STABILITY CRITERIA FOR SWITCHED AND HYBRID SYSTEMS

– DRAFT –

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Abstract. The study of the stability properties of switched and hybrid systems gives rise to a number of interesting and challenging mathematical problems. The objective of this paper is to outline some of these problems, to review progress made in solving these problems in a number of diverse communities , and to review some problems that remain open. An important contribution of our work is to bring together material from several areas of research and to present results in a unified manner. We begin our review by relating the stability problem for switched linear systems and a class of linear differential inclusions. Closely related to the concept of stability are the notions of exponential growth rates and converse Lyapunov theorems, both of which are discussed in detail. In particular, results on common quadratic Lyapunov functions and piecewise linear Lyapunov functions are presented, as they represent constructive methods for proving stability, and also represent problems in which significant progress has been made. We also comment on the inherent difficulty of determining stability of switched systems in general which is exemplified by NP-hardness and undecidability results. We then proceed by considering the stability of switched systems in which there are constraints on the switching rules; be it through dwell time requirements or state dependent switching laws. Also in this case the theory of Lyapunov functions and the existence of converse theorems is reviewed. We briefly comment on the classical Lure' problem and on the theory of stability radii, both of which encapture many of the features of switched systems and are rich sources of practical results on the topic. Finally, both as an application, and an introduction to stochastic positive switched systems, a switched linear model of TCP dynamics is derived and several results presented.

Key words. hybrid systems, switched systems, stability, growth rates, converse Lyapunov theorem, common quadratic Lyapunov functions, dwell time, Lure' problem, stability radii, TCP.

AMS subject classifications. 93-01, 93D09, 93D30, 34D08, 34D10, 37B25, 37C75

1. Motivation. The past decade has witnessed an enormous interest in systems whose behaviour can be described mathematically using a mixture of logic based switching and difference/differential equations. This interest has been primarily motivated by the realisation that many man-made systems, and some physical systems, may be modelled using such a framework. Examples of such systems include the Multiple-Models , Switching and Tuning paradigm from adaptive control [118], Hybrid Control Systems [66], and a plethora of techniques that arise in Event Driven Systems. Due to the ubiquitous nature of these systems, there is a growing demand in industry for methods to model, analyse, and to understand systems with logic-based and continuous components. Typically, the approach adopted is to describe and analyse these systems is to employ theories that have been developed for differential equations whose parameters vary in time. Differential equations whose parameters are time-varying have been the subject of intense study in several communities for the best part of the last century. While major advances in this topic have been made in the Mathematics, Control Engineering, and more recently, the Computer Science communities, many important questions that relate to their behaviour still remain unanswered, even for linear systems. Perhaps the most important of these relate to

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the stability of such systems. The objective of this paper is to review the theory of stability for linear systems whose parameters vary abruptly with time and to outline some of the most pressing questions that remain outstanding.

The study of differential equations whose parameters vary discontinuously has been the subject of study in the Mathematics community for the past 100 years. While research on this topic has progressed at a steady pace, the past decade in particular has witnessed a developing interest in this topic in several other fields of research. The multi-disciplinary research field of Hybrid Systems that has emerged as a result of this interest lies at the boundaries of computer science, control engineering and applied mathematics.

A Hybrid System is a dynamical system that is described using a mixture of continuous/discrete dynamics and logic based switching. The classical view of such systems is that they evolve according to mode dependent continuous/discrete dynamics, and experience transitions between modes that are triggered by 'events'. The following examples show that this feature occurs in diverse areas of application.

EXAMPLE 1.1. *[76]* **Automobile with a manual gearbox :** *The motion of an car that travels along a fixed path can be characterised by two continuous states: velocity v and position s. The system has two inputs: the throttle angle (u) and the engaged gear (g). It is evident that the manner in which the velocity of the car responds to the throttle input depends on the engaged gear. The dynamics of the automobile can be therefore be thought of as being hybrid in nature: in each mode (engaged gear) the dynamics evolve in a continuous manner according to some differential equation. Transitions between modes are abrupt and are triggered by driver interventions in the form of gear changes.*

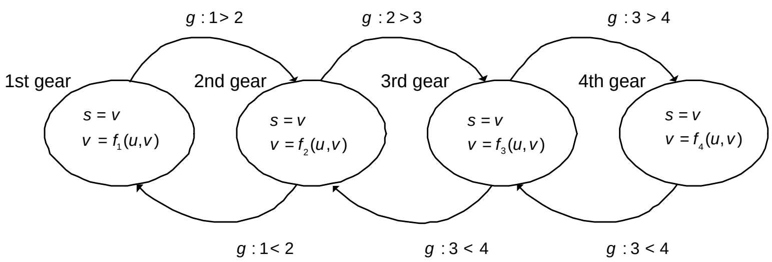

Fig. 1.1: A hybrid model of a car with a manual gearbox [76].

EXAMPLE 1.2. *[65]* **Network congestion control :** *The Transmission Control Protocol (TCP) is the protocol of choice for end-to-end packet delivery in the internet. TCP is an acknowledgement based protocol. Packets are sent from sources to destinations and destinations inform sources of packets that have been successfully received. This information is then used by the i'th source to control the number of unacknowledged packets (belonging to source i) in the network at any one time (w_i). This basic mechanism provides TCP sources operating in congestion avoidance mode [97] with a method for inferring available network bandwidth and for controlling congestion in the network. Upon receipt of a successful acknowledgement, the variable w_i is updated according to the rule: $w_i \leftarrow w_i + a$ where a is some positive number and a new packet is inserted into the network. This is TCP's self clocking mechanism. If w_i exceeds* *an integer threshold, then another packet is inserted into the network to increase the number of unacknowledged packets by one. TCP deduces from the detection of lost packets that the network is congested and responds by reducing the number of unacknowledged packets in the network according to:* $w_i \leftarrow \beta w_i$ *where* β *is some number between zero and* 1*.*

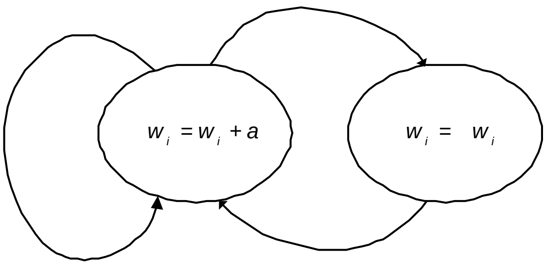

Fig. 1.2: A hybrid model of TCP operation in congestion avoidance mode.

EXAMPLE 1.3. **Biology :** *Genetic regulatory networks [9, 43, 42] consist of a family of interacting genes, each of which produces a specific protein through a process known as gene expression. These proteins can then influence the rates at which the genes in the network are expressed. In fact, the rate of expression of a given gene typically depends on the concentration of a regulatory protein in a switch-like manner, with the rate changing abruptly if the protein's concentration crosses some threshold value. If we consider a network of* N *genes and denote the concentration of the corresponding proteins by* $x_1, \ldots, x_N$*, then the threshold values for the proteins in the network naturally define a partition of the state space into a number of regulatory sub-regions. Within each sub-region the network evolves continuously according to a system of differential equations. However, the system dynamics change abruptly whenever the concentration of some protein crosses a threshold, giving rise to switched or hybrid dynamics. Note that in this case, the switching rule is state-dependent with a mode-switch triggered by the crossing of some threshold value for a regulatory protein.*

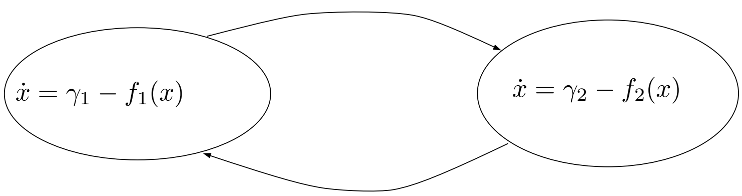

Fig. 1.3: Hybrid dynamics in genetic regulatory networks

It should be clear from the above examples that Hybrid Systems provide a convenient method for modelling a wide variety of complex dynamical systems. Unfortunately, while the modelling paradigm itself is quite straightforward, the analysis of even relatively simple hybrid dynamical systems remains a highly non-trivial task. The basic difficulty in their analysis is that even simple hybrid systems may exhibit extremely complicated non-linear behaviour. A simple intuitive example that illustrates one type of behaviour that emerges as a result of switching is the 'Car in the desert' example from [155]. In this example the driver of a car in the desert is given the task of returning to the oasis, as depicted in Figure 1.4. If the driver follows either trajectory 'a' or 'b' then the car will arrive safely at the oasis. However, if the car continually switches between these 'stable' trajectories, then instead of moving towards the oasis, the car will follow trajectory 'c' away from the oasis.[1]

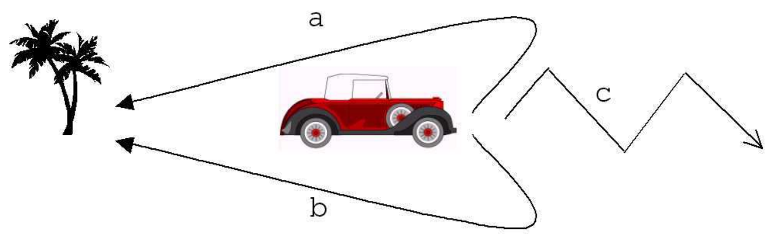

Fig. 1.4: 'The car in the desert'

The above discussion should hopefully indicate that while switched and hybrid systems provide an attractive paradigm for modelling a variety of practical situations, the analysis of such systems is far from straightforward. In fact, the study of switched systems has raised a number of challenging mathematical problems, that remain to a large extent, unanswered. Many of these problems are related to stability issues that arise in Hybrid dynamical systems and give rise to a number of basic questions, some of which we now list.

(i) **Arbitrary switching:** *Is it possible to determine verifiable conditions on a family of constituent systems that guarantee the stability of the associated switched system under arbitrary switching laws? Much of the work on this problem has been focussed on the question of common Lyapunov function existence.*

(ii) **Dwell time:** *If we switch between a family of individually stable systems sufficiently slowly then the overall system will be stable also [94, 117]. This raises the question of determining how fast we may switch while still guaranteeing stability. In other words what is the minimum length of time that must elapse between successive switches to ensure that the system remains stable? This problem is usually referred to as the dwell time problem.*

(iii) **Stabilisation:** *While switching can introduce instability when switching between stable systems, on the other hand, it is sometimes possible to stabilize*

[1]Even though it is beyond the scope of the work presented in this paper, it is worth mentioning that more complex behaviour may also be found in systems of dimension three or greater; see the excellent paper by Chase *et al.* [37] for a discussion of chaotic behaviour arising as a result of switching between several three stable LTI systems.

a family of individually unstable systems by switching between them appropriately. Based on this observation, several authors have worked on the problem of determining such stabilizing switching laws [52, 173].

(iv) **Chaos:** *Even though it is beyond the scope of the present paper, Chase, Serrano and Ramadge presented an example in [37] to illustrate how chaotic behaviour car arise when switching between low dimensional linear vector fields. This raises the question as to whether it is possible to determine if a switched system can exhibit chaotic behaviour for a given set of constituent vector fields?*

(v) **Complexity:** *Other problems that have been considered include questions relating to the complexity and decidability of determining the stability of switched systems [24, 22], and the precise nature of the connection between stability under arbitrary switching and stability under periodic switching rules (periodic stability) [21, 178, 90, 39, 60].*

In view of the above basic questions, the objective of this article is to review the major progress that has been made on a number of these and related questions, over the past number of years. As part of this process we will attempt to outline the major outstanding issues that have yet to be resolved in the study of switched linear systems.

2. Definitions and mathematical preliminaries. Throughout this paper our primary concern shall be with the stability properties of the switched linear system

$$\Sigma_S : \dot{x}(t) = A(t)x(t) \quad A(t) \in \mathcal{A} = \{A_1, \ldots, A_m\}, \tag{2.1}$$

where $\mathcal{A}$ is a set of matrices in $\mathbb{R}^{n\times n}$, and $t \to A(t)$ is a piecewise constant[2] mapping from the non-negative real numbers, $\mathbb{R}_+$, into $\mathcal{A}$. For each such mapping, there is a corresponding piecewise constant function σ from $\mathbb{R}_+$ into $\{1, \ldots, m\}$ such that $A(t) = A_{\sigma(t)}$ for all $t \geq 0$. This mapping σ is known as the *switching signal* , and the points of discontinuity, $t_1, t_2, \ldots$, of $A(t)$ (or $\sigma(t)$) are known as the *switching instances*. We denote the set of switching signals by $\mathcal{S}$. A function $x : \mathbb{R}_+ \to \mathbb{R}^n$ is called a solution of (2.1) if it is absolutely continuous and if there is a switching signal σ such that

$$\dot{x}(t) = A_{\sigma(t)}x(t)\,,$$

for all t except at the switching instances of σ. By convention we only consider right continuous switching signals. This does not affect the set of solutions, as going over to left continuous switching signals only changes the differential equation on a set of measure zero.

