D-Stability and Delay-Independent Stability of Homogeneous Cooperative Systems

Vahid Samadi Bokharaie , Oliver Mason and Mark Verwoerd

Appeared in IEEE Transactions on Automatic Control, 55(12), pp 2882-2885, 2010.
DOI: 10.1109/TAC.2010.2076334

A CORRECTION to the results of Section V appeared in IEEE Transactions on Automatic Control 1489 - 1489, Volume: 56 Issue: 6, June 2011, DOI: 10.1109/TAC.2011.2122790

Abstract

We introduce a nonlinear definition of D-stability, extending the usual concept for positive linear time-invariant (LTI) systems. We show that globally asymptotically stable, cooperative systems, homogeneous of any order with respect to arbitrary dilation maps are D-stable. We also prove a strong stability result for delayed cooperative homogeneous systems. Finally, we show that both of these results also hold for planar cooperative systems without the restriction of homogeneity.

I. Introduction

Due to their practical importance, Positive Systems have been the focus of a significant research effort in the Engineering, Applied Mathematics and Computational Sciences communities. The theory of positive linear time-invariant (LTI) systems is now well understood; however, for many applications of positive systems, factors such as nonlinearities, uncertainties and delays need to be taken into account. The work of this note is concerned with extending aspects of the stability theory of positive LTI systems to classes of nonlinear and delayed systems. Specifically, we shall

Hamilton Institute, NUI Maynooth, Ireland

Hamilton Institute, NUI Maynooth, Ireland, Corresponding Author - email: oliver.mason@nuim.ie

Hamilton Institute, NUI Maynooth, Ireland

show that two key stability properties of positive LTI systems extend directly to *cooperative systems* defined by vector fields that are homogeneous with respect to an arbitrary dilation map. The LTI system $\dot{x}(t) = Ax(t)$ is positive if and only if the matrix A is *Metzler*, meaning that all of its off-diagonal elements are non-negative. It is well known [8] that a positive LTI system is globally asymptotically stable (GAS) if and only if $\dot{x}(t) = DAx(t)$ is asymptotically stable for any diagonal matrix D with positive diagonal entries. This latter property is usually referred to as *D*-stability.

For positive time-delayed systems, it was shown in [5] that the delayed positive linear system $\dot{x}(t) = Ax(t) + Bx(t - \tau)$, where A is Metzler and B is nonnegative, is GAS for all values of the delay $\tau \geq 0$ provided the system with zero delay $\dot{x}(t) = (A + B)x(t)$ is GAS. In this regard, interesting results providing similar stability conditions for classes of positive systems defined by functional and integrodifferential equations have recently appeared in [10], [9].

Recently, it was shown in [3] that the results for positive LTI systems mentioned in the previous paragraph also hold for cooperative systems that are homogeneous of degree zero with respect to the standard dilation map on $\mathbb{R}^n$. The principal contribution of the current note is to further extend these results to cooperative systems that are homogeneous of any degree with respect to an arbitrary dilation map. It should be noted that the definition of D-stability considered here is considerably more general than that investigated in [3]. In particular, this allows the results of the current paper to be applied to cooperative systems that are not necessarily homogeneous. In the same vein, we show that the assumption of homogeneity is not necessary for planar cooperative systems. Removing this assumption for higher dimensional systems is the subject of ongoing work by the authors.

The layout of the note is as follows. In Section II we introduce notation, standard definitions and the key results needed for our later analysis. In Section III we introduce a nonlinear extension of the concept of D-stability and demonstrate that GAS homogeneous cooperative systems have this property. A strong stability result for delayed systems is then given in Section IV. In Section V we show that the homogeneity assumption is not required for planar systems and finally, in Section VI we present our conclusions.

