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Nash Equilibria for Competitive Information Diffusion on Trees

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Abstract

We consider the game theoretic model of competitive information diffusion recently introduced in [1]. We show that for the case of 2 competing agents, there exists a Nash Equilibrium for this game on any tree. We also present an example to show that this is not necessarily true for 3 or more agents.

Keywords: Competitive Information Diffusion; Game Theory; Nash Equilibrium; Trees

1. Introduction

In the recent papers [1, 2] a deterministic model for competitive information diffusion on social networks was introduced and studied. The model considers the diffusion process as a game played on the network by external agents. In contrast to many other game theoretic models for the diffusion of information and innovation [5, 3], it considers competition between different innovations rather than the spread of a single innovation in a network. The main result of [1] claimed that any network of diameter 2 possesses a pure strategy Nash Equilibrium (N.E.). Unfortunately, as pointed out in [2], this result is not true without additional technical assumptions. In fact, even for the case of 2 competing agents on a network of diameter 2, it is possible that the game introduced in [1] does not possess a N.E.
Rather than restricting the diameter of the network, we focus instead on hierarchical structures, which can be represented by a tree. We show that the model of [1] with 2 competing agents always possesses a N.E. when the underlying graph is a tree. While the tree structure is clearly restrictive, it is worth noting that many social networks, including the online example twitter, are hierarchical in nature with 'leaders' and 'followers'. Moreover, our result makes no assumption concerning the diameter of the network; thus it opens an alternative line of

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research into the model of [1]. The layout of the note is as follows. In Section 2, we recall the basic model and set our notation. In Section 3, we present and prove our result, while in Section 4 we present our conclusions.

2. Preliminaries

All the graphs considered in this note are simple and undirected. For a graph G, we use $V(G)$ to denote its vertex set and $E(G)$ to denote its edge set. We denote an edge between v, w in $V(G)$ by vw. For $v \in V(G)$, the neighbours of v are given by $N(v) = \{w \in V(G) \mid vw \in E(G)\}$. If $vw \in E(G)$, $G - vw$ denotes the graph obtained by removing the edge vw from $E(G)$. Similarly if $vw \notin E(G)$, $G + vw$ denotes the graph obtained by adding the edge vw to the edge set of G. Also for $v \in V(G)$, $G - v$ denotes the graph obtained by removing v and all of its incident edges from G. For a set X, $|X|$ denotes the cardinality of X.
A tree T is a connected acyclic graph. It is well known that in a tree T, there exists a *unique* path between every pair of distinct vertices v, w in $V(T)$. Also, for every edge $vw \in E(T)$, $T - vw$ is disconnected.
Our primary interest in the current paper is to study the game theoretic model of competitive information diffusion introduced in [1] on trees. We now recall the model of competitive information diffusion from [1].

The Game D

Let G be a graph with $|V(G)| = N$ and a set of agents $[1, n] = \{1, \ldots, n\}$ be given. Initially ($t = 0$), each agent i selects one vertex, x_i, in $V(G)$, which is labelled i. The n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$ is known as a *strategy profile.* We only consider strategy profiles in which all of the x_i are distinct. All other nodes are initially labelled 0 (corresponding to white nodes in [1]). We use -1 to denote grey nodes. In keeping with the original model of [1], grey nodes *do not propagate.* The basic paradigm is as follows. At time $t \geq 1$, if a white vertex v (labelled 0) neighbours two vertices with distinct labels in $[1, n]$ at time $t - 1$, then v is labelled -1 (grey). If v neighbours a vertex labelled $i \in [1, n]$ at time $t - 1$ and has no neighbours labelled $j \in [1, n] \backslash \{i\}$, then v is labelled i. Otherwise, v's label is unchanged.
Throughout the note, we refer to the above process as the game **D**. The process terminates in a finite number of steps when no further vertices can be labelled $i \in [1, n]$. The *utility* $U_i(\mathbf{x})$ of agent i is the total number of vertices labelled i when the process terminates. In a slight abuse of notation, we shall occasionally write $U_i(x_1, \ldots, x_n)$ for $U_i(\mathbf{x})$. A strategy profile $\mathbf{x} = (x_1, \ldots, x_n)$ is a Nash Equilibrium $N.E.$ for **D** if no agent can unilaterally improve their utility by changing to another starting vertex. Formally if

$$U_i(x_1, \ldots, x_n) \geq U_i(x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_n)$$

for all $v \in V(G) \backslash \{x_1, \ldots, x_n\}$.

