda_i \tag{15}
$$

$$
E(d_i^2) \leq \frac{1}{3} \times 9 + \frac{1}{3} \times 1 + \frac{1}{3} \times 0 = \frac{10}{3} \tag{16}
$$

Substituting equations (15) and (16) into (14) gives:

$$
\begin{aligned}
\sum_{i=1}^{2} \lambda_i E(q_i) &\leq \frac{\lambda_1 + \lambda_2 + 2 \times \frac{10}{3}}{2\epsilon} \\
&= \frac{\lambda_1 + \lambda_2 + \frac{20}{3}}{2\epsilon}
\end{aligned} \tag{17}
$$

Finally, the sum of the queue lengths can be upper-bounded from equation (17) as follows:

$$\sum_{i=1}^{2} E(q_i) \leq \frac{\lambda_1 + \lambda_2 + \frac{20}{3}}{2min\{\lambda_1, \lambda_2\}\epsilon} \tag{18}$$

Question 6

Assuming that the second moments of the arrival rate process and scheduling process are finite, we have the relationship given in the notes:

$$E(\triangle V(k) \,|\, q(k) = q) \leq K + \sum_{l=1}^{L} q_l(\lambda_l - E(M_l|q)) \tag{19}$$

where $E(M_l|q)$ represents our probabilistic scheduling policy. We are assuming that the second moments of the arrival rate and the scheduling policy are finite.

Case 1: Suppose $q \in B^c_{\delta,\epsilon}$.

$$E(\triangle V(k) \,|\, q(k) = q) \leq K + \sum_{l=1}^{L} q_l\lambda_l - \sum_{l=1}^{L} q_l E(M_l|q) \tag{20}$$

where our probabilistic scheduling policy $E(M_l|q)$, is with probability $1-\delta$ within $1-\epsilon'$ of the max-weight scheduling policy. Therefore (20) can be rewritten as:

$$E(\triangle V(k) \,|\, q(k) = q) \leq K + \sum_{l=1}^{L} q_l\lambda_l - \sum_{l=1}^{L} q_l u^*_l (1-\epsilon')(1-\delta) \tag{21}$$

where u^*_l is the max-weight scheduling policy. We know that for some $\epsilon > 0$, $\lambda(1+2\epsilon) \in \mathcal{C}$, where $\mathcal{C}$ represents the capacity region of the ad hoc wireless network and 2ϵ represents the largest distance between the coordinates of $(\lambda_1, \lambda_2, ..., \lambda_L)$ and the boundary of the capacity region. Moreover, since u^* is the max-weight solution, we must have:

$$\sum_{l=1}^{L} q_l u^*_l \geq (1+2\epsilon) \sum_{l=1}^{L} q_l \lambda_l \tag{22}$$

Substituting (22) into (21) gives:

$$E(\triangle V(k) \,|\, q(k) = q) \leq K + \sum_{l=1}^{L} q_l\lambda_l - \sum_{l=1}^{L} q_l\lambda_l(1+2\epsilon)(1-\epsilon')(1-\delta) \tag{23}$$

Choose $\epsilon' > 0$ and $\delta > 0$ such that:

$$(1-\epsilon')(1-\delta) = \frac{1+\epsilon}{1+2\epsilon} \tag{24}$$

Substituting (24) into (23) gives:

$$E(\triangle V(k)\,|\,q(k)=q) \leq K - \epsilon \sum_{l=1}^{L} q_l \lambda_l \tag{25}$$

Now we have from (25):

$$E(\triangle V(k)\,|\,q(k)=q) \leq -\epsilon K, \qquad q \in \left\{ \sum_{l=1}^{L} q_l \lambda_l > \frac{2K}{\epsilon} \right\} \bigcap B^c_{\delta,\epsilon} \tag{26}$$

$$E(\triangle V(k)\,|\,q(k)=q) < \infty, \qquad q \in \left\{ \sum_{l=1}^{L} q_l \lambda_l \leq \frac{2K}{\epsilon} \right\} \bigcap B^c_{\delta,\epsilon} \tag{27}$$

Note that the queue lengths in (27) belong to a finite set. What happens when $q \in B_{\delta,\epsilon}$?

Case 2: Suppose $q \in B_{\delta,\epsilon}$**.** From (19) and the fact $B_{\delta,\epsilon}$ is a compact set of queue lengths it follows that:

$$\begin{aligned} E(\triangle V(k)\,|\,q(k)=q) &\leq K + \sum_{l=1}^{L} q_l \lambda_l \\ &< \infty \end{aligned} \tag{28}$$

Putting (26), (27), and (28) together gives the following:

$$\begin{aligned} E(\triangle V(k)\,|\,q(k)=q) &\leq -\epsilon K, \qquad && q \in \left\{ \sum_{l=1}^{L} q_l \lambda_l > \frac{2K}{\epsilon} \right\} \bigcap B^c_{\delta,\epsilon} \\ &< \infty, && q \in \left\{ \sum_{l=1}^{L} q_l \lambda_l \leq \frac{2K}{\epsilon} \right\} \bigcup B_{\delta,\epsilon} \end{aligned}$$

So by the Lyapunov-Foster Theorem, the randomized algorithm is also throughput optimal.