II. Mathematical Background

Throughout the paper, $\mathbb{R}$ and $\mathbb{R}^n$ denote the field of real numbers and the vector space of all n-tuples of real numbers, respectively. $\mathbb{R}^{n\times n}$ denotes the space of $n\times n$ matrices with real entries. For $x \in \mathbb{R}^n$ and $i = 1, \dots, n$, x_i denotes the i^{th} coordinate of x. Similarly, for $A \in \mathbb{R}^{n\times n}$, a_{ij} denotes the $(i,j)^{th}$ entry of A. Also, for $x \in \mathbb{R}^n$, $diag(x)$ is the $n \times n$ diagonal matrix in which $d_{ii} = x_i$.

Throughout the paper, we shall be concerned with *positive systems* and with the stability properties of the equilibrium at the origin. For this reason, when we say that a system is Globally Asymptotically Stable, GAS for short, we mean that the origin is a GAS equilibrium of the system with respect to *initial conditions in the non-negative orthant* $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$.

The interior of $\mathbb{R}^n_+$ is denoted by $int(\mathbb{R}^n_+)$ and its boundary by $bd(\mathbb{R}^n_+) := \mathbb{R}^n_+ \backslash int(\mathbb{R}^n_+)$. For vectors $x, y \in \mathbb{R}^n$, we write: $x \geq y$ if $x_i \geq y_i$ for $1 \leq i \leq n$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x_i > y_i, 1 \leq i \leq n$.

Cooperative Homogeneous Systems

Given an n-tuple $r = (r_1, \dots, r_n)$ of positive real numbers and $\lambda > 0$, the *dilation map* $\delta^r_\lambda(x) : \mathbb{R}^n \to \mathbb{R}^n$ is given by $\delta^r_\lambda(x) = (\lambda^{r_1}x_1, ..., \lambda^{r_n}x_n)$. For an $\alpha \geq 0$, the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *homogeneous* of degree α with respect to $\delta^r_\lambda(x)$ if

$$\forall x \in \mathbb{R}^n, \lambda \geq 0, \quad f(\delta^r_\lambda(x)) = \lambda^\alpha \delta^r_\lambda(f(x)).$$

Throughout the paper, all vector fields $f : \mathcal{W} \to \mathbb{R}^n$ are defined on a neighbourhood $\mathcal{W}$ of $\mathbb{R}^n_+$. f is said to be cooperative on $\mathcal{U} \subseteq \mathcal{W}$ if it is differentiable on $\mathcal{U}$ and the Jacobian matrix $\frac{\partial f}{\partial x}(a)$ is Metzler for all $a \in \mathcal{U}$.

We shall call f ***irreducible*** if for $a \in int(\mathbb{R}^n_+)$, $\frac{\partial f}{\partial x}(a)$ is irreducible; (ii) for $a \in bd(\mathbb{R}^n_+) \setminus \{0\}$, either $\frac{\partial f}{\partial x}(a)$ is irreducible or $f_i(a) > 0 \ \forall i : a_i = 0$ [4].

We shall call $f : \mathcal{W} \to \mathbb{R}^n$ *non-decreasing* if $f(x) \geq f(y)$ whenever $x \geq y$ for $x, y \in \mathbb{R}^n_+$.

It is well known that cooperative systems are monotone [1], [4]. Formally, if $f : \mathcal{W} \to \mathbb{R}^n$ is cooperative on $\mathcal{W}$ and we denote by $x(t, x_0)$ the solution of $\dot{x}(t) = f(x(t))$ satisfying $x(0) = x_0$, then $x_0 \leq y_0$ implies $x(t, x_0) \leq x(t, y_0)$ for all $t \geq 0$. Moreover, as the origin is automatically an equilibrium of a homogeneous cooperative system, it follows that such systems are positive which means $\mathbb{R}^n_+$ is an invariant set for these systems.

III. STABILITY AND D-STABILITY OF HOMOGENEOUS COOPERATIVE SYSTEMS

Throughout this section, we are concerned with extending results on D-stability for linear positive systems to the nonlinear system

$$\dot{x}(t) = f(x(t)). \tag{1}$$

Throughout the section, $f : \mathcal{W} \to \mathbb{R}^n$ satisfies the following assumption unless explicitly stated otherwise.