3. Main Result

In this section, we show that for 2 agents competing on a tree, there always exists a Nash equilibrium (N.E.) for the game $\mathbf{D}$. Moreover, our result characterises the N.E. on a tree T. We first note that for the general game with $n > 2$ agents, this conclusion does not necessarily hold.

Example 3.1. *Consider the game $\boldsymbol{D}$ on the tree below with 7 vertices and 3 agents. Consider, without loss of generality, a strategy profile $\mathbf{x} = (v_i, v_j, v_k)$*

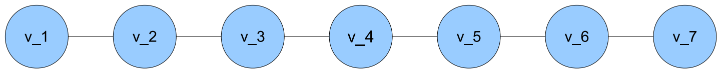

Figure 1: Three agents and no Nash Equilibrium on T

where $i < j < k$.
If $v_i v_j \notin E(T)$ then $U_1(\mathbf{x})$ is increased by player 1 switching to vertex v_{j-1}. If $v_j v_k \notin E(T)$, $U_3(\mathbf{x})$ is increased by player 3 choosing vertex v_{j+1}.
If $v_i v_j \in E(T)$ and $v_j v_k \in E(T)$ then $U_2(\mathbf{x}) = 1$, and it is always possible for player 2 to improve their utility by moving to either v_{i-1} or v_{k+1}. Thus there is no N.E. for $\boldsymbol{D}$ on T.

For the remainder of this section, we consider the game $\mathbf{D}$ with $n = 2$ agents on a tree T with vertex set V and edge set E.
The next lemma notes that once one agent selects an initial vertex v, the optimal choice for the other agent is a neighbour of v.

Lemma 3.1. *Let $v \in V$ be given. There exists $w \in N(v)$ such that*

$$U_2(v, w) = \max_{x \in V} U_2(v, x).$$

Proof. Suppose the degree of v is d. As T is a tree, it is readily seen that $T - v$ has d connected components $C_1, .., C_d$. Let u_i denote the neighbour of v that is in C_i for $1 \leq i \leq d$.
If $x \in V(C_i)$, $y \in V(C_j)$, with $i \neq j$, the unique path in T from x to y includes v. Hence, agent 2 cannot colour vertices in more than one component of $T - v$. Thus $\max_{x \in V} U_2(v, x) = \max_{1 \leq i \leq d} |V(C_i)|$.
Let k be such that $|V(C_k)| = \max_{1 \leq i \leq d} |V(C_i)|$. Choose $w \in N(v) \cap V(C_k)$. For any $u \in V(C_k)$ the unique path (in T) from v to u must contain w. It follows that all vertices in C_k are labelled 2. Hence,

$$U_2(v, w) = |V(C_k)| = \max_{x \in V} U_2(v, x)$$

as claimed. □

Lemma 3.2. *Let $\mathbf{x} = (v, w)$ be a strategy profile such that $vw \in E(T)$. Let $u \in N(v) \backslash \{w\}$ and consider the strategy profile $\mathbf{x}' = (v, u)$. Then $U_2(\mathbf{x}') < U_1(\mathbf{x})$*

Proof. As T is a tree, $T - vw$ has two connected components. Write C_v, C_w for the component of $T - vw$ containing v, w respectively. It is clear that all vertices in C_v will be labelled 1 and all the vertices in C_w are labelled 2. Hence, $U_1(\mathbf{x}) = |V(C_v)|$.
Now consider the profile $\mathbf{x}' = (v, u)$ where $u \in N(v)\backslash\{w\}$. Then $u \in V(C_v)$. Hence, for the strategy profile $\mathbf{x}'$, no vertices in C_w are labelled 2, since the path to any $c \in V(C_w)$ must contain v. It follows that the vertices labelled 2 form a subset of $V(C_v)\backslash\{v\}$. Thus

$$U_2(\mathbf{x}') \leq |V(C_v)| - 1 < U_1(\mathbf{x})$$

□

The following lemma shows that if a strategy profile consists of two neighbouring vertices, then all vertices are either labelled 1 or 2 when the process terminates.

Lemma 3.3. *Consider a strategy profile* $\mathbf{x} = (v, w)$ *such that* $vw \in E(T)$*. Then* $U_1(\mathbf{x}) + U_2(\mathbf{x}) = |V(T)|$*.*

Proof. As v, w are neighbours, they have no common neighbour as T is a tree. Further, $T - vw$ consists of two connected components, C_v and C_w. By similar arguments to those employed in the previous results, it can be seen that every vertex in C_v is labelled 1 while every vertex in C_w is labelled 2. So $U_1(\mathbf{x}) = |V(C_v)|$, $U_2(\mathbf{x}) = |V(C_w)|$ and hence $U_1(\mathbf{x}) + U_2(\mathbf{x}) = |V(T)|$. □

We can now state the main result of the note, which shows that the game **D** with 2 agents has a N.E. on any tree of any diameter.