For $1 \leq i \leq m$, the i^{th} constituent system of of the switched linear system (2.1) is the linear time-invariant (LTI) system

$$\Sigma_{A_i} : \dot{x} = A_i x. \tag{2.2}$$

We can think of the system (2.1) as being constructed by switching between the constituent LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$, with mode switches occurring at the switching

[2]We recall that by definition piecewise constant maps only have with only finitely many discontinuities in any bounded time-interval.

instances, and the precise nature of the switching pattern being determined by the switching signal.

For a given switching signal $\sigma \in \mathcal{S}$, the switched linear system Σ_S evolves in the same way as an LTI system between any two successive switching instances. Thus for each switching signal σ, and initial condition $x(0)$, there exists a unique continuous, piecewise differentiable solution $x(t)$ which is given by

$$x(t) = [e^{A(t_k)(t-t_k)} e^{A(t_{k-1})(t_k - t_{k-1})} \dots e^{A(t_1)(t_2-t_1)} e^{A(0)(t_1)}]x(0),$$

where $t_1, t_2, \dots,$ is the sequence of switching instances and t_k is the largest switching instance smaller than t.

It should be noted that the stability theory of switched linear systems has close links with the corresponding theory for linear differential inclusions (LDIs) [5, 164]. The linear differential inclusion related to the set $\mathcal{A} = \{A_1, \dots, A_m\}$ is denoted by

$$\dot{x}(t) \in \{Ax(t) \mid A \in \mathcal{A}\} . \tag{2.3}$$

A solution of this inclusion is an absolutely continuous function x satisfying $\dot{x}(t) \in \{Ax(t) \mid A \in \mathcal{A}\}$ almost everywhere. By an application of Filippov's theorem this is equivalent to saying that there exists a measurable map $A : \mathbb{R}_+ \to \mathcal{A}$ such that

$$\dot{x}(t) = A(t)x(t) , \quad \text{almost everywhere} ,$$

see [53] for details. So studying the differential inclusion (2.3) amounts to extending the set of switching signals to the class of measurable functions. If we are studying the system for arbitrary switching sequences, the effect of this is often negligible. In fact, if we consider the convex hull of $\mathcal{A}$, denoted by $\operatorname{conv} \mathcal{A}$ and the convexified differential inclusion

$$\dot{x}(t) \in \{Ax(t) \mid A \in \operatorname{conv} \mathcal{A}\} , \tag{2.4}$$

then the solution sets of the three systems we have now defined are closely related. To make this statement precise, we denote by $\mathcal{R}_t^{\text{switch}}(x)$ the set of points that can be reached from an initial condition x at time t by solutions of (2.1), i.e.

$$\mathcal{R}_t^{\text{switch}}(x) := \{y \in \mathbb{R}^{n \times n} \mid \exists \text{ switching signal } \sigma \text{ such that } y = x(t; x, \sigma)\} .$$

Similarly, we introduce the notation $\mathcal{R}_t^{\text{ldi}}(x), \mathcal{R}_t^{\text{conv ldi}}(x)$ for reachable sets of (2.3) and (2.4), respectively, then we have for all $t \geq 0, x \in \mathbb{R}^n$ that

$$\mathcal{R}_t^{\text{switch}}(x) \subset \mathcal{R}_t^{\text{ldi}}(x) \subset \mathcal{R}_t^{\text{conv ldi}}(x) = \operatorname{cl} \mathcal{R}_{\rm t}^{\rm switch}(\rm x) , \tag{2.5}$$

see e.g. [5, 53]. For an in-depth investigation of the structure of the signal set and its interplay with the dynamics of (2.3), we refer to [38]. We also note that systems equivalent to (2.3) are often studied under the name of *linear parameter-varying systems*. We will make brief reference of the relation between this literature and switched systems where appropriate in the sequel.

2.1. Discrete-time systems. Thus far we have only considered switched linear systems in continuous-time. However, as in the example of TCP congestion control discussed in the last section, it is also of interest to study discrete-time switched linear systems. In discrete-time, a switched linear system is a system of the form

$$\Sigma_S : x(k+1) = A(k)x(k) \quad A(k) \in \mathcal{A} = \{A_1, \dots, A_m\}, \tag{2.6}$$

where as before $\mathcal{A}$ is a set of matrices in $\mathbb{R}^{n\times n}$ and $k \to A(k)$ is a mapping from the non-negative integers into $\mathcal{A}$. The notions of switching signal, switching instances and constituent systems are defined analogously to the continuous-time case. The existence of solutions to (2.6) is straightforward. In the discrete time case, the analysis of (2.6) is equivalent to that of the discrete linear inclusion

$$x(t+1) \in \{Ax(t) \mid A \in \mathcal{A}\}\,. \tag{2.7}$$

On the other hand the solution set is significantly enlarged when going over to the convexified inclusion

$$x(t+1) \in \{Ax(t) \mid A \in \operatorname{conv}\mathcal{A}\}\,. \tag{2.8}$$

It is of interest to note, that the exponential stability of (2.6), (2.7) and (2.8) is equivalent nonetheless, as we shall discuss in the sequel.

2.2. Exponential Growth Rates. One of the basic properties of switched linear systems is that a growth rate may be defined as in the case of linear time-invariant systems. The definition proceeds similarly in continuous and discrete time. There are several approaches to defining the exponential growth rate, all of which turn out to be equivalent. A trajectory based definition considers Lyapunov exponents of individual trajectories, which are defined by

$$\lambda(x_0,\sigma) := \limsup_{t\to\infty} \frac{1}{t}\log\|x(t;x_0,\sigma)\|\,.$$

The exponential growth rate of the switched system is then defined by the maximal Lyapunov exponent

$$\kappa(\mathcal{A}) := \max\{\lambda(x_0,\sigma) \mid x_0 \neq 0, \sigma\in\mathcal{S}\}\,.$$

A different point of view is to consider the evolution operators corresponding to the system equations (2.1), resp. (2.6). In continuous time these are defined as the solution of

$$\dot{\Phi}_\sigma(t,s) = A_{\sigma(t)}\Phi_\sigma(t,s)\,, \quad \Phi_\sigma(s,s) = I\,.$$

Similarly, in discrete time we have for $t \geq s$

$$\Phi(t+1,s) = A_{\sigma(t)}\Phi_\sigma(t,s)\,, \quad \Phi_\sigma(s,s) = I\,.$$

The growth of the system can then also be measured by considering the maximal growth of the norms or the spectral radii of the operators $\Phi(t,0)$ as $t\to\infty$. Ultimately, these definitions coincide. More precisely, it is known that

$$\kappa(\mathcal{A}) = \limsup_{t\to\infty}\frac{1}{t}\log\max_{\sigma\in\mathcal{S}} r(\Phi_\sigma(t,0)) = \limsup_{t\to\infty}\frac{1}{t}\log\max_{\sigma\in\mathcal{S}}\|\Phi_\sigma(t,0)\|\,. \tag{2.9}$$

The previous equality has been obtained by two different avenues. In discrete-time this was first shown by Berger and Wang [14] and alternative ways of proving the result have been presented by Elsner [49] and Shih, Wu and Pang [147]. On the other hand the result has also been implicit in the Russian literature. Namely, Pyatnitskii and Rapoport [139] show that if system (2.1) has an unbounded trajectory, then there exists a *periodic* switching signal $\sigma_j \in \mathcal{S}$ of period T such that $r(\Phi_{\sigma_j}(T,0)) = 1$, i.e. if we can find a *periodic* switching signal for which the system is marginally stable. This implies in particular, that absolute stability is equivalent to

$$\bar{\kappa} := \limsup_{t\to\infty} \frac{1}{t} \log \max_{\sigma\in\mathcal{S}} r(\Phi_\sigma(t,0)) < 0\,.$$

On the other hand Barabanov [11] shows that absolute stability is equivalent to

$$\hat{\kappa} := \limsup_{t\to\infty} \frac{1}{t} \log \max_{\sigma\in\mathcal{S}} \|\Phi_\sigma(t,0)\|\,.$$

as both $\bar{\kappa}, \hat{\kappa}$ are additive with respect to spectral shifts, i.e. $\bar{\kappa}(\mathcal{A}-\alpha I) = \bar{\kappa}(\mathcal{A}) - \alpha$, for all $\alpha \in \mathbb{R}$, they have to satisfy $\bar{\kappa} = \hat{\kappa}$.

In the area of discrete-time systems the growth rate of discrete inclusions is often defined as $\rho(\mathcal{A}) := e^{\kappa(A)}$. This quantity has become notorious under the name of joint spectral radius or generalized spectral radius.

There are numerous approaches to the computation of growth rates, either in their guise as maximal Lyapunov exponents or as joint spectral radii. We cannot cover these methods here but refer the reader to the various methods presented in [56, 101, 12, 57].

3. Stability for switched linear systems. As with general linear and nonlinear systems, numerous different concepts of stability have been defined for switched linear systems, including uniform stability, uniform attractivity, uniform asymptotic stability and uniform exponential stability. We now recall the definitions of uniform stability and uniform exponential stability.

Definition 3.1. *The origin is a uniformly stable equilibrium point of* Σ_S *if given any* $\epsilon > 0$*, there is some* $\delta > 0$ *such that* $\|x(0)\| < \delta$ *implies* $\|x(t)\| < \epsilon$ *for* $t \geq 0$ *for all solutions* $x(t)$ *of the system.*

Definition 3.2. *The origin is a uniformly exponentially stable equilibrium of* Σ_S *if there exist real constants* $M \geq 1$*,* $\beta > 0$ *such that*

$$\|x(t)\| \leq Me^{-\beta t}\|x(0)\|, \tag{3.1}$$

for $t \geq 0$*, for all solutions* $x(t)$ *of* Σ_S*.*

Uniform exponential stability is often called *absolute stability* especially in the Russian literature. It is known that the related concepts of attractivity and asymptotic stability together are equivalent to exponential stability for switched linear systems [41, 11, 40].

In a slight abuse of notation we shall often speak of the stability or exponential stability of the system Σ_S itself. One of the major topics discussed here is the problem of establishing when the system (2.1) is exponentially stable for arbitrary switching signals. In this case, Definition 3.2 requires the existence of constants $M \geq 1$, $\beta > 0$ such that (3.1) is satisfied for every piecewise continuous switching signal $\sigma(t)$. When considering the question of exponential stability under arbitrary switching, it is necessary to assume that the matrices $A_1, \ldots, A_m$ in the set $\mathcal{A}$ are all *Hurwitz* (all of their eigenvalues lie in the open left half of the complex plane), thus ensuring that each of the constituent LTI systems is exponentially stable.

In certain situations, it is not necessary to guarantee stability for every possible switching signal and a number of authors have considered questions related to the stability of switched linear systems under restricted switching regimes. One important example of this kind is *state-dependent* switching, where the rule that determines when a switch in system dynamics occurs is determined by the value of the state-vector x. The example of a genetic regulatory network discussed in the previous section was of this type. Other results on stability for restricted switching signals consider the question of determining restricted classes of switching signals for which a switched linear system is guaranteed to be stable.

In all of the above definitions, we are concerned with the stability properties of the internal system (2.1). In practice however, it is often necessary to consider systems with inputs and outputs of the form

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x. \end{aligned} \tag{3.2}$$

In this context the notion of bounded-input bounded-output (BIBO) stability arises. Formally, the input-output system (3.2) is uniformly BIBO stable if there exists a positive constant η such that for any essentially bounded input signal u, the zero-state response y satisfies

$$\sup_{t \geq t_0} \|y(t)\| \leq \eta \sup_{t \geq t_0} \|u(t).\|$$

Essentially, if a system is BIBO stable, this means that an input signal cannot be amplified by a factor greater than some finite constant η in passing through the system. While we shall not consider BIBO stability explicitly here, it should be noted that if the system (2.1) is uniformly exponentially stable, then the corresponding input-output system (3.2) is BIBO stable provided the matrices $B(t), C(t)$ are uniformly bounded in time, [111], which is the case when they switch between a finite family of matrices.

4. Arbitrary switching. The arbitrary switching problem is concerned with obtaining verifiable conditions on the matrices in $\mathcal{A}$ that guarantee the exponential stability of the switched system (2.1) for any switching signal. This problem has been the subject of interest from the research community in recent years and a number of general approaches to it have been investigated. Many of these rely on the construction of common quadratic and non-quadratic Lyapunov functions for the constituent systems of (2.1). In this context, it has been established that the existence of a common Lyapunov function is necessary and sufficient for the exponential stability of a switched linear system. In particular, a number of authors have derived *converse theorems* that prove the existence of common Lyapunov functions under the assumption of exponential stability. We begin by describing some of these theorems and then proceed by reviewing results on the common Lyapunov function existence problem for switched linear systems. Many of the mature results in this area concern the existence of common quadratic Lyapunov functions (CQLFs) and this part of our review reflects this fact. Nevertheless, some results are also presented concerning the existence of common non-quadratic Lyapunov functions.