Assumption 3.1: (i) f is continuous on $\mathcal{W}$ and C^1 on $\mathcal{W} \setminus \{0\}$;

(ii) f is homogeneous of degree α with respect to the dilation map δ_λ^r;

(iii) f is cooperative in $\mathbb{R}^n_+ \setminus \{0\}$.

These conditions ensures the existence and uniqueness of solutions [4].

Definition 3.1: We say that the system (1) is *D-stable* if

$$\dot{x}(t) = diag(d(x))f(x(t)) \tag{2}$$

is GAS for all C^1 mappings $d = (d_1, \dots, d_n) : \mathbb{R}^n \to \mathbb{R}^n$ satisfying:

(i) for $1 \leq i \leq n$, $d_i(x) \gg 0$ for all $x \in \mathbb{R}^n_+$ with $x_i > 0$;

(ii) $\frac{\partial d_i}{\partial x_j}(a) = 0$ for all $a \in \mathbb{R}^n_+$ and $i \neq j$; in other words, $d_i(x) = d_i(x_i)$.

Remark: The standard definition of D-stability for linear systems and the one used in [3] assumes that the function d above is constant. Furthermore, in the definition considered here, homogeneity of a vector field f is not necessarily preserved after pre-multiplication by $diag(d(x))$. This is in contrast to the situation in [3].

The main result of this section shows that a GAS cooperative homogeneous system is D-stable in the above sense. We first recall the following theorem for irreducible systems, which is a restatement of Theorem 5.2 in [4].

Theorem 3.1: Let $f : \mathcal{W} \to \mathbb{R}^n$ satisfy Assumption 3.1; further assume that f is irreducible. Then there exists $x^* \in int(\mathbb{R}^n_+)$ and $\gamma_{x^*} \in \mathbb{R}$ such that $f(x^*) = \gamma_{x^*} diag(r)x^*$. In addition $\gamma_{x^*} < 0$ if and only if the system (1) is GAS.

We now use the above result to prove the following proposition, which plays a key role in the proof of the main result of this section.

Proposition 3.1: Let the system (1) be GAS. Then for any $x_0 \in \mathbb{R}^n_+$, there exists a $v \gg x_0$ with $f(v) \ll 0$.

Proof: If f were irreducible, this result would be an immediate consequence of Theorem 3.1. The main step in the proof is to show that we can find an irreducible, homogeneous cooperative vector field f_1 such that $f_1(x) \geq f(x)$ for all $x \in \mathbb{R}^n_+$ and such that $\dot{x} = f_1(x)$ is GAS. Consider the vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ given by:

$$g_i(x) = \left((x_1^2)^{\frac{M}{r_1}} + (x_2^2)^{\frac{M}{r_2}} + ... + (x_n^2)^{\frac{M}{r_n}}\right)^{(r_i+\alpha)/M} \quad \text{for all} \quad 1 \leq i \leq n \tag{3}$$

where M is a real number such that $M/r_i > 1$ for $i = 1, ..., n$. It can be easily checked that:

$$\frac{\partial g_i}{\partial x_j} = 2((r_i + \alpha)/r_j) x_j^{(2M/r_j)-1} \left(x_1^{2M/r_1} + ... + x_n^{2M/r_n}\right)^{(r_i+\alpha/M)-1} \tag{4}$$

It follows from (3) and (4) that:

- $g(a) \geq 0$ and $\frac{\partial g_i}{\partial x_j}(a) \geq 0$ for all $a \geq 0$ and $i \neq j$;
- g is continuous on $\mathbb{R}^n$ and C^1 on $\mathbb{R}^n \setminus \{0\}$;
- g is irreducible;
- g is homogeneous of degree α with respect to δ^r_λ.

