Homework 3

Exercise 1. Let $0 < p < 1$ and $q = 1 - p$. Compute the stationary distributions associated to the following transition matrices and for each of them, describe for which values of p they are reversible.

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & p & q & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & p & 0 & q \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ p & 0 & q & 0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} q & 0 & 0 & p \\ 0 & 0 & p & q \\ 0 & p & q & 0 \\ p & q & 0 & 0 \end{pmatrix}.$$

Exercise 2. Let $0 < p < 1$ be such that $p \neq 1/2$, and $q = 1 - p$. Consider the gambler's ruin problem on $\{0, \dots, N\}$ with transition probabilities

$$p_{0,0} = 1, \quad p_{i,i+1} = p, \quad p_{i,i-1} = q, \quad \text{for } i = 1, \dots, N-1, \quad p_{N,N} = 1,$$

that is, at each step, the gambler wins one franc with probability p and loses one franc with probability q. The game stops when either the gambler loses everything or reaches a fortune of N francs.

a) What is the probability that, starting from a fortune of i francs, $1 \leq i \leq N - 1$, the gambler loses everything before reaching N francs?

b*) [not required] What is the average duration of the game?

Hint: The vector $(x_i,\ 0 \leq i \leq N)$ solution of the system of equations

$$b\, x_i = a\, x_{i+1} + c\, x_{i-1}, \quad 1 \leq i \leq N-1, \quad \text{(with additional boundary conditions for } x_0,\ x_N)$$

can be found trying a vector of the form $x_i = y^i$ for $1 \leq i \leq N - 1$. In order to compute the possible values of y, notice that

$$b\, y^i = a\, y^{i+1} + c\, y^{i-1} \quad \text{implies} \quad b\, y = a\, y^2 + c,$$

so there are two possible values for y: the two roots y_1, y_2 of the above quadratic equation. In case these two roots are distinct, the general form for the solution of the above equation reads

$$x_i = \alpha y_1^i + \beta y_2^i,$$

where α, β are constants to be determined using the boundary conditions.

Exercise 3. A golf game is modeled as follows: at a given time instant, the distance between the ball and the hole is a number between 0 and N. The initial distance is N. From position $1 \leq k \leq N$, each time the ball is played, it reaches uniformly one of the positions between 0 and $k - 1$.

a) Draw the transition graph of this Markov chain.

b) What is the average number of times it takes until the ball reaches the hole?

c) Does this average number of times decrease if the player refines its strategy such that from position k, the ball reaches either the hole or position $\lfloor k/2 \rfloor$ with equal probability 1/2? (consider $N = 2^M$ for simplicity, and assume N to be large)