4.1. Converse theorems. Lyapunov theory played a key role in the stability analysis of both linear and non-linear systems for much of the last century [100, 133, 119, 80]. The key idea of this approach is that the stability of a dynamical system can be established through demonstrating the existence of a positive definite, norm-like, function that decreases along all trajectories of the system as time evolves. Much of the recent research on the stability of switched linear systems has been directed towards applying similar ideas to the class of systems (2.1); relating the stability of such systems to the existence of positive definite functions, $V(x)$, on $\mathbb{R}^n$ such that $V(x(t))$ is a decreasing function of t for all solutions $x(t)$ of (2.1). Before discussing results that have been derived for specific forms of Lyapunov functions, we first present a number of more general facts about Lyapunov theory as it relates to the stability of switched linear systems.

First of all, note that if a positive definite function $V(x(t))$ decreases along all trajectories of the system (2.1) for arbitrary switching signals, then this certainly must be true for constant switching signals. Hence, any such function $V(x)$ would have to be a *common* Lyapunov function for each of the constituent LTI systems of (2.1). It is well established [115, 94, 40] that if a common Lyapunov function exists for the constituent systems of a switched linear system, then the system is uniformly exponentially stable for arbitrary switching signals. We shall now discuss the work of a variety of authors who have considered the problem of deriving *converse theorems* to establish the necessity of common Lyapunov function existence for uniform exponential stability under arbitrary switching.

In [115] Molchanov and Pyatnitskiy established that the uniform exponential stability of the system (2.1) under arbitrary switching is equivalent to the existence of a common Lyapunov function $V(x)$ for its constituent LTI systems. Formally they derived the following result.

THEOREM 4.1. *The system (2.1) is uniformly exponentially stable for arbitrary switching signals if and only if there exists a strictly convex, positive definite function $V(x)$, homogeneous of degree 2, of the form*

$$\begin{aligned} V(x) &= x^T \mathcal{L}(x) x \qquad \text{where } \mathcal{L}(x) \in \mathbb{R}^{n \times n} \text{ and} \\ \mathcal{L}(x)^T &= \mathcal{L}(x) = \mathcal{L}(cx) \qquad \text{for all non-zero } c \in \mathbb{R}, x \in \mathbb{R}^n, \end{aligned}$$

such that

$$\max_{y \in \mathcal{A}x} \frac{\partial V(x)}{\partial y} \leq -\gamma \|x\|^2$$

for some $\gamma > 0$, where $\mathcal{A}x = \{A_1 x, \ldots, A_m x\}$ and

$$\frac{\partial V(x)}{\partial y} = \inf_{t>0} \frac{V(x+ty) - V(x)}{t}$$

is the usual directional derivative of the convex function $V(x)$ [143].

A number of points about the results presented in [115] are worth noting.

(i) The converse theorem in [115] was derived for linear differential inclusions. However, combining (2.5) with the original results of [115] shows that common Lyapunov function existence is also necessary for the uniform exponential stability of the switched linear system (2.1) under arbitrary switching.

(ii) The common Lyapunov function whose existence is established in [115] is strictly convex and homogeneous of degree two, but not necessarily continuously differentiable.
(iii) Furthermore, Molchanov and Pyatnitskiy have shown that if a switched linear system is uniformly exponentially stable under arbitrary switching, then its constituent LTI systems will have both a piecewise quadratic and piecewise linear common Lyapunov function.

A closely related result was later derived by Dayawansa and Martin in [40], where it was established that the uniform exponential stability of the switched linear system (2.1) is equivalent to the existence of a continuously differentiable common Lyapunov function for its constituent LTI systems. This has already been noted by Brockett in [34]. Moreover, it was also shown that while this Lyapunov function can be chosen to be C^1 and homogeneous of degree two, it cannot always be chosen to be a quadratic form.

Thus, it is not in general necessary for the stability of (2.1) that there exists a common *quadratic* Lyapunov function for its constituent systems.

In the context of converse Lyapunov theorems, the work of Brayton and Tong, described in [31], is also worthy of mention. These authors established that the existence of a common Lyapunov function for the constituent systems of a discrete-time switched linear system is equivalent to the uniform stability of the system under arbitrary switching. Independently, Barabanov [10] showed that for an exponentially stable linear discrete inclusion there is always a norm that is a Lyapunov function. In particular, this implies by the convexity of norms, that if the set $\mathcal{A}$ generates an exponentially stable discrete linear inclusion, then so does $\operatorname{conv}\mathcal{A}$.

More recently, building on the work of Lin, Sontag and Wang in [95] on non-linear systems subject to disturbances, Mancilla-Aguilar and Garcia have derived converse Lyapunov theorems for non-linear switched systems [102], as well as some related results for input to state stability, a notion that we do not discuss here [103]. Finally, we note the recent result of Mason that shows that while a common Lyapunov function always exists for systems that are exponentially stable under arbitrary switching, its level curves may in fact be arbitrarily complex [108]. Thus searching for such a function using numerical techniques is not easy.

4.2. The CQLF existence problem. Quadratic Lyapunov functions play a central role in the study of linear time-invariant systems. Their existence is well understood in this context and consequently, studying the existence of such functions is a natural starting point in the study of switched linear systems. At the heart of the CQLF existence problem is the desire to find useful criteria to determine whether a given collection of Hurwitz matrices $\{A_1, \ldots, A_m\}$ has a CQLF. The main purpose of this Section is to survey the known results in this area, and indicate the different lines of attack that have been used. Despite the considerable work done so far, there are still some open questions that remain to be resolved.

4.2.1. Definitions. Recall that $V(x) = x^T Px$ is a quadratic Lyapunov function (QLF) for the LTI system $\Sigma_A : \dot{x} = Ax$ if (i) P is symmetric and positive definite, and (ii) $PA+A^T P$ is negative definite. Now suppose that $\{A_1, \ldots, A_m\}$ is a collection of $n \times n$ Hurwitz matrices, with associated stable LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$. Then the function $V(x) = x^T Px$ is a common quadratic Lyapunov function (CQLF) for these systems if $V(x)$ is a QLF for each individual system. Given a set of matrices $\{A_1, \ldots, A_m\}$, the CQLF existence problem is to determine whether such a matrix P exists. A secondary question is to construct a CQLF when one is known to exist. It is a standard fact that an LTI system Σ_A has a QLF if and only if the matrix A is Hurwitz. This property is also equivalent to the exponential stability of the system Σ_A, so for a single LTI system there is no gap between the existence of a QLF and exponential stability. Therefore a simple spectral condition determines completely the stability of the LTI system Σ_A.

For a collection of Hurwitz matrices the situation is more complicated in several respects. Firstly, in general, CQLF existence is only a sufficient condition for the exponential stability of a switched linear system under arbitrary switching. Secondly, no correspondingly simple condition is known which can determine the existence of a CQLF for a family of LTI systems, although progress has been made in some special cases. The rest of this section will describe a variety of approaches which have been used to attack this problem, and outline some of the open problems.

In many cases it is useful to analyse the mapping $P \mapsto PA + A^T P$ as a linear function on the space of matrices. To set up the notation, let $S^{n\times n}$ denote the linear vector space of real symmetric $n \times n$ matrices. A matrix $P \in S^{n\times n}$ is positive definite (written $P > 0$) if $x^T P x > 0$ for all $x \neq 0 \in \mathbb{R}^n$, and P is positive semidefinite ($P \geq 0$) if $x^T P x \geq 0$ for all $x \in \mathbb{R}^n$. Similarly P is negative definite if $-P > 0$, or $P < 0$, and P is negative semidefinite if $-P \geq 0$, or $P \leq 0$. Finally recall that P is positive definite if and only if $\mathrm{Tr}\, PQ > 0$ for all positive definite Q, where Tr is the usual matrix trace.

The Lyapunov map defined by the real $n \times n$ matrix A is

$$\mathcal{L}_A : S^{n\times n} \to S^{n\times n}, \qquad \mathcal{L}_A(H) = HA + A^T H \tag{4.1}$$

The following properties of $\mathcal{L}_A$ are well-known, [73]. (i) If A has eigenvalues $\{\lambda_i\}$ with associated eigenvectors $\{v_i\}$, then $\mathcal{L}_A$ has eigenvalues $\{\lambda_i + \lambda_j\}$, with eigenvectors $\{v_i v_j^T + v_j v_i^T\}$, for all $i \leq j$. It follows immediately that $\mathcal{L}_A$ is invertible whenever A is Hurwitz, since in this case $\lambda_i + \lambda_j$ cannot be zero. (ii) A is Hurwitz if and only if there exists $P > 0$ such that $\mathcal{L}_A(P) < 0$. Note that in this case $x^T P x$ is a QLF for the system Σ_A.

Now define $\mathcal{P}_A$ to be the collection of all positive definite matrices which provide QLF's for the system Σ_A, that is

$$\mathcal{P}_A = \{P > 0 \,:\, \mathcal{L}_A(P) < 0\} \tag{4.2}$$

Clearly $\mathcal{P}_A$ is an open convex cone in $S^{n\times n}$. The above results concerning the Lyapunov map show that $\mathcal{P}_A$ is nonempty if and only if A is Hurwitz. In this language, the CQLF existence problem for a collection of matrices $\{A_1, \ldots, A_k\}$ is the problem of determining whether the intersection of the cones $\mathcal{P}_{A_1} \cap \cdots \cap \mathcal{P}_{A_k}$ is non-empty.

There are some straightforward observations that can be made at this point. Firstly, for $A \in \mathbb{R}^{n\times n}$, the cones $\mathcal{P}_A$ and $\mathcal{P}_{A^{-1}}$ are identical. Thus, there exists a CQLF for the systems $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$ if and only if there is a CQLF for the systems $\Sigma_{A_1^{\epsilon_1}}, \ldots, \Sigma_{A_m^{\epsilon_m}}$ where $\epsilon_i = \pm 1$ for $i = 1, \ldots, m$. Secondly, CQLF existence is invariant under a change of coordinates. That is, if $R \in \mathbb{R}^{n\times n}$ is non-singular, then

$$\mathcal{P}_{R^{-1}AR} = R^T \; \mathcal{P}_A \; R \equiv \{R^T P R \,:\, P \in \mathcal{P}_A\} \tag{4.3}$$

Therefore CQLF existence for the family of systems $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$ is equivalent to CQLF existence for the transformed family $\Sigma_{R^{-1}A_1R}, \ldots, \Sigma_{R^{-1}A_mR}$.

4.2.2. Dual formulation. The QLF and CQLF existence problems have dual formulations which will play an important role in some of our later discussions. To set up the notation, define $\widehat{\mathcal{L}}_A$ to be the adjoint of the Lyapunov map with respect to the standard inner product $\langle X, Y \rangle = \operatorname{Tr} X^T Y$ on $S^{n\times n}$, that is

$$\langle X, \mathcal{L}_A(Y) \rangle = \langle \widehat{\mathcal{L}}_A(X), Y \rangle \tag{4.4}$$

for all $X, Y \in S^{n\times n}$. It follows that

$$\widehat{\mathcal{L}}_A : S^{n\times n} \to S^{n\times n}, \qquad \widehat{\mathcal{L}}_A(H) = AH + HA^T = \mathcal{L}_{A^T}(H) \tag{4.5}$$

We will use the following formulation of duality, which can be found for example in [81]: given a collection of Hurwitz matrices $\{A_1, \ldots, A_k\}$, there exists a CQLF if and only if there do not exist positive semidefinite matrices $X_1, \ldots, X_k$ (not all zero) satisfying $\sum_{i=1}^k \widehat{\mathcal{L}}_{A_i}(X_i) = 0$. That is,

$$\begin{aligned} &\exists\ P > 0 \ \text{ such that } \ \mathcal{L}_{A_i}(P) < 0 \text{ for all } i = 1, \ldots, k \\ \iff &\nexists\ X_1, \ldots, X_k \geq 0 \ \text{ (not all zero) such that } \sum_{i=1}^{k} \widehat{\mathcal{L}}_{A_i}(X_i) = 0 \end{aligned} \tag{4.6}$$

4.3. Numerical approaches to the CQLF problem. While we shall concentrate here on theoretical results obtained on the CQLF existence problem, it should be noted that numerical methods are also available for testing for CQLF existence. Recent advances in computational technology along with the development of efficient numerical algorithms for solving problems in the field of convex optimization have resulted in the widespread use of linear matrix inequality (LMI) techniques throughout systems theory. For details on the various applications of LMI methods in systems and control consult [28, 48, 55]. In this section, we focus on one specific aspect of this development; the use of LMI methods to test for the existence of a CQLF for a number of stable LTI systems.

The conditions for $V(x) = x^T P x$ to be a CQLF for the asymptotically stable LTI systems Σ_{A_i}, $i \in \{1, \ldots, m\}$ define a system of linear matrix inequalities (LMIs) in P, namely

$$P = P^T > 0, \ (A_i^T P + P A_i) < 0 \text{ for } i \in \{1, \ldots, m\} \tag{4.7}$$

The system of LMIs (4.7) is said to be *feasible* if a solution P exists; otherwise the LMIs (4.7) are *infeasible*. Thus, determining whether or not the LTI systems Σ_{A_i}, $i \in \{1, \ldots, m\}$ possess a CQLF amounts to checking the feasibility of a system of LMIs. LMIs are built on convex optimization algorithms developed over the past two decades which are capable of solving this type of problem with considerably more speed than was possible using previous techniques. Conversely, it is also possible to verify that no CQLF exists for the LTI systems Σ_{A_i} via the use of LMI techniques. More specifically [28], there is no CQLF for the LTI systems Σ_{A_i} if there exist matrices $R_i = R_i^T$, $i \in \{1, \ldots, m\}$ satisfying

$$R_i > 0, \qquad \Sigma_{i=1}^m (A_i^T R_i + R_i A_i) > 0 \tag{4.8}$$

While LMIs provide an effective way of verifying that a CQLF exists for a family of LTI systems, there are also a number of drawbacks to the method which should be mentioned.

