Homework 2

Exercise 1. a) Consider the Markov chain described in Exercise 3, Homework 1. Is this Markov chain ergodic? In case the answer is yes, compute the stationary distribution of the chain, but before doing any computation, do you expect this distribution to be uniform?

b) On average, how many red balls can be found in the first urn?

c) On 1000 steps in total, how many times does it happen (on average) that the first urn contains no red ball?

Exercise 2. In a rainy country of Northern Europe (where official statistics report 135 days of rain on average per year), a student attending classes at the university decides to adopt the following strategy. He first goes to a shop and buys a stock of N umbrellas. Then, every morning, he checks whether it is raining. If yes, he takes an umbrella with him (provided that he has one at disposal) before going to the university. If it does not rain, he does not take one. On evenings, he adopts the same strategy before going back home from his office at the university.

Let us now assume that, independently of the weather of the other days, it rains every morning and evening with the same probability $0 < p < 1$ (and that these two events are also independent). Let us further assume that it has been a while that the student has been living in the country.

a) On a given day (morning or evening), what is the probability that the student gets wet? (i.e. that he finds himself in the situation where it is raining, but all his umbrellas are at the office while he is at home, or vice-versa)

Hint: Let X_n denote the number of umbrellas available to the student at time n (n even = morning, n odd = evening). Compute first the transition probabilities of this Markov chain (is it time-homogeneous?) and then the stationary distribution of the chain (is it ergodic?).

b) Compute the probability of getting wet in the specific case where $N = 5$ and $p = 135/365 \approx 0.37$. What is the average number of times the student gets wet during a period of four months (that is, 120 mornings and 120 evenings)?

Exercise 3. Let $N \geq 1$ and $(S_n,\, n \geq 0)$ denote the random walk on $\{0, \dots, N\}$ with transition probabilities

$$p_{k,k+1} = p, \qquad p_{k+1,k} = q = 1 - p, \quad \text{for} \quad k = 0, \dots, N-1, \quad \text{and} \quad p_{0,0} = q, \quad p_{N,N} = p.$$

Is this Markov chain ergodic? In case the answer is yes, compute the stationary distribution of the chain.

Exercise 4. Let $0 < p < 1$, $q = 1-p$ and $(X_n,\, n \geq 1)$ be the random walk on $S = \mathbb{N} = \{0, 1, 2, \dots\}$ with transition probabilities

$$p_{k,k+1} = p, \quad p_{k+1,k} = q, \quad \text{for} \quad k \geq 0, \quad \text{and} \quad p_{0,0} = q.$$

a) For which values of p is this chain transient / null recurrent / positive recurrent?

b) In the case where it is positive recurrent, compute its stationary distribution π^*.

Hint: You can first try the detailed balance equation, betting on the fact that the chain is reversible.