We now claim that $f + \epsilon_1 g$ is GAS for some $\epsilon > 0$. We prove this by contradiction. For all $\epsilon > 0$, we know that $(f + \epsilon g)$ is irreducible and satisfies Assumption 3.1. Further, $(f + \epsilon g)(v) \geq f(v)$ for all $v \geq 0$ because $g(v) \geq 0$ for all $v \geq 0$. If there is no $\epsilon_1 > 0$ such that the system $\dot{x} = (f + \epsilon_1 g)(x)$ is GAS, Theorem 3.1 implies that for every $\epsilon > 0$, there exists a non-zero $w_\epsilon \geq 0$ such that $(f + \epsilon g)(w_\epsilon) \geq 0$. We could then pick a sequence $\epsilon_n \to 0$, such that there exists a corresponding sequence $w_n \geq 0$, $w_n \neq 0$ with $(f + \epsilon_n g)(w_n) \geq 0$ for all n. By homogeneity, we can normalize all w_n such that $\|w_n\| = 1$. Choosing a subsequence, if necessary, we can assume that $w_n \to w'$ with $w' \geq 0$ and $\|w'\| = 1$. Since $\epsilon_n \to 0$, we know that

$$\lim_{n \to \infty} (f + \epsilon_n g)(w_n) = f(w') \geq 0$$

Since $\|w_n\| = 1$ and $w_n \geq 0$, it follows immediately from Proposition 3.2.1 in [1], that $x(t, w) \geq w > 0$ for all $t \geq 0$ which contradicts the fact that (1) is GAS. Therefore there must exist an $\epsilon_1 > 0$, such that $f + \epsilon_1 g$ is GAS.

Theorem 3.1 implies that there is a vector $v_1 \gg 0$ such that $(f + \epsilon_1 g)(v_1) = f(v_1) + \epsilon_1 g(v_1) \ll 0$ and since $g(v_1) \geq 0$, $f(v_1) \ll 0$. To conclude the proof, simply choose $\lambda > 0$ such that $v := \delta^r_\lambda(v_1) \gg x_0$; the homogeneity of f implies that $f(v) \ll 0$. This completes the proof.

Remark: In [3], the construction of the dominating vector field f_1 is more straightforward, as a linear positive mapping will satisfy the requirements of the function g. The main contribution of the above proposition is to show that this is also possible for an arbitrary dilation map. In the present context, we should note that a version of the above result for vector fields that can be expressed in a nonlinear matrix-vector form (not necessarily homogeneous) has recently appeared in [7].

Before stating the main result of this section, we recall the following fact, which is Proposition 3.2.1 in [1].

Lemma 3.1: Let $f : \mathcal{W} \to \mathbb{R}^n$ satisfy (i) and (iii) of Assumption 3.1. Suppose that $v \geq 0$ is such that $f(v) \leq 0$ ($f(v) \geq 0$). Then the trajectory $x(t, v)$ of the system (1) is non-increasing (non-decreasing) for $t \geq 0$.

We are now in a position to prove the main result of this section.

Theorem 3.2: If the system (1) is GAS then it is D-stable.

Proof: Let $d : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the condition of Definition 3.1 and let $x_0 \in \mathbb{R}^n_+$ be given. It follows from Proposition 3.1 that there exists $v \gg x_0$ with $diag(d(v))f(v) \ll 0$. Lemma 3.1 immediately implies that the trajectory $x(t, v)$ is non-increasing and bounded. Theorem 1.2.1 of [1] implies that it must converge to an equilibrium. Thus, the theorem will be proven provided we can show that the origin is the only equilibrium of the system

$$\dot{x} = diag(d(x))f(x) \tag{5}$$

To this end, note that since f is homogeneous, we know that 0 is an equilibrium of (5). We shall show that it is the only equilibrium of the system by way of contradiction. Suppose that there is some $e := [e_1, e_2, ..., e_n]^T > 0$ satisfying $diag(d(e))f(e) = 0$. Let $x(t, x_0)$ denote the solution of (5) with initial condition x_0.