(i) Firstly, situations have arisen where known analytic results can be used to show that a CQLF definitely exists (or does not exist), but the (commonly used) LMI toolbox for MATLAB fails to give a definitive answer to the existence question. For instance, it is well known that any set of systems Σ_{A_i} where the system matrices A_i are all upper triangular and Hurwitz must possess a CQLF (see section 4.1 on triangular systems). However it is possible to construct a set of two 2×2 Hurwitz triangular matrices where the LMI toolbox will be unable to find a CQLF. For more detailed examples of this type, consult [107].

(ii) Secondly, because of the numerical nature of the approach, it provides little insight into why a CQLF may or may not exist for a set of LTI systems, and does not add to our understanding of the precise relationship between CQLF existence and the dynamics of switched linear systems. A particular problem that arises in this context, and for which numerical approaches are unsuitable, is that of determining specific classes of systems for which CQLF existence is equivalent to exponential stability under arbitrary switching. Such system classes do exist and we shall mention some later in this section.

(iii) The question of weak CQLF existence, where the strict inequalities in the system (4.7) are replaced with non-strict inequalities, is not amenable to solution by LMI methods. Here a more theoretical approach is required.

4.4. Special structures of matrices that guarantee existence of a CQLF. Some special cases are known where the structure of the matrices $\{A_1, \dots, A_m\}$ by itself guarantees the existence of a CQLF for the associated LTI systems, provided of course that the matrices are Hurwitz. We now review these cases.

4.4.1. Matrices with Lyapunov function $x^T x$. The condition that a system Σ_A have the Lyapunov function $x^T x$ is

$$\mathcal{L}_A(I) = A^T + A < 0 \tag{4.9}$$

where I is the $n \times n$ identity matrix. If $\{A_1, \dots, A_m\}$ is a collection of matrices which all satisfy the condition (4.9), then $x^T x$ must be a QLF for every individual system, and hence must be a CQLF for the collection. The condition (4.9) is satisfied in the following cases:

(i) A is symmetric and Hurwitz,
(ii) A is normal (i.e. $AA^T = A^T A$) and Hurwitz,
(iii) if a matrix S is skew-symmetric, (i.e. if $S^T = -S$) then if A satisfies (4.9), then so does $A + S$.

4.4.2. Triangular and related systems. If the Hurwitz matrices $\{A_i\}$ are all in upper triangular form, then it was shown by Shorten and Narendra [155], and independently by Mori, Mori and Kuroe in [116], that the collection of systems Σ_{A_i} always has a CQLF, and furthermore that the matrix P which defines the CQLF can be chosen to be diagonal. This result extends to the case where there is a non-singular matrix R for which the matrices $\{R^{-1} A_i R\}$ are all upper triangular, by the remarks in Section 4.2.1.

One interesting application of this result arises when the matrices $A_1, \dots, A_m$ all commute with each other. In this case there is a unitary matrix U such that $U^* A_i U$

is in upper triangular form for each $i = 1, \ldots, m$ [73], and it then follows that the systems $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$ have a CQLF [118]. This result has an interesting extension to a class of systems with non-commuting matrices [60]. To explain this class, let $\mathfrak{g} = \{A_1, \ldots, A_m\}_{LA}$ denote the Lie algebra generated by the matrices $\{A_1, \ldots, A_m\}$, that is the collection of all matrices of the form $\{A_i, [A_i, A_j], [A_i, [A_j, A_k]], \ldots\}$ and so on. If $\mathfrak{g}$ is solvable, then it follows from a well-known theorem of Lie [64] that the matrices $\{A_1, \ldots, A_m\}$ can be put into upper triangular form by a nonsingular transformation. Recall that a Lie algebra is said to be *solvable* if $\mathfrak{g}^k = \{0\}$ for some finite k, where the sequence of Lie algebras $\mathfrak{g}^0, \mathfrak{g}^1, \mathfrak{g}^2, \ldots$ are defined recursively by $\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k]$, and $\mathfrak{g}^0 = \mathfrak{g}$. The basic example of a solvable Lie algebra is the Lie algebra generated by a set of upper triangular matrices, where it can be seen that the nonzero entries of $\mathfrak{g}^k$ retreat further from the main diagonal at each step.

Using this result, and the Shorten-Narendra result about upper triangular matrices, it follows that if $\mathfrak{g} = \{A_1, \ldots, A_m\}_{LA}$ is solvable, then the systems $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$ have a CQLF. The most general result along these lines is the following theorem due to Agrachev and Liberzon [2]. The theorem describes the type of Lie algebra which can be generated by a collection of Hurwitz matrices that share a CQLF. Their result also shows that if $\mathfrak{g}$ is not of this type, then it could be generated by a collection of matrices whose LTI systems are individually stable, but for which the corresponding switching system can be made unstable with some switching sequence. So the theorem describes the most general conclusions about the CQLF existence question which can be reached using only the Lie algebra structure generated by the collection $\{A_1, \ldots, A_m\}$.

THEOREM 4.2. *[2] Let $A_1, \ldots, A_m$ be Hurwitz matrices, and $\widehat{\mathfrak{g}} = \{I, A_1, \ldots, A_m\}_{LA}$ where I is the identity matrix. Let $\widehat{\mathfrak{g}} = \mathfrak{r} \oplus \mathfrak{s}$ be the Levi decomposition, where $\mathfrak{r}$ is the radical, and suppose that $\mathfrak{s}$ is a compact Lie algebra. Then the systems $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$ have a CQLF. Furthermore, if $\mathfrak{s}$ is not compact, then there is a set of Hurwitz matrices which generate $\widehat{\mathfrak{g}}$, such that the corresponding switched linear system is not uniformly exponentially stable.*

Given the body of literature that has been dedicated to, and continues to be dedicated to, triangular system, a few further comments are in order.

(i) In essence, triangular switching systems obey the same stability laws as LTI systems; namely the exponential stability of the system is equivalent to the condition that the eigenvalues of $A(t)$ lie in the open left-half of the complex plane for all t. This is not surprising; systems of this form can be represented as cascades comprised of first order sub-systems or very benign second order sub-systems. Each such sub-system has the quadratic Lyapunov function, $V(x) = x^T x$, and is thus exponentially stable. The exponential stability of the overall systems follows.

(ii) It is important to appreciate that the property of a family of matrices being simultaneously triangularisable is not robust, and that this requirement is only satisfied by a very limited class of systems.

(iii) From a practical viewpoint, the requirement of simultaneous triangularisability imposes unrealistic conditions on the matrices in the set $\mathcal{A}$. It is therefore of interest to extend the results derived by [116] with a view to relaxing this requirement. In this context several authors have recently published new conditions for exponential stability of the switching system. Typically, the approach adopted is to bound the maximum allowable perturbations of the matrix parameters from a nominal (triangularisable) set of matrices, thereby guaranteeing the existence of a CQLF; see [116]. An alternative approach is presented in [160, 161]; rather than assuming maximum allowable perturbations from nominal matrix parameters, it is explicitly assumed that no single non-singular transformation T exists that simultaneously triangularises all of the matrices in $\mathcal{A}$. Instead, the authors assume that a collection of non-singular matrices T_{ij} exists, such that for each pair of matrices $\{A_i, A_j\}$ in $\mathcal{A}$, the pair of matrices $\{T_{ij}A_iT_{ij}^{-1}, T_{ij}A_jT_{ij}^{-1}\}$ are upper triangular.

(iv) Several papers in the area of switching systems are related to the simultaneous triangularisation of a set of matrices. For example, matrices that commute are simultaneously triangularisable. Hence, the commuting vector field result of Narendra and Balakrishnan [118] is a special case of the above discussion [152][3].

(v) To apply the above results, it is necessary to be able to determine if a non-singular T exists such that for all A_i in a set of matrices $\mathcal{A}$, TA_iT^{-1} is upper triangular. McCoy's theorem provides useful insights in this context; see [89, 155], as does the work relating Lie algebras and simultaneous triangularisation by Liberzon *et al.* [93].

4.5. Necessary and sufficient conditions for special classes. One long-standing goal in the field of switched systems has been to find simple algebraic conditions for existence of a CQLF which can be checked explicitly just from knowledge of the matrices $\{A_1, \ldots, A_m\}$. In the discrete-time case it is known by the work of Kozyakin [86], that exponential stability is not a property that can be described by finitely many algebraic constraints in the set of pairs of 2×2 matrices. In this section we describe several cases where such conditions are known.

4.5.1. Two second order systems. For a pair of second order systems there is a complete solution to the CQLF existence problem. We quote the following result from [157].

THEOREM 4.3. *Let A_1 and A_2 be 2×2 Hurwitz matrices. Then the two LTI systems Σ_{A_1} and Σ_{A_2} have a CQLF if and only if the matrix products A_1A_2 and $A_1A_2^{-1}$ have no negative real eigenvalues.*

Theorem 4.3 provides an extremely simple and elegant solution to the CQLF problem for the case of two matrices in $\mathbb{R}^{2\times 2}$. It is known that CQLF existence is a conservative criterion for the stability of second order systems; however, the simplicity of Theorem 4.3 demonstrates the usefulness of using CQLF methods to analyse stability, and it provides insights into the precise relationship between CQLF existence and stability. In particular, using this result, it can be shown that if a CQLF fails to exist for a pair of LTI systems Σ_{A_1}, Σ_{A_2}, with $A_1, A_2 \in \mathbb{R}^{2\times 2}$ Hurwitz, then at least one of the related switched linear systems

$$\dot{x} = A(t)x \quad A(t) \in \{A_1, A_2\} \tag{4.10}$$

$$\dot{x} = A(t)x \quad A(t) \in \{A_1, A_2^{-1}\} \tag{4.11}$$

fails to be exponentially stable for arbitrary switching signals. Moreover, Theorem 4.3 has been used in [61, 105] to show that for second order positive switched linear

[3] It is worth noting that it has been shown by Shim *et. al.* [148] that a quadratic Lyapunov function exists for commuting vector fields even if the vector fields are themselves non-linear

systems [4] with two stable constituent systems, CQLF existence is in fact equivalent to exponential stability under arbitrary switching.

No simple spectral condition is known when there are more than two matrices in $\mathbb{R}^{2\times 2}$, although the following result provides some useful information in this case [157]. Suppose that Σ_{A_i} are stable LTI systems in the plane. If any subset of three of these systems has a CQLF, then there is a CQLF for the whole family. This can be viewed as a consequence of Helly's theorem from convex analysis [143] in combination with the discussion of intersection of the cones $\mathcal{P}_{A_j}$ in Section 4.2.1.

4.5.2. Two systems with a rank one difference. Suppose that $A \in \mathbb{R}^{n\times n}$ is Hurwitz, and $b, c \in \mathbb{R}^n$. If the pair (A, b) is completely controllable, then there is again a simple spectral condition which is equivalent to CQLF existence for the systems $\{\Sigma_A, \Sigma_{A-bc^T}\}$. This condition was originally derived as a frequency domain condition using the SISO Circle Criterion [119], however it was later realised [158] that the condition has the following natural and elegant formulation as a condition similar to Theorem 4.3. This result was then generalised to the case of a pair of Hurwitz matrices whose difference is rank 1 in [153].

THEOREM 4.4. *Let A_1 and A_2 be Hurwitz matrices in $\mathbb{R}^{n\times n}$, where the difference $A_1 - A_2$ has rank one. Then the two LTI systems Σ_{A_1} and Σ_{A_2} have a CQLF if and only if the matrix product $A_1 A_2$ has no negative real eigenvalues.*

Theorem 4.4 provides a simple spectral condition for CQLF existence for a pair of exponentially stable LTI systems whose system matrices differ by a rank one matrix. Further, it follows from this result that for a switched linear system

$$\dot{x} = A(t)x \quad A(t) \in \{A_1, A_2\}, \tag{4.12}$$

where $A_1, A_2 \in \mathbb{R}^{n\times n}$ are Hurwitz and $\text{rank}(A_2 - A_1^{-1}) = 1$, CQLF existence is *equivalent* to exponential stability under arbitrary switching signals.

4.6. Sufficiency. In addition to the results discussed above, several authors have developed tests for CQLF existence which provide sufficient conditions. In some cases these tests allow explicit computations, and therefore can be useful in practical applications.

4.6.1. Lyapunov operator conditions. In a series of papers [124, 122, 123, 125], Ooba and Funahashi derived conditions involving the Lyapunov operators $\mathcal{L}_A$ defined in (4.1). The key idea in their work is the observation that Σ_{A_1} and Σ_{A_2} have a CQLF if and only if there is some positive definite Q such that $\mathcal{L}_{A_1}\mathcal{L}_{A_2}^{-1}(Q)$ is also positive definite. This leads to their following result [122]. Recall that for an operator $\mathcal{L}$ on the space of symmetric matrices $S^{n\times n}$, $\widehat{\mathcal{L}}$ denotes the adjoint of $\mathcal{L}$ with respect to the usual inner product on $S^{n\times n}$.