Theorem 3.1. *Let* $\mathbf{x} = (v, w)$ *be a strategy profile on a tree* T *such that* $vw \in E(T)$ *and*

$$|U_1(v, w) - U_2(v, w)| = \min_{yz \in E(T)} |U_1(y, z) - U_2(y, z)|.$$

Then $\mathbf{x}$ *is a Nash Equilibrium.*

Proof. Without loss of generality, assume $U_2(\mathbf{x}) \geq U_1(\mathbf{x})$. By Lemma 3.1, we know that there is some $u \in N(v)$ such that

$$U_2(v, u) \geq U_2(v, y)$$

for all $y \in V(T)$. However, it follows from Lemma 3.2 that

$$U_2(v, u) \leq U_1(v, w) \leq U_2(v, w).$$

Thus agent 2 certainly cannot increase their utility by unilaterally changing strategy.
Next suppose that there exists some $u \in V(T)$ such that $U_1(u, w) > U_1(v, w)$. Lemma 3.1 implies that there exists such a u in $N(w)$. From Lemma 3.3 we know that

$$U_1(u, w) + U_2(u, w) = |V(T)| = U_1(v, w) + U_2(v, w).$$

It follows immediately that $U_2(u,w) < U_2(v,w)$. We now show that this leads to a contradiction. There are three cases to consider.

Case 1: $U_1(u,w) \geq U_2(v,w)$
From Lemma 3.2, we know that $U_1(u,w) \leq U_2(v,w) - 1$ so this cannot happen.

Case 2: $U_1(u,w) < U_2(v,w)$, and $U_2(u,w) > U_1(u,w)$
We know that $U_2(v,w) > U_2(u,w)$, and $U_1(u,w) > U_1(v,w)$. Thus, $U_2(v,w) + U_1(u,w) > U_1(v,w) + U_2(u,w)$. Rearranging, we see that

$$U_2(v,w) - U_1(v,w) > U_2(u,w) - U_1(u,w) > 0$$

which contradicts our initial assumption, that $|U_2(v,w) - U_1(v,w)|$ is minimal.

Case 3: $U_1(u,w) < U_2(v,w)$, and $U_2(u,w) \leq U_1(u,w)$
We know from Lemma 3.2 that $U_1(u,w) \leq U_2(v,w) - 1$. It similarly follows that $U_2(u,w) \geq U_1(v,w) + 1$. Taken together, these observations imply that

$$\begin{aligned} |U_1(u,w) - U_2(u,w)| &= U_1(u,w) - U_2(u,w) \\ &\leq U_2(v,w) - U_1(v,w) - 2 \\ &< U_2(v,w) - U_1(v,w). \end{aligned}$$

This is again a contradiction.
Putting the above arguments together, we see that $\mathbf{x} = (v,w)$ is a N.E. as claimed. □

Example 3.2. *In the tree T in Figure 2 below there is a unique Nash Equilibrium, $\mathbf{x} = (v,w)$. $|U_1(\mathbf{x}) - U_2(\mathbf{x})| = |7-5| = 2$ is clearly minimal over all pairs of neighbours.*

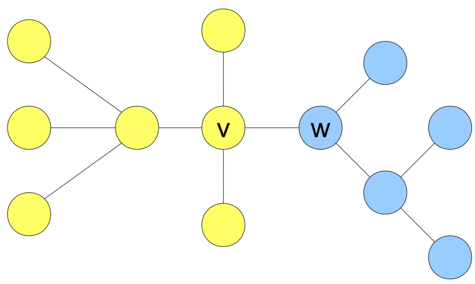

Figure 2: The tree T

4. Concluding Remarks

b As highlighted by the work of [1, 2] and [3, 4], identifying conditions for the existence of N.E. for games on graphs is a difficult problem. We have shown that the model of competitive information diffusion studied in [1, 2] always admits a N.E. on a tree when the number of agents is 2. While trees are a restrictive class of graphs, they can serve as an idealisation of hierarchical structures that arise in many social networks. Identifying other structures which guarantee the existence of a N.E. and characterising these when they exist remains a challenging question for future research.

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