Choose some $v \gg 0$ with $f(v) \ll 0$. It is immediate that $diag(d(v))f(v) \ll 0$. Define $\kappa = \max\{(\frac{e_i}{v_i})^{(1/r_i)} : 1 \leq i \leq n\}$ and let $j \in \{1, ..., n\}$ be such that $(\frac{e_j}{v_j})^{1/r_j} = \kappa$. Note that as $e \neq 0$, $\kappa > 0$. It follows from the definition of κ that $e \leq \delta^r_\kappa(v)$ and that $e_j = \delta^r_\kappa(v)_j$. As f is homogeneous, we have that $f(\delta^r_\kappa(v)) \ll 0$ and hence $diag(d(\delta^r_\kappa(v)))f(\delta^r_\kappa(v)) \ll 0$. Thus, we can pick t_1, such that for all $0 < t < t_1$,

$$x(t, \delta^r_\kappa(v)) \ll \delta^r_\kappa(v)$$

In particular, $x(t, \delta^r_\kappa(v))_j < \kappa^{r_j} v_j = e_j$. But as $e \le \delta^r_\kappa(v)$ and the system (5) is monotone, we must have $x(t, \delta^r_\kappa(v))_j \ge e_j$ for all $t \ge 0$. This contradiction shows that the origin is the only equilibrium of (5) as claimed. This completes the proof.

Remark: As has been emphasised above, the definition of D-stability we are considering here is more general than that used in [3]. In the earlier paper, it was only required that $\dot{x}(t) = Df(x(t))$ is GAS for diagonal matrices D with positive diagonal entries. In this case, it is immediate that if (1) is GAS, then the origin is the only equilibrium of $\dot{x}(t) = Df(x(t))$ as the equilibria of the two systems are in one-to-one correspondence. However, in the case considered here, some components of $d(x)$ could be zero at boundary points of $\mathbb{R}^n_+$, potentially leading to non-trivial equilibria of (5). The above result establishes that this cannot happen. Furthermore, the arguments used in [3] rely explicitly on the fact that the system $\dot{x} = Df(x)$ is still homogeneous and the proof of asymptotic convergence made use of the fact that for homogeneous systems of degree 1, $x(t, \lambda x_0) = \lambda x(t, x_0)$. This relation becomes more complicated for higher order systems and the original arguments of [3] cannot be applied directly to the more general case. The arguments in the present paper only rely on the boundedness and monotonicity of trajectories $x(t, v)$ with $f(v) \ll 0$ and the *uniqueness* of the equilibrium at the origin. As such the methods of proof, though on the surface similar, are quite distinct.

The proof of the previous result shows that a cooperative, positive system $\dot{x} = f(x)$ with equilibrium at the origin is GAS if for any $x_0 \in \mathbb{R}^n_+$, there exists a $v \gg x_0$ with $f(v) \ll 0$. In [6], under the additional assumption of irreducibility, it was shown that the existence of $v \gg 0$ such that $f(v) < 0$, was sufficient for GAS. In [11] a similar sufficient condition for *almost complete stability* of cooperative systems with inputs was presented. Another result of this type for discrete time systems appeared in [12].

Example 3.1: Consider the system

$$\dot{x}(t) = f(x(t))$$

where

$$f(x_1, x_2, x_3) = \begin{pmatrix} -2x_1^{5/3} + x_3 \\ x_1^2 - 2x_2^{3/2} + x_3 x_1^{1/3} \\ x_1 x_2 + x_2^{7/4} - 5x_3^{7/5} \end{pmatrix}$$

It can be easily checked that this system is cooperative and homogeneous of degree 2 with respect to the dilation map δ_λ^r, with $r = (3, 4, 5)$. Also, $f(0.5, 0.5, 0.3) = (-0.33, -0.22, -0.38) \ll 0$, which implies that the system is GAS and in fact D-stable. If we consider d given by:

$$d(x_1, x_2, x_3) = \left(\frac{x_1^2}{x_1^2 + 1}, x_2, 1 + \sin^2(x_3) \right)^T,$$

d satisfies all the conditions of Definition (3.1) and it follows that $\dot{x}(t) = diag(d(x)).f(x)$ is GAS. Note that the vector field $diag(d(x))f(x)$ is not homogeneous in this case.