THEOREM 4.5. *Let A_1 and A_2 be $n \times n$ Hurwitz matrices, and suppose that*

$$\widehat{\mathcal{L}}_{A_2-A_1}\mathcal{L}_{A_2-A_1} - (\widehat{\mathcal{L}}_{A_1}\mathcal{L}_{A_1} + \widehat{\mathcal{L}}_{A_2}\mathcal{L}_{A_2}) < 0 \tag{4.13}$$

Then Σ_{A_1} and Σ_{A_2} have a CQLF.

A second similar, but independent condition is presented in [123] involving the Lyapunov operators of the commutators of the matrices A_1 and A_2. In one of their

[4] A positive dynamical system is one where non-negative initial conditions imply that the state vector remains in the non-negative orthant for all time.

other papers [124], Ooba and Funahashi derive sufficient conditions which involve minimal eigenvalues computed using the Lyapunov operators. Given a collection of Hurwitz matrices $\{A_1, \ldots, A_m\}$ in $\mathbb{R}^{n\times n}$, define

$$\mu_{ij} = \lambda_{\min}\left(\mathcal{L}_{A_i}\mathcal{L}_{A_j}^{-1}(I)\right), \quad i,j = 1,\ldots,m \tag{4.14}$$

where I is the $n \times n$ identity matrix, and where $\lambda_{\min}$ is the smallest eigenvalue, and define the $m \times m$ matrix

$$M = (\mu_{ij})_{i,j=1,\ldots,m}. \tag{4.15}$$

Then we have the following result.

THEOREM 4.6. *Suppose that the matrix M defined in (4.15) is semipositive, meaning that there is a vector $x \in \mathbb{R}^n$ with $x_i \geq 0$ for all i, such that $(Mx)_i > 0$ for all i. Then the systems Σ_{A_i}, $1 \leq i \leq m$, have a CQLF.*

4.7. Necessary and sufficient conditions for the general case. In this section we review a new approach to deriving necessary and sufficient conditions for existence of a CQLF, based on the duality condition (4.6). This relation states that $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$ do NOT have a CQLF if and only if there are positive semidefinite matrices $X_1, \ldots, X_m$ (not all zero) which satisfy the equation

$$\sum_{i=1}^{m} A_i X_i + X_i A_i^T = 0 \tag{4.16}$$

The main idea is to rewrite (4.16) in the following form.

THEOREM 4.7. *Suppose that equation (4.16) holds, and let $d = \mathrm{rk}(X_1 + \cdots + X_m)$. Then there are positive semidefinite $d \times d$ matrices $Y_1, \ldots, Y_m$, with $rk(Y_i) = rk(X_i)$ for all i, and a skew-symmetric $d \times d$ matrix S such that*

$$\det\Big(\sum_{i=1}^{m} A_i \otimes Y_i + I \otimes S\Big) = 0 \tag{4.17}$$

where I is the $n \times n$ identity matrix. Conversely, if (4.17) holds for some positive semidefinite matrices $\{Y_i\}$ and skew symmetric matrix S, then the equation (4.16) holds, with $\{X_i\}$ positive semidefinite and not all zero, and $rk(X_i) \leq rk(Y_i)$ for all i.

The key to deriving (4.17) is to select a basis $v_1, \ldots, v_d$ for the range of $X_1 + \cdots + X_M$, and express the matrices X_i with respect to this basis as

$$X_i = \sum_{p,q=1}^{d} (Y_i)_{pq} v_p v_q^T \tag{4.18}$$

The $d \times d$ matrix Y_i is also positive semidefinite, and has the same rank as X_i.

Using (4.17), four necessary and sufficient conditions were derived for non-existence of a CQLF for a pair of 3×3 Hurwitz matrices [85]. Three of the conditions can be expressed as singularity conditions for some convex combinations of A_i and A_i^{-1}. For example one of the conditions says that some convex combination of A_1, A_2 and $(xA_1 + (1-x)A_2)^{-1}$ is singular for some $0 \leq x \leq 1$. Testing this condition involves searching over a three-parameter space, so it is quite infeasible. The main importance of the conditions lies in the possibility that they can lead to new insights into the CQLF problem.

4.8. Stability radii. Associated to the existence of Lyapunov functions or common quadratic Lyapunov functions is the property that exponential stability is a robust property of switched systems, that is, small perturbations of the systems data does not destroy stability. One is often interested in quantifying this robustness and this is the aim in the study of stability radii.

We assume we are given a nominal asymptotically stable system, which we take for the sake of simplicity to be time-invariant. It is thus of the form

$$\dot{x} = A_0 x\,. \tag{4.19}$$

Due to imprecise modelling it may be expected that the system of interest does not have the same dynamics as the nominal system, but can be interpreted as a particular system in the class

$$\dot{x}(t) = \left(A_0 + \sum_{k=1}^{m} \delta_k(t) A_k \right) x(t) \tag{4.20}$$

Here the matrices $A_k, k = 1, \ldots, m$ are prescribed, modelling the expected perturbations of the systems, while $\delta(t) = (\delta_1(t), \ldots, \delta_m(t)$ is an *unknown*, essentially bounded perturbation. The question is how large this perturbation may be without destroying stability. To measure this size we prescribe a norm $\|\cdot\|$ in $\mathbb{R}^m$, and denote by $\|\cdot\|_\infty$ the corresponding norm on bounded functions $\delta : \mathbb{R}_+ \to \mathbb{R}^m$.

The stability radius in a switched systems sense is then given by

$$r_{Ly}(A, (A_i)) = \inf\{\|\delta\|_\infty \mid (4.20) \text{ is not exponentially stable for } \delta\}\,. \tag{4.21}$$

Stability radii of this type are discussed in [38, 68]. In particular, the interested reader will find an in depth discussion of related literature in these references. In particular, the calculation of stability radii has been studied in [175] in the discrete time case and in [58] in continuous time. Of course, this is again a difficult problem, as already the determination of the growth rate is NP-hard. We note that if the set

$$\left\{ A_0 + \sum_{k=1}^{m} \delta_k A_k \mid \|\delta\| \le 1 \right\},$$

is a polytope, then we are in the case of the switched system (2.1) again, as by (2.5) the exponential stability of the inclusion (2.3) and (2.1) are equivalent.

We note that in the theory of stability radii there is an elegant interpretation of the CQLF problem. This applies to the special case, that perturbations are measured in the spectral norm $\|\cdot\|2$ and the perturbation structure is determined by structure matrices $B \in \mathbb{R}^{n\times l}, C \in \mathbb{R}^{q\times n}$. We are thus considering perturbed systems of the form

$$\dot{x}(t) = (A + B\Delta(t)C)\, x(t) \tag{4.22}$$

where $\Delta(t) \in \mathbb{R}^{l\times q}$ is an unknown perturbation. In this case three different stability radii may be defined corresponding to real constant, real time-varying, and complex constant perturbations may be defined. They are given by

$$\begin{aligned}
r_{\mathbb{R}}(A,B,C) &:= \inf\left\{\|\Delta_0\|_2 \mid \Delta_0 \in \mathbb{R}^{l\times q} : (4.22) \text{ is not exp. stable for } \Delta(t) \equiv \Delta_0\right\},\\
r_{Ly}(A,B,C) &:= \inf\left\{\|\Delta\|_\infty \mid \Delta : \mathbb{R} \to \mathbb{R}^{l\times q} : (4.22) \text{ is not exp. stable for } \Delta\right\},\\
r_{\mathbb{C}}(A,B,C) &:= \inf\left\{\|\Delta_0\|_2 \mid \Delta_0 \in \mathbb{C}^{l\times q} : (4.22) \text{ is not exp. stable for } \Delta(t) \equiv \Delta_0\right\}.
\end{aligned}$$

The relation between these stability radii is

$$r_{\mathbb{R}}(A,B,C) \leq r_{Ly}(A,B,C) \leq r_{\mathbb{C}}(A,B,C)\,. \qquad (4.23)$$

In particular, we have

THEOREM 4.8. *Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz and $B \in \mathbb{R}^{n\times l}, C \in \mathbb{R}^{q\times n}$. The following statements are equivalent:*

(i) $\rho < r_{\mathbb{C}}(A,B,C)$,

(ii) there exists a CQLF for the set of matrices

$$\{A + B\Delta C \mid \|\Delta\|_2 \leq \rho\}\,.$$

In view of (4.23) it is of course interesting to find conditions that guarantee $r_{\mathbb{R}}(A,B,C) = r_{\mathbb{C}}(A,B,C)$, because in this case the intrinsically difficult problem of calculating the stability radius of switched systems reduces to the calculation of the complex stability radius. For the latter problem there exist quadratically convergent algorithms if stability radii are constructed with respect to the spectral norm. One interesting case where this can be done concerns the area of positive systems. In fact, for this system class the problem turns out to be particularly simple.

A matrix $B \in \mathbb{R}^{n\times m}$ is called nonnegative, if all its entries are nonnegative numbers. We denote this property by $B \geq 0$. Recall, that a matrix $A \in \mathbb{R}^{n\times n}$ is called *Metzler*, if all offdiagonal entries are nonnegative numbers. Metzler matrices are precisely the matrices for which $e^{At} \geq 0$ for all $t \geq 0$. For this system class the following result has been shown in [54] for the more general case of positive systems in infinite dimensions.

THEOREM 4.9. *Let $A \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz. Assume $B \in \mathbb{R}^{n\times l}, C \in \mathbb{R}^{q\times n}$ are nonnegative. Then*

$$r_{\mathbb{R}}(A,B,C) = r_{Ly}(A,B,C) = r_{\mathbb{C}}(A,B,C) = \|CA^{-1}B\|_2^{-1}\,.$$

The previous result shows that determining the stability radius or stability margin of switched systems is quite easy for positive systems subjected to a particular perturbation structure.

4.9. Decidability and computability issues. At this point some readers may wonder whether Lyapunov theory is not overkill for analysing switched *linear* systems. After all, explicit solutions to any given differential equation of this form can be constructed by piecing together solutions of the constituent linear time-invariant systems, and the stability properties of such solutions can be determined, for any given switching sequence, can be easily deduced. We shall see that this comment is naive and that determining the properties of all such solutions is a computational impossibility.

In the discrete-time case, the computational complexity and decidability of problems regarding stability properties of linear inclusions have been actively investigated. The problem can be described as follows. Consider a finite set of matrices $\mathcal{A} = \{A_1, \ldots, A_m\} \subset \mathbb{R}^{n\times n}$ and the associated switched system

$$x(t+1) = A(t)x(t)\,, \quad A(t) \in \mathcal{A}\,, \quad t \in \mathbb{N}\,. \qquad (4.24)$$

One might be tempted to ask for a good algorithmic procedure for deciding, whether the set $\mathcal{M}$ defines an exponentially stable or stable system.

An easy answer would be possible, if the question can be decided by checking a finite number of algebraic inequalities, as one does in the Schur-Cohn test for single matrices. To formulate the problem we consider m-tuples of matrices $M = (A_1, \ldots, A_m) \in (\mathbb{R}^{n\times n})^m$ and we denote by $\Sigma(A_1, \ldots, A_m)$ the system (4.24) corresponding to the set $\mathcal{A}$ of distinct matrices in M.

Definition 4.10. *A set Y is called* semi-algebraic *in $\mathbb{R}^p$, if it can be represented as a finite union of sets, that are each described by a finite number of polynomial equalities and inequalities.*

The first negative result is then

Theorem 4.11 (Kozyakin, [86]). *Given $n, m \geq 2$, the sets*

$$\{(A_1, \ldots, A_m) \in (\mathbb{R}^{n\times n})^m \mid \Sigma(A_1, \ldots, A_m) \textit{ is exponentially stable.}\},$$

$$\{(A_1, \ldots, A_m) \in (\mathbb{R}^{n\times n})^m \mid \Sigma(A_1, \ldots, A_m) \textit{ is stable.}\}.$$

are not semialgebraic.

In fact, the proof of Kozyakin even shows that both sets are not subanalytic, a notion of real analytic geometry, that we cannot discuss here, see [87]. Summarizing, this shows, that there is no simple description of the set of stable systems in algebraic or even analytic terms, which suggests that the problem of deciding whether a given system is stable is a computationally difficult one, in general.

To investigate the problem further, recall that a computational problem is called of class P, if there exists an algorithm on a Turing machine that solves the problem in a time that depends in a polynomial manner on the amount of information needed to describe a particular instance of the problem. A problem is in class NP if a nondeterministic Turing machine can solve the problem in polynomial time. In particular, any problem in P is in NP. A problem is termed NP-hard, if by its solution any other problem in the class NP can be solved, so that it is at least as hard as any other NP problem. It is one of the fundamental open questions, whether $P = NP$, but assuming this is not the case, this means that for any NP hard problem there is no algorithm that computes the answer to this question in a time that is a polynomial function of the size of the data.

