Homework 1

Exercise 1. Show that any sequence of independent random variables $(X_n,\, n \geq 0)$ taking values in a countable set S is a Markov chain. Under which additional condition is the chain time-homogeneous?

Exercise 2. Let $(X_n,\, n \geq 1)$ be i.i.d. random variables such that $\mathbb{P}(X_n = +1) = \mathbb{P}(X_n = -1) = 1/2$ for all $n \geq 1$. Which of the following discrete-time processes are (time-homogeneous) Markov chains? For the cases for which the answer is positive, compute the transition matrix of the chain.

a) $(S_n,\, n \geq 0)$, where $S_0 = 0$, $S_n = X_1 + \ldots + X_n$, $n \geq 1$.

b) $(T_n,\, n \geq 0)$, where $T_n = \max(S_0, \ldots, S_n)$, $n \geq 0$.

c) $(Y_n,\, n \geq 0)$, where $Y_0 = 0$, $Y_n = S_{2n}$, $n \geq 1$.

d) $(Z_n,\, n \geq 0)$, where $Z_0 = 0$, $Z_1 = X_1$, $Z_{n+1} = X_{n+1} + X_n$, $n \geq 1$.

Exercise 3. Three red balls and three green balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state $k \in \{0, 1, 2, 3\}$ if the first urn contains k red balls. At each step, we draw one ball from each urn at random and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let X_n denote the state of the system at time n. Explain why $(X_n,\, n \geq 0)$ forms a Markov chain and compute its transition matrix.

Exercise 4. Let $i, j, k \in S$ be three states of a time-homogeneous Markov chain $(X_n,\, n \geq 0)$. Show that if state i communicates with state j and state j communicates with state k, then state i communicates with state k.

Exercise 5. Let P be the transition matrix of a time-homogeneous Markov chain X with state space S. Let us moreover assume that the rows of P are all equal (i.e. that $p_{ij} = p_j$ for all $i, j \in S$).

a) Show that $P^2 = P$.

b) Compute $\mathbb{P}(X_n = j)$ for $n \geq 1$, $j \in S$. Does this probability depend on n? Does it depend on the initial distribution of the Markov chain $\pi^{(0)}$?

Exercise 6. For each of the following transition matrices, draw the corresponding transition graph of the underlying time-homogeneous Markov chain. Deduce from this the equivalence classes of the chain, and establish for each of these classes whether the class is recurrent or transient, periodic or not.

$$P_1 = \begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 4/5 & 0 & 1/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 3/4 & 0 & 1/4 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$