IV. Stability Independent of Delay

We now consider the delayed system

$$\dot{x}(t) = f(x(t)) + g(x(t - \tau)) \quad \tau \geq 0, \tag{6}$$

where $f : \mathcal{W} \to \mathbb{R}^n$ and $g : \mathcal{W} \to \mathbb{R}^n$ satisfy the following properties.

- f and g are continuous in $\mathcal{W}$ and C^1 on $\mathcal{W} \setminus \{0\}$;
- f and g are homogeneous of degree α with respect to the dilation map δ_λ^r;
- f is cooperative in $\mathbb{R}_+^n \setminus \{0\}$ and g is non-decreasing in $\mathbb{R}_+^n$.

In the main result stated below, we show that (6) is GAS for any fixed delay $\tau \geq 0$ if the system with $\tau = 0$ is GAS, thus extending the main result of [5] to a broad class of nonlinear systems. Initial conditions for (6) are elements ϕ of $C([-\tau, 0], \mathbb{R}_+^n)$, and the state $x_t, t \geq 0$ of (6) is the history segment $x_t(\theta) = x(t-\theta)$ for $\theta \in [0, \tau]$. For a vector $v \in \mathbb{R}_+^n$, we define $\hat{v} \in C([-\tau, 0], \mathbb{R}_+^n)$ by $\hat{v}(\theta) = v$ for $\theta \in [-\tau, 0]$.

The order relation on $C([-\tau, 0], \mathbb{R}_+^n)$ is defined in the usual manner with respect to the cone $\{\phi : \phi(s) \geq 0 \quad \forall s \in [-\tau, 0]\}$. For $\phi \in C([-\tau, 0], \mathbb{R}_+^n)$, let $x(t, \phi)$, $x_{t,\phi}$ denote the trajectory and state of (6) respectively. Then for any $\phi, \psi \in C([-\tau, 0], \mathbb{R}_+^n)$, with $\phi \leq \psi$, it follows that $x(t, \phi) \leq x(t, \psi)$ for all $t \geq 0$ [1].

As noted in Chapter 5 of [1], the equilibria of the system (6) correspond exactly with the equilibria of the undelayed system given by:

$$\dot{x}(t) = (f + g)(x(t)) \tag{7}$$

Formally, $e \in \mathbb{R}_+^n$ is an equilibrium of (7) if and only if $\hat{e}$ is an equilibrium of (6).

The following fact is the analogue for delayed systems of Lemma 3.1 and follows immediately from Corollary 5.2.2 of [1].

Lemma 4.1: Consider the system (6). Suppose there exists a vector $v \geq 0$ with $(f+g)(v) \leq 0$. Then the trajectory $x(t, \hat{v})$ is non-increasing.

Theorem 4.1: Consider the system (6). If the system (7) is GAS then the system (6) is GAS for all $\tau \geq 0$.

Proof: Note that as the equilibria of (6) and (7) are identical and (7) is GAS by assumption, the origin is the unique equilibrium of (6).

For any initial condition $\phi \in C([-\tau, 0], \mathbb{R}^n_+)$, it is a simple consequence of Proposition 3.1 that there exists some $v \gg 0$ in $\mathbb{R}^n_+$ with $\phi \ll \hat{v}$ and $(f+g)(v) \ll 0$. It now follows from Lemma 4.1 that the solution $x(t, \hat{v})$ of (6) is non-increasing and bounded. Hence, Theorem 1.2.1 of [1] implies that it converges to an equilibrium which must be the origin. Finally, the monotonicity of (6) implies that the solution $x(t, \phi)$ of (6) also converges to 0 as $t \to \infty$

Example 4.1: Consider the system

$$\dot{x}(t) = f(x(t)) + g(x(t-\tau))$$

where

$$f(x_1, x_2, x_3) = \begin{pmatrix} -5x_1^3 + x_1 x_2 \\ x_1 x_3 - 7x_2^2 \\ x_1^3 x_2 + 0.5x_1^2 x_3 - 10x_3^{5/3} \end{pmatrix}$$

$$g(x_1, x_2, x_3) = \begin{pmatrix} x_1^3 + 2x_3 \\ 0.5x_1^2 x_2 + x_1 x_3 + 1.5x_2^2 + 2x_3^{4/3} \\ x_1 x_2^2 + x_3 x_2 + 2x_3^{5/3} + x_1^5 \end{pmatrix}$$