Theorem 4.12 (Tsitsiklis, Blondel, [167]). *Unless $P = NP$, there is no algorithm, that approximates the joint spectral radius ρ in polynomial time for all finite sets of matrices $\{A_1, \ldots, A_m\}$, $n, m \geq 2$.*

Definition 4.13. *A problem is called (algorithmically) undecidable, if there is no algorithm, that takes any set of data from a prescribed set and terminates after a finite number of computations and gives an answer.*

A more fundamental question is whether checking exponential stability is algorithmically decidable. Here, a problem is called (algorithmically) undecidable, if there is no algorithm, that takes any set of data from a prescribed set and terminates after a finite number of computations and gives an answer. As we have seen exponential stability is equivalent to $\rho(\mathcal{M}) < 1$ and stability implies $\rho(\mathcal{M}) \leq 1$. The following theorem states that determining the maximum spectral of a switched linear system is algorithmically undecidable.

THEOREM 4.14 (Blondel, Tsitsiklis, [23]). *The problem, whether $\rho(\mathcal{M}) \leq 1$ is algorithmically undecidable, even when restricted to sets $\mathcal{M}$ containing only 2 elements. Furthermore, the problem of determining, whether $\mathcal{M}$ is stable, is undecidable.*

It is an open question, if it is algorithmically decidable whether a discrete linear inclusion is exponentially stable, that is, if $\rho(\mathcal{M}) < 1$, see [19].

4.10. Periodic systems and switched systems. One class of switching system for which easily verifiable conditions for stability are known is the class of periodic switched linear systems. For this system class, necessary and sufficient conditions are available from Floquet theory [113, 145]; the growth rate of these systems is determined by the spectral radius of the evolution operator evaluated at the period (and suitably normalized). Since any general system may be thought of as a periodic system whose period is infinite, and notwithstanding decidability issues that we have just discussed, it is natural to question the precise relationship between switched linear systems with arbitrary switching sequences and those with periodic switching sequences. In view of the equality (2.9) we already know that periodic switching signals can have growth rate arbitrarily close to the uniform exponential growth rate of the system. However, this does not answer the question, whether it is possible to realize the growth rate with one particular periodic switching signal.

Consider the system $\dot{x} = A(t)x$, $A(t) \in \{A_1, ..., A_m\}$. Suppose now that the switching system is exponentially stable for all periodic switching signals σ. Does this imply that the system is exponentially stable for arbitrary switching signals?

This question has been studied by many authors; see, for example, [139] and Blondel *et al.* [20] and the references therein. In principle, if it were true, it would provide a method for testing the exponential stability of any given switching system.

When considering this problem an interesting difference occurs between the discrete time and the continuous time case. In discrete time, this question is equivalent to the *finiteness conjecture* that was introduced by Lagarias and Wang [90]. The conjecture has been disproved by Bousch and Mairesse [27], so in discrete time the answer to the above question is no. Blondel, Theys and Vladimirov have presented an alternative analysis of this example [20]. In particular, in the latter paper the existence of a switching system is shown where all periodic switching signals results in transition matrices with spectral radius strictly less than one, whereas the joint spectral radius $\rho = \exp(\kappa) = 1$. Hence, it would appear that periodic stability does not generally imply absolute stability of switched linear systems. We note that the counter example relies crucially upon the switched system operating at the boundary of stability; namely, the system is characterised by a limiting generalised spectral radius of 1.

While the counter example is certainly interesting, it merely states that switching systems that are marginally stable (neither divergent nor convergent), need not be characterised by periodic motions at the boundary of stability. However, by introducing a robustness margin, namely by insisting that $r(\Phi_\sigma(T, 0)) < 1 - \varepsilon$, for a suitable $\varepsilon > 0$, and for all periodic switching signals σ, we can conclude that robust periodic stability does indeed imply asymptotic stability.
The following sufficient condition for stability is due to Gurvits [60, Theorem 2.3] (in discrete time), see also [178] for the continuous time version. Note that the result is

an improvement on (2.9).

THEOREM 4.15. *The switched linear system (2.1) is absolutely stable if and only if there exists an* $\varepsilon > 0$ *such that* $r(\Phi_\sigma(T,0)) < 1-\varepsilon$ *for all periodic switching signals* σ.

In other words, if the switched system is periodically stable with any finite robustness margin ε, then it is also asymptotically stable for arbitrary switching signals. We point out that this result is true in discrete as well as in continuous time. Thus if the switched system is periodically stable with some finite robustness margin ε, then it is exponentially stable for arbitrary switching signals. In principle, the above theorem gives a practical method for testing the stability of any given switching system.

For continuous time systems the situation is a bit different than in discrete time, in that there are positive results available at least for systems of dimension two or three. This problem has been studied by Pyatnitskii and Rapoport [137, 138] as well as by Barabanov [13]. In particular, for second order systems of the form $\dot{x} = A(t)x, A(t) \in \{A_1, A_2\}$, where $\text{rank}(A_1 - A_2) = 1$, stability may be tested by considering all periodic switches with two switches per period (*worst case switching*). Similar results have been obtained for third order systems, as well as for higher oder systems, that leave a proper cone invariant [140]. Whether similar statements concerning worst case switching signals hold for general higher dimensional systems is an open question.

4.10.1. Describing functions for switched systems. A simple argument that identifies the existence (or non-existence) of unstable periodic switching sequences is given in [135, 152]. Here, the authors consider systems of the form

$$\begin{aligned} \dot{x} &= A(t)x,\ A(t) \in \{A, A + bc^T\}, \\ &= (A + \sigma(t)bc^T)x,\ \sigma(t) \in \{0,1\}, \end{aligned} \tag{4.25}$$

where $A, A + bc^T$ are $n \times n$ companion matrices, $b, c \in \mathbb{R}^{n\times 1}$, and where the switching signal $\sigma(t)$ is assumed to be periodic. Introducing the output y and setting $x^T = [y, \dot{y}, ..., y^{(n)}]$, then systems of this form can be conveniently represented in the frequency domain. The key to the analysis in [135, 152] revolves around finding conditions under which a sinusoid, at a particular critical frequency ω_c, undergoes an amplitude magnification of unity, and an effective net phase shift of 2π as it traverses the feedback loop in Figure 4.1 (i.e. by assuming the that the systems destabilises via a sinusoidal limit cycle). Clearly, the existence of such an output signal, $y_c(t) = A\sin(\omega_c t + \theta)$, constitutes the existence of a marginally stable (unstable) limit cycle as a result of switching. If $\sigma(t)$ is assumed to be periodic with period T_σ, then

$$\sigma(t) = \sum_{-\infty}^{\infty} k_n e^{jn\omega_\sigma t},$$

where the k_i are the Fourier coefficients of $\sigma(t)$ and $\omega_\sigma = \frac{2\pi}{T_\sigma}$. Then, by applying the principle of harmonic balance, the condition for the existence of $y_c(t)$ is that,

$$Y_c(j\omega) = G(j\omega) \sum_{-\infty}^{+\infty} k_n Y_c(j(\omega + n\omega_\sigma)). \tag{4.26}$$

Clearly, finding conditions under which (4.26) is satisfied is not simple. However, by assuming the typical low-pass characteristic of $G(j\omega)$ enables us to neglect the effect

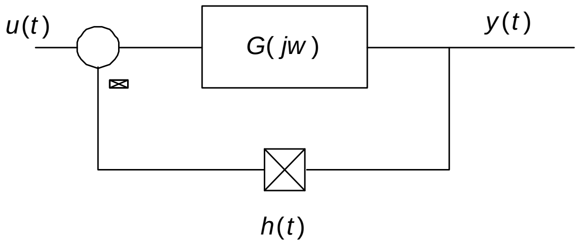

Fig. 4.1: $G(s) = c^T(sI - A)^{-1}b$

of frequency components in $\sigma(t)$ with the exception of $n = 0$ and $n = -1$, and by assuming that the system destabilises via a sinusoidal limit cycle whose frequency is half that of the switching signal $\sigma(t)$, then one obtains using Describing Function-like [110] arguments the following approximate condition for the existence of $y_c(t)$

$$-\frac{1}{G(j\omega)} = k_0 + k_1 e^{-2\omega t_0} \tag{4.27}$$

where $t_0 \in \mathbb{R}$ is some arbitrary time-origin. Equation (4.27) implies that the intersection of the inverse Nyquist plot of $-G(j\omega)$ and one of the family of circles centred at $(k_0, 0)$ with radius k_1 for some frequency ω_c, for all k_0, k_1, implies the existence of a periodic switching signal $\sigma(t)$ with fundamental frequency $2\omega_c$ such that the system is marginally unstable.

Example 4.16. *Consider the system (4.25) with*

$$A_1 = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -10 & -2 \end{pmatrix}.$$

It is easy to verify that $G(s) = \frac{8}{s^2+2s+10}$. *The plot of* $-\frac{1}{G(j\omega)}$ *is depicted in Figure 4.2, and 10 circles at radius* $(0, k_0)$ *with radius* k_1 *are depicted for* $\sigma(t)$ *with a single switch per period and with duty cycle increasing in steps of* 0.1. *For* $\sigma(t)$ *of this form, the above analysis does not indicate the existence of marginally stable sinusoidal signals of the form discussed above.*

Remark 4.17. *Note that due to the approximation implied in condition* (4.27) *the method does not yield reliable results neither in terms of stability nor instability. However it can serve as a convenient engineering tool for the system design. Including higher harmonics ca improve accuracy and even approximate the smallest feedback gain for which a periodic solution is obtained [177].*

To conclude we state a convenient result, formulated in terms of matrix cones, that identifies unstable periodic switching sequences in switched linear systems arising due to unstable chattering.

Theorem 4.18. *[154, 162] A sufficient condition for the existence of a switching sequence, such that the system (1) is unstable, is that there exists non-negative constants* $\{\alpha_1, \alpha_2, ..., \alpha_m\}$, *with* $\sum_{i=1}^{m} \alpha_i > 0$ *such that* $\sum_{i=1}^{m} \alpha_i A_i$ *has an eigenvalue with*

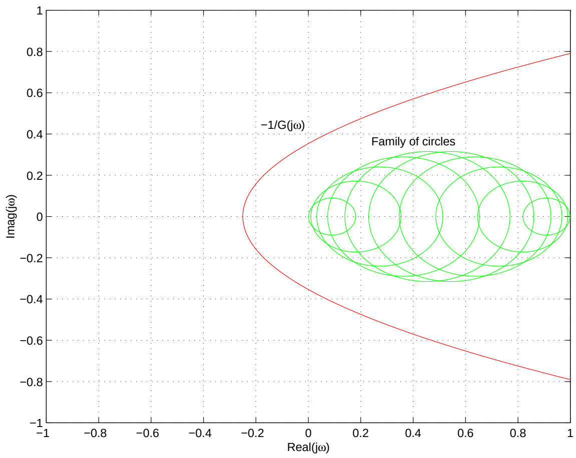

Fig. 4.2: Example

a positive real part.

REMARK 4.19. *We note that similar results, albeit in another context, are obtained in [173, 52]. In these papers the authors consider the quadratic stabilisation of switched linear systems. In [173] it is shown that a switched linear system is quadratically stabilisable if* $\sum_{i=1}^{m} \alpha_i A_i$ *is Hurwitz for some* $\{\alpha_1, ..., \alpha_m\}$*; in [52] it is shown that this condition is both necessary and sufficient for the quadratic stabilisation of switched linear systems where switching takes place between two LTI systems. In both cases the authors use arguments from Lyapunov theory to obtain their results. While it may be possible to use the Lyapunov-based arguments in [173, 52] to obtain the previous result, it is shown in [162] that this result follows immediately from the nature of the solution to (2.1) using only arguments from Floquet theory. A direct consequence of this result is the existence of a periodically destabilising switching sequence; this is entirely consistent with the more general, but also more abstract results presented in [139].*

REMARK 4.20. *The conditions of Theorem 4.18 guarantee the existence of a periodic switching sequence such that the system (2.1) is unstable. More specifically, given a positive sum that has an eigenvalue with a positive real part for some non-negative constants* $\{\alpha_1, ..., \alpha_m\}$*, an unstable switching sequence for (2.1) is constructed as follows: (a) scale the positive constants* α_i *such that* $\sum_{i=1}^{m} \alpha_i = 1$*; (b) let the matrix* A_i *describe the dynamics of (2.1) for* $\alpha_i T$ *seconds. Then, for sufficiently small* T*, the periodic switching sequence so defined results in an unbounded solution to (2.1) irrespective of initial condition* x_0*.*

Theorem 4.18 has a number of interesting consequences for the switched system (2.1):

(i) It is well known that a necessary condition for the existence of a common quadratic Lyapunov function (CQLF), $V(x) = x^T Px$, $P = P^T > 0$, for the LTI systems Σ_{A_i}, $i \in \{1, ..., m\}$, with $\dot{V}(x) < 0$, is that $\sum_{i=1}^{m} \alpha_i A_i$ is Hurwitz for all $\alpha_i \geq 0$, with $\sum_{i=1}^{m} \alpha_i > 0$. Hence the condition that this sum has no eigenvalues with positive real part is necessary for the existence of a CQLF.

It follows from Theorem 4.18 that this condition is a much stronger necessary condition; namely, it is also necessary for the the existence of any type of common Lyapunov function for the systems Σ_{A_i}.

(ii) Often design laws based upon the existence of a CQLF place unnecessarily conservative restrictions on the switching system. However that this is not necessarily true for second order systems. It follows from Theorem 4.3 that one of the following positive sums is singular (and hence non-Hurwitz) for some $\alpha_0 \in [0,1]$ if a CQLF does not exist for Σ_{A_1} and Σ_{A_2}. [5]

$$\alpha_0 A_1 + (1-\alpha_0)A_2,$$
$$\alpha_0 A_1 + (1-\alpha_0)A_2^{-1},$$

Hence, as we mentioned above in Section 4, from Theorem 4.18, the non-existence of a CQLF for (2.1) implies that an unstable, or a marginally unstable[6] switching sequence exists for at least one of the dual switching systems

$$\dot{x} = A(t)x, \quad A(t) \in \{A_1, A_2\}, \tag{4.28}$$
$$\dot{x} = A(t)x, \quad A(t) \in \{A_1, A_2^{-1}\}. \tag{4.29}$$

Although this observation is not true for $m > 2$ matrices [156], it is somewhat surprising since it implies a strong connection between the stability problem for (2.1), and the CQLF existence problem for the constituent systems Σ_{A_i}, namely:

> *given two stable second order LTI systems for which a CQLF does not exist, it follows that an unstable, or marginally unstable, switching sequence exists for one of the associated switching systems (4.28) and (4.29).*

4.11. Common piecewise linear Lyapunov functions. Most of the available results for the arbitrary switching problem are related to the existence of common quadratic Lyapunov functions. However it is not difficult to construct a switched linear system that is asymptotically stable for arbitrary switching sequences where the constituent systems do not have a common quadratic Lyapunov function (see for example [40]). A rapidly maturing area of research is concerned with determining conditions for the existence of non-quadratic Lyapunov functions.

It follows from the converse theorem of Molchanov and Pyatnitski that a common piecewise quadratic, or a common piecewise linear Lyapunov function always exists provided that the underlying switched linear system is asymptotically stable for arbitrary switching. Motivated by this result, a number of authors have sought to develop verifiable conditions for the existence of piecewise linear Lyapunov functions (PLLF) of the form

$$V(x) = \max_{1 \le i \le N} \{w_i^T x\} \tag{4.30}$$

[5]The stated implication can be obtained as follows. Suppose that A_1A_2 (respectively $A_1A_2^{-1}$) has a negative real eigenvalue, $-\lambda$. Then $det[\lambda I + A_1A_2] = 0$. Since A_2 is Hurwitz, this implies that $det[\lambda A_2^{-1} + A_1] = 0$; hence, the matrix $\lambda A_2^{-1} + A_1$ is singular.

[6]By marginally unstable we mean a switching sequence for which there is a solution that does not converge to 0.

where $w_i \in \mathbb{R}^n$, $i = 1, \ldots, N$ and the linear functions $w_i^T x$ are called generators of the piecewise linear Lyapunov function. The function (4.30) is induced by a polyhedral set of the form

$$\mathcal{P} = \left\{ x \in \mathbb{R}^n : \; w_i^T x \leq c, \; i = 1, \ldots, N, \; c \in \mathbb{R}^+ \right\} .$$

Such functions can be shown to be proper and locally Lipschitz [121] and decompose the state-space into a number of convex cones with disjoint interior. The polyhedral set $\mathcal{P}$ is called positively invariant with respect to the trajectories of a dynamical system if for all $x(0) \in \mathcal{P}$ the solution of $x(t) \in \mathcal{P}$ for $t > 0$. A complete survey of properties of positively invariant sets and their usage for a series of problems in control theory can be found in [17].

If the polyhedron $\mathcal{P}$ is bounded and centrally symmetric, then it describes a polytope and the Lyapunov function $V(x)$ can be expressed as

$$V(x) = ||Wx||_\infty \;=\; \max_{1 \leq i \leq N} \{|w_i x|\} \tag{4.31}$$

where $W \in \mathbb{R}^{N \times n}, N \geq n$ has full rank n. Functions of the form (4.31) are radially unbounded, have a unique minimum, and the onesided derivative exists [115].

The existence of piecewise linear Lyapunov functions has been considered in a number of papers for establishing the stability of nonlinear time-varying systems and numerical techniques for the calculation of such functions have been developed. The existence question for piecewise linear Lyapunov functions can be traced back to the sixties to a series of papers by Rosenbrock [144] and Weissenberger [171] on Lur'e type systems. However, despite several decades of research, powerful algebraic tools for the existence of piecewise linear Lyapunov functions remain scarce. Although the class of PLFs appears powerful in theory, the computational requirements necessary to establish their existence represents a serious bottleneck in practice. The main reason is that a complex representation (with a large number of parameters) is usually required for a solution to be found rendering the techniques applicable to low-dimensional problems only.
Indeed few theoretical tools exist to support the development of numerical or analytical tests for checking the existence of such functions. One notable exception is the following results that was obtained in [114, 115]:

Theorem 4.21. *The function $V(x) = ||Wx||_\infty$ is a common piecewise linear Lyapunov function for the switched system* (2.1) *if and only if there exist $Q_i \in \mathbb{R}^{N \times N}$, $i = 1, \ldots, m$ such that*

$$q_{kk}^{(i)} + \sum_{\substack{l=1 \\ l \neq k}}^{N} |q_{lk}^{(i)}| < 0 \tag{4.32}$$

and

$$WA_i - Q_i W = 0 \tag{4.33}$$

for $i = 1, \ldots, m$. Here, $q_{jk}^{(i)}$ denotes the (j, k) entry in the matrix Q_i.

A generalisation of this result for norm-based Lyapunov functions of the form $V(x) = ||Wx||_p$ can be found in [83, 96].

A particular problem for the lack of results in this area is that it appears to be difficult to specify *a-priori* the number N of generators that are necessary for the construction of a common Lyapunov function (4.30) for a given switched system. We note that recently some progress on this question has been made in the context of LTI systems [26, 25]. In this work, the authors relate the number of faces of the PLLF (4.31) to the location of the spectrum of the system matrix A. The results in these papers may serve as a starting point for the derivation of conditions for the existence of a common PLLF for a set of LTI systems. We can deduce the non-existence of the common piecewise linear Lyapunov function with N generators if the spectrum of the convex sum of the system matrices does not satisfy the conditions on the spectrum of LTI systems.

In [179] the existence of a PLLF with four faces ($N = 4$) is considered for second-order switched systems with two subsystems.

THEOREM 4.22. *Given the switched linear system* (2.1) *with* $A_1, A_2 \in \mathbb{R}^{2\times 2}$ *and* $\sigma(A_i) \in \mathbb{R}^-$, $i = 1, 2$ *where* $\sigma\big(\alpha A_1 + (1-\alpha)A_2\big) \notin \mathbb{R}$ *for some* $\alpha \in (0, 1)$ *then there exits a common piecewise linear Lyapunov function* (4.31) *with* $N = 4$ *if and only if the real part of the spectrum of*

$$\sigma\big(\alpha A_1 + (1-\alpha)A_2\big),$$

is always greater than its imaginary part for all $\alpha \in [0, 1]$.

Finally we note that a number of attempts have been made to develop numerical techniques for the construction of such Lyapunov functions. In [31] and [32] Brayton and Tong develop an algorithm for difference inclusions which calculates a series of balanced polytopes converging to the level set of a common PLLF after a finite number of steps. Barabanov [12] proposed another technique for checking asymptotic stability of a linear differential inclusion. An algorithm is constructed which calculates the Lyapunov exponent and a common PLLF in a finite number of steps. This idea has been developed initially for difference inclusions and requires a sufficiently dense discretisation and progressive refinements. Again convex hull computations increase the computational load significantly, rendering the techniques applicable to planar systems, as evidenced by the examples in [31], [32], [12].

In a series of publications, Polański has described an algorithm to construct a common piecewise linear Lyapunov function (4.31) with a given number of generators for the LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$ [130, 131, 132]. Here the algebraic stability condition (4.33) and a scaling idea are used to formulate the search for PLLFs as a linear program. Similar numerical difficulties with high complexity arise and the technique is applicable to planar systems. In [132] an improved formulation using polytope vertices and scaling makes the technique applicable to three-dimensional problems. Thus there are cases, even in three dimensions, in which instability cannot be inferred even when a solution cannot be found. Furthermore, the question of the number N generators required remains unsolved.

This problem is partially avoided in the ray-gridding method developed by Yfoulis and his co-authors in [184, 186, 187]. The approach is based on uniform partitions of the state-space in terms of ray directions which allow refinable families of polytopes of adjustable complexity. The technique provides two important advantages. Firstly, the optimisation problem can be solved much more efficiently such that a complete treatment of the three-dimensional case is feasible; and secondly, by applying a refinement technique there is no prior knowledge about the numbers of generators required.

5. Restricted Switching. In earlier sections we dealt with the problem of determining conditions on the set of matrices $\{A_1, ..., A_m\}$ such that the resulting switched system

$$\dot{x} = A(t)x, \; A(t) \in \{A_1, ..., A_m\}, \tag{5.1}$$

is exponentially stable for all switching sequences. While it is true that this problem has received most of the recent attention in the switching systems literature, a number of other stability problems stand out as being worthy of attention. Among these, the problem of determining the stability of (5.1) in the case where the switching action is constrained in some manner is a problem that arises in a number of important applications [94, 165].

Example 5.1. *An example of how constrained switching can arise in practical situations is given in [169], where the problem of designing a controller to deliver prescribed handling behaviour for a four-wheel steering vehicle is considered. The controller described in [169] operates by manipulating the front and rear steering angles of the vehicle to achieve the desired behaviour. Due to physical considerations, the steering angle of the rear tyres is subject to a tight constraint, and when the maximal allowed steering angle is reached, a change of control action is required leading to an abrupt switch in the overall system dynamics.*

Roughly speaking, research on systems in which switching is constrained has proceeded along two distinct lines of enquiry. The first of these involves the study of systems in which constraints on the switching action are induced by the evolution of the state vector x, as in the example above. The second body of research is concerned with systems in which one seeks to impose constraints on the rate at which switching takes place between the constituent subsystems so as to ensure the stability of the overall system. It should be noted that the classical Lur'e system studied by Popov [120] may be viewed as an example of the former system class, whereas classical Floquet theory developed for the study of periodic systems may be viewed as an example of the latter [113].

One further important problem that arises in the context of this discussion is the following; namely, given a set of non-Hurwitz matrices, determine whether or not it is possible to develop a state dependent switching law such that the system (5.1) is globally uniformly asymptotically stable [92]. We shall briefly discuss this problem later in the paper. The interested reader is referred to [92] and the references therein for a discussion of some of the approaches that have been employed in the study of this and the other problems concerned with constrained switching.

5.1. Constraints on the rate of switching. If all of the matrices in the switching set $\{A_1, ..., A_m\}$ are Hurwitz, then it is possible to ensure the stability of the associated switched system by switching sufficiently slowly between the asymptotically stable constituent LTI systems. This means that instability arises in (5.1) as a result of rapid switching between these vector fields. Given this basic fact, a natural and obvious method to ensure the stability of (5.1) is to somehow constrain the rate

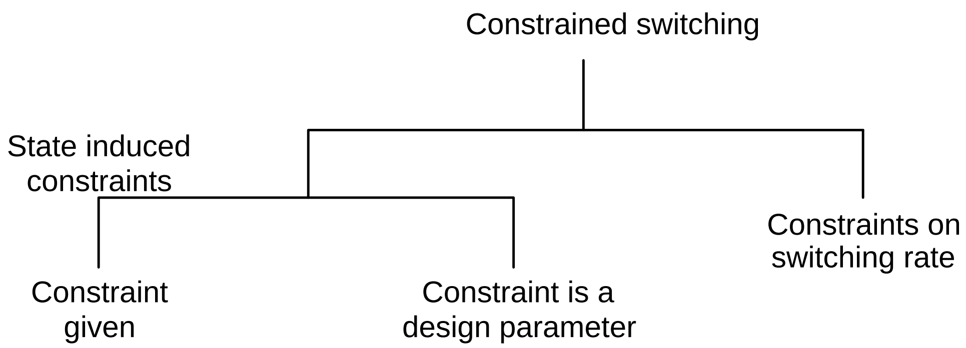

Fig. 5.1: Types of constraints on switching laws that arise in practical applications

at which switching takes place.