It can be easily checked that both f and g are homogeneous of degree 2 with respect to the dilation map δ_λ^r with $r = (1, 2, 3)$. Moreover, f is cooperative and g is non-decreasing for $x \geq 0$. Note that $(f+g)(0.5, 0.3, 0.1) = (-0.15, -0.26, -0.02) \ll 0$. Therefore, we can conclude that the origin is the unique equilibrium of this system and the delayed system is globally asymptotically stable for every non-negative delay.

V. Non-homogeneous Planar Systems

In this section we prove that the two main results of this paper do not require the assumption of homogeneity in the case of 2-dimensional cooperative systems. Throughout the section, f :

$\mathcal{W} \to \mathbb{R}^2$ is assumed to satisfy (i) and (iii) of Assumption 3.1, while g satisfies (i) of Assumption 3.1. The main fact we shall need is the following result.

Theorem 5.1: Assume that the system $\dot{x}(t) = f(x(t))$ is GAS. Then given any $x_0 \in \mathbb{R}^2_+$, there exists $v \gg x_0$ with $f(v) \ll 0$.

Proof: As f is cooperative and GAS, it follows from Lemma 3.1 that there cannot exist a non-zero vector $w \geq 0$ with $f(w) \geq 0$. Write $\Omega := \{x \in \mathbb{R}^2 : x \gg x_0\}$ and define $\Omega_1 := \{x \in \Omega : f_1(x) < 0\}$, $\Omega_2 := \{x \in \Omega : f_2(x) < 0\}$. Now Ω is clearly a connected set and Ω_1, Ω_2 are open subsets of Ω with $\Omega = \Omega_1 \cup \Omega_2$. As Ω is connected, it follows that $\Omega_1 \cap \Omega_2$ is non-empty; but this means that there exists some $v \gg x_0$ with $f(v) \ll 0$ as claimed.

Theorem 5.2: If the system $\dot{x}(t) = f(x(t))$ is GAS, then it is D-stable.

Proof: Let $d : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 mapping satisfying conditions (i) and (ii) of Definition 3.1. We first prove that the system (2) has a unique equilibrium at the origin. As f is GAS, $f(x) = 0$, $x \geq 0$ implies $x = 0$. Hence, as $d_i(x) > 0$ for any x with $x_i > 0$, it is immediate that (2) can have no equilibrium in the interior of $\mathbb{R}^2_+$. Suppose now that $diag(d(a))f(a) = 0$ for some $a = (0, a_2)$ with $a_2 > 0$. (The case $a = (a_1, 0)$ with $a_1 > 0$ is handled similarly.) As $d_2(a) > 0$ by assumption, we must have $f_2(0, a_2) = 0$. Hence as (1) is GAS, $f_1(0, a_2) < 0$ (otherwise $f(a) \geq 0$ which contradicts that f is GAS). However, $f_1(0, 0) = 0$ and

$$\frac{\partial f_1}{\partial x_2}(s) \geq 0$$

for all $s \in \mathbb{R}^2_+$, which implies that $f_1(0, a_2) \geq 0$. This is a contradiction and we can conclude that (2) has a unique equilibrium at the origin as claimed.

Theorem 5.1 implies that for any $x_0 \in \mathbb{R}^2_+$, there exists some $v \gg x_0$ with $diag(d(v))f(v) \ll 0$. It now follows from Lemma 3.1 that the trajectory $x(t, v)$ of (2) is non-increasing and bounded from below. Hence as 0 is the only equilibrium of the system, $x(t, v) \to 0$ as $t \to \infty$. It follows immediately from the monotonicity of (2) that $x(t, x_0) \leq x(t, v)$ also converges to the origin. This completes the proof.