The basic idea of constraining the switching rate has appeared in many studies on time varying systems over the past number of decades [59, 191, 74]. One of the best known and most informative of these studies was given by Charles Desoer in 1969 in his study of slowly varying systems [46]. The basic problem considered by Desoer was to find conditions on the switching rate that would ensure the stability of an unforced system of the form $\dot{x} = A(t)x$, where $A(t)$ is a matrix valued continuous function such that $sup_{t\geq 0} \parallel A(t) \parallel < \infty$, and $A(t)$ is Hurwitz for all fixed t (here $\parallel . \parallel$ can be any norm on $\mathbb{R}^{n\times n}$.) Using an argument based on quadratic Lyapunov functions, Desoer demonstrated that there exists some constant $K > 0$ such that the solution $x(t)$ satisfies $\parallel x(t, t_0) \parallel < me^{\lambda}(t - t_0) \parallel x(t_0) \parallel$ for some $m > 0$ and $\lambda < 0$ provided $sup \parallel \dot{A}(t) \parallel = K$. There are two key points to emphasize here; firstly, the stability of the time-varying system can be ensured by suitably constraining the rate of variation of $A(t)$, and secondly the constraint on $\dot{A}(t)$ is determined by a Lyapunov function associated with the system.

Recently, similar ideas been exploited in the Hybrid and Switched Systems community [94, 45, 91, 112, 182]. However, when dealing with switched linear systems, one may approach the problem of constraining the rate at which switching takes place in at least two ways. The first, an indirect method, is close to the method suggested by Desoer; one constrains switching indirectly by ensuring negative definiteness of the derivative of a certain Lyapunov function. An alternative to this approach is to use knowledge of the form of the solutions to (5.1) to ensure stability. While the latter approach is difficult for general time-varying linear systems, the explicit form of the solution to (5.1), makes such an approach possible in the case of switched linear systems and gives rise to the following basic problem in the study of switched linear systems.

Given the system (5.1), let $\sigma_\tau(t)$ denote any switching signal with the property that $t_{k+1} - t_k > \tau$ for all $k > 0$. Let $\mathcal{S}[\tau]$ denote the class of all such signals. One may then pose the following question: Find the minimum τ for which (5.1)is uniformly exponentially stable for all $\sigma_\tau \in \mathcal{S}[\tau]$.

The above problem, often referred to in the literature as the *Dwell-time Problem*, poses a fundamental question in the study of switched systems. However, rather surprisingly, very little progress has been made on this and related problems, and to the best of our knowledge few papers have appeared in the recent literature that deal with this topic; the most notable of those to have appeared are [45, 91, 112, 182]. For convenience we report here on the work developed by Hespanha and his coauthors in [67] as this appears at the present time to be the most complete treatment of the Dwell-time problem to have appeared .

DEFINITION 5.2 (Dwell-time). *[67] Given a positive constant τ_D then $\mathcal{S}[\tau_D] \subset \mathcal{S}$ denotes the set of all switching signals $\sigma(t) \in \mathcal{S}$ where the intervals between consecutive discontinuities are no shorter than τ_D. The constant τ_D is called (fixed) dwell-time.*

For switched linear systems all of whose subsystems, Σ_{A_i}, are Hurwitz stable, Morse [117] established the (unsurprising) fact that the switching system (2.1) is asymptotically stable provided the dwell-time τ_D is chosen to be sufficiently large. This result was extended by Hespanha in [67], where the notion of *average* dwell-time $\bar{\tau}_D$ was introduced. This concept allows some switching intervals to be of length less than $\bar{\tau}_D$ provided that, in a sense to be made precise below, the average dwell-time is at least $\bar{\tau}_D$.

Formally, for a switching signal σ, and real numbers t_1, t_2 with $t_2 > t_1 > 0$, let $N_\sigma(t_1, t_2)$ denote the number of discontinuities of σ in the interval (t_1, t_2). Then given a positive real number $\bar{\tau}_D > 0$ and a positive integer $N_0 > 0$, define $\mathcal{S}[\bar{\tau}_D, N_0]$ to be the set of switching signals $\sigma \in \mathcal{S}$ such that

$$N_\sigma(t_1, t_2) \leq N_0 + \frac{t_2 - t_1}{\bar{\tau}_D} \tag{5.2}$$

for all $t_2 > t_1 > 0$. The parameter N_0 is referred to as the *chatter bound* and $\bar{\tau}_D$ is known as the *average dwell-time*. Note that for $\bar{\tau}_D > 0$, $\mathcal{S}[\bar{\tau}_D] \subset \mathcal{S}[\bar{\tau}_D, 1]$. Using these concepts, Hespanha and Morse derived the following sufficient condition for stability in [67].

THEOREM 5.3. *Consider the switching system (2.1) and suppose that $A_i \in \mathcal{A}$ is Hurwitz for $1 \leq i \leq m$. Further, let $\lambda_0 > 0$ be such that $A_i + \lambda_0 I$ is Hurwitz for $1 \leq i \leq m\}$. Then, for any chosen $\lambda \in [0, \lambda_0)$, there is a finite constant $\bar{\tau}_D^*$ such that (2.1) is exponentially stable, with decay rate λ, for all switching signals $\sigma(t) \in \mathcal{S}[\bar{\tau}_D, N_0]$ for $\bar{\tau}_D \geq \bar{\tau}_D^*$ and any chatter bound $N_0 > 0$.*

In other words, the system is stable if we switch *on average* more slowly than the rate corresponding to $\bar{\tau}_D^*$. This lower bound on the average dwell-time can be calculated by first selecting: $\lambda_0 > 0$ such that $A_i + \lambda_0 I$ is Hurwitz for all $i \in \{1, \ldots, m\}$. We then calculate a quadratic Lyapunov function, $V_i(x) = x^T P_i x$, for each subsystem by solving the Lyapunov equations

$$P_i(A_i + \lambda_0 I) + (A_i + \lambda_0 I)^T P_i = -I$$

for $i = 1, \ldots, m$ and calculate

$$\mu = \sup_{1 \leq i,j \leq m} \frac{\rho_{max}[P_i]}{\rho_{min}[P_j]}.$$

Finally, one chooses a stability margin λ for the switched system and obtain the average dwell-time by

$$\bar{\tau}_D^* = \frac{\log \mu}{2(\lambda_0 - \lambda)}$$

Note that the results on average dwell-time in [67] were derived for a compact (not necessarily finite) set of Hurwitz matrices $\mathcal{A}$ and that a version of Theorem 5.3 was also derived for certain classes of switched nonlinear systems. A number of authors have further extended this work and derived analogous results for other classes of switched nonlinear systems [127, 44]. Further developments have also included the work of Zhai *et al.* [189] in which the authors modify this result so that the lowest average dwell-time $\bar{\tau}_D^*$ ensures that the switched system achieves a chosen L_2 gain, while several other authors have extended the results to allow for switching between stable and unstable subsystems, e.g. [190], [183].

REMARK 5.4. *The principal difficulty associated with the dwell-time problem is to obtain a tight lower bound on τ_D. It is quite remarkable that a non-conservative estimate of τ_D, even for simple classes of switching systems, has to-date eluded the research community.*

5.2. Converse Lyapunov Theorems for systems with dwell time. In the present context, it is important to note that converse theorems for the existence of Lyapunov functions also exist for switched systems with a restriction on the dwell time. Before we discuss these results, we note that in general it is unreasonable to expect one Lyapunov function as a function of the state to suffice for capturing stability. Consider again a set of matrices $\mathcal{A} = \{A_1, \ldots, A_m\}$ and assume we are given a dwell time restriction τ_D. If there exists a Lyapunov function V such that $t \to V(x(t))$ is strictly decreasing for all nonzero solutions of the switched system (5.1), then this implies that the system (2.1) with arbitrary switching is also exponentially stable. In general, there is of course a distinction between stability under arbitrary switching and switching with a dwell time restriction.

A converse theorem for the case of dwell times is presented in [176, Cor. 6.5]. In the following statement we tacitly use the same symbol v for a norm defined on $\mathbb{R}^n$ and the induced matrix norm on $\mathbb{R}^{n \times n}$.

THEOREM 5.5. *The system* (5.1) *with fixed dwell time τ_D is exponentially stable, if and only if there are norms $v_1, \ldots, v_m$ on $\mathbb{R}^n$ and a constant $\beta > 0$ such that*

(i) for all $i = 1, \ldots, m$ it holds that

$$v_i(e^{A_i t}) \leq e^{-\beta t}, \textit{ for all } t \geq 0,$$

(ii) for all $i, j = 1, \ldots, m$ it holds that

$$v_j(e^{A_j t} x) \leq e^{-\beta t} v_i(x), \textit{ for all } x \in \mathbb{R}^n, t \geq \tau_D.$$

Additionally, if the set of matrices $\mathcal{A}$ is irreducible[7] then the norms in the previous theorem may be chosen, such that β is equal to the exponential growth rate of (5.1)

[7] A set of matrices is called irreducible, if only the trivial subspaces $\{0\}$ and $\mathbb{R}^n$ are invariant under all matrices in the set.

with fixed dwell time τ_D. In this manner the result is an extension of the results of Molchanov and Pyatnitski, and Barabanov on the existence of nonquadratic Lyapunov functions in the case of arbitrary switchings.

5.2.1. Indirectly induced constraints: Multiple Lyapunov functions and slowly varying systems. An important paper in the recent evolution of stability theory of switched systems was published in 1994 by Michael Branicky. Branicky made the observation, as Desoer had done in the 1960's, that Lyapunov functions could be used to derive laws to constrain the rate of switching in such a way as to guarantee stability. However, as was suggested by our converse theorem for dwell time systems, rather than using a single Lyapunov function to achieve this as Desoer had done, Branicky suggested the use of multiple Lyapunov functions (one for each mode of the system), to guarantee exponential stability.

> *Branicky's basic idea was to define a Lyapunov function for each mode i of the system Σ. One then uses these functions to construct a stabilising switching signal $\sigma(t)$ by only allowing the system to switch into mode i if the value of the corresponding Lyapunov function $V_i(x)$ is less than it was when the system last left mode i.*

It is important to note that the use of multiple Lyapunov functions to select a stable switching sequence for switched systems had been suggested by a number of authors prior to Branicky's original paper in 1994; in particular Peleties & DeCarlo [126] deserve credit for promoting the original idea for switched linear systems. However, Branicky's paper, which extended the basic idea to the nonlinear case, has had a great impact on the community and in the following discussion we adopt the notation and arguments given in the original papers [30].

Branicky considers a general autonomous switched non-linear system

$$\dot{x}(t) = f(x,t) \quad f(x,t) \in \{f_1(x), \dots, f_m(x)\} \;\; t \geq 0, \tag{5.3}$$

where each mode, f_i, is assumed to be globally Lipschitz continuous and exponentially stable; and where the switching strategy is chosen in such a way that there are finite switches in finite time. While there are several similar versions of Branicky's result, we quote the following from [29] which can be stated with a minimum of mathematical formalism.

THEOREM 5.6. *Suppose that we have a finite number of Lyapunov functions $V_i(x)$ associated with the continuous-time vector fields $\dot{x} = f_i(x)$. Let $S_k = i_0, i_1, \dots, i_k, \dots$ denote the switching sequence of the system, and $T_k = t_0, t_1, \dots, t_k, \dots$ denote the sequence of corresponding switching instances for the system. If, for each instant t_j when we switch into mode i, with corresponding Lyapunov function V_i, we have that*

$$V_i(x(t_j)) \leq V_i(x(t_k)) \tag{5.4}$$

where $t_k < t_j$ and t_k is the last time that we switched out of mode i, then the system is stable in the sense of Lyapunov.

Theorem 5.6 gives a simple rule for the construction of a stable slowly varying switching system. It states that when the system enters mode i, the value of the Lyapunov function associated with this mode must be less than the value it attained when the system last left mode i. For the purpose of illustration, consider a general non-linear system with two modes, $\dot{x} = f_i(x),\ i \in \{1, 2\}$. The profile of typical Lyapunov functions associated with modes 1 and 2, for a switching strategy constructed according to Theorem 5.6, is depicted in Figure 5.2.

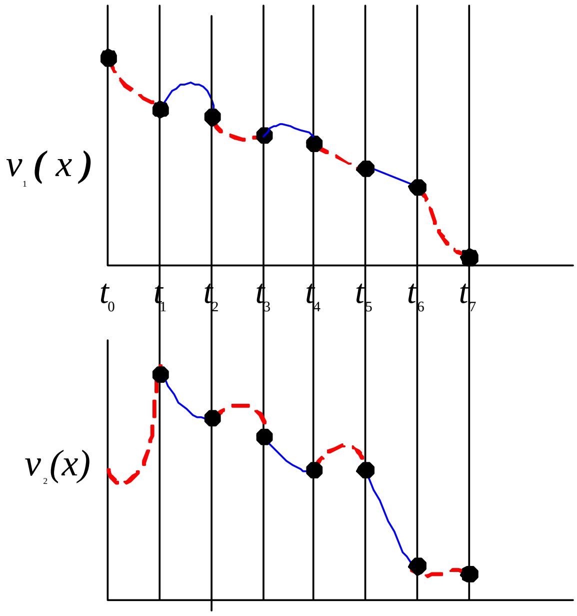

Fig. 5.2: Multiple Lyapunov functions

The approach of multiple Lyapunov functions can be varied in several ways. For instance, we can relax the requirement that the functions V_i, $1 \leq i \leq m$, are proper Lyapunov functions in the sense that $\dot{V}_i(x) \leq 0$ along every entire trajectory $x(t)$ of the i^{th} subsystem. Clearly, we may achieve less