Theorem 5.3: If the system $\dot{x}(t) = (f + g)(x(t))$ is GAS, then the delayed system

$$\dot{x}(t) = f(x(t)) + g(x(t - \tau))$$

is GAS for any $\tau > 0$.

Proof: Theorem 5.1 implies that for any initial condition $\phi \in C([-\tau, 0], \mathbb{R}^2_+)$, there exists some $v \in \mathbb{R}^2_+$ with $\phi(s) \ll v$ for all $s \in [-\tau, 0]$ and $(f + g)(v) \ll 0$. Further as the equilibria of

(7) coincide with those of (6), it follows that (6) has a unique equilibrium at zero. These facts combined with Lemma 4.1 imply that $x(t, \phi) \leq x(t, \hat{v})$ tends to zero as $t \to \infty$.

VI. Conclusions

We have extended the notion of D-stability to nonlinear systems and shown that GAS cooperative homogeneous systems are D-stable. We have also presented a strong stability result for delayed systems of this class. Our results extend earlier work for linear systems and cooperative systems homogeneous of degree one with respect to the standard dilation map. The assumption of homogeneity is not needed for planar systems and it is the authors' opinion that this assumption is not required for higher dimensional systems either. This conjecture is the subject of ongoing work, the results of which we hope to report in the future.

Acknowledgements

This work has been supported by Science Foundation Ireland (SFI) grant 08/RFP/ENE1417 and by the Irish Higher Education Authority (HEA) PRTLI Network Mathematics grant. The authors would like to thank the anonymous reviewers for helpful comments which have improved the paper. The authors would also like to thank Professor Richard Middleton for several helpful conversations related to the subject matter of the paper.

References

[1] Smith H.A. (1995) Monotone Dynamical Systems. An introduction to the Theory of Competitive and Cooperative Systems American Mathematical Society, Providence, Rhode Island, USA.

[2] Horn R.A., Johnson C.R. (1985) Matrix Analysis Cambridge University Press, Newyork, NY, USA.

[3] Mason O., Verwoerd M. Observations on the Stability Properties of Cooperative System. *Systems and Control Letters*, 58 (2009) 461-467.

[4] Aeyels D., Leenheer P. D. Extension of The Perron-Frobenius Theorem to Homogeneous Systems. *SIAM Journal of Control Optimization*, Vol. 41, No. 2, pp 563-582, 2002.

[5] Haddad W. M., Chellaboina V. Stability theory for nonnegative and compartmental dynamical systems with time delay. *Systems and Control Letters*, 51 (2004) 355-361.

[6] Leenheer P.D., Aeyels D. Stability Properties of Equilibria of Classes of Cooperative Systems. *IEEE Transactions on Automatic Control*, 46(12):1996–2001, 2001.

[7] Ruffer, B. S., Kellett, C. M. and Weller, S. R. Connection between cooperative positive systems and integral input-to-state stability of large-scale systems *Automatica*, 46 (2010) 1019-1027.

[8] Farina L., Rinaldi S. (2000) Positive Linear Systems. Theory and Applications. Pure and Applied mathematics. John Wiley & Sons, Inc., New York

[9] Ngoc P.H.A., Murakami S., Naito T., Shin T.S., Nagabuchi Y. On Positive Linear Volterra-Stieltjes Differential Systems. *Integr. equ. oper. theory*, 64 (2009) 325-255.

[10] Ngoc P.H.A., Naito T., Shin J.S., Murakami S. On Stability and Robust Stability of Positive Linear Volterra Equations. *SIAM Journal of Control Optimization*, Vol. 47, No. 2, pp 975-996, 2008.

[11] Piccardi C., Rinaldi S. Remarks on Excitability, Stability and Sign of Equilibria in Cooperative Systems. *Systems and Control Letters*, 46 (2002) 153-163.

[12] S. Dashkovskiy, B.S. Rüffer, and F. Wirth. An ISS small-gain theorem for general networks. *Mathematics of Control, Signals and Systems*, 19:93-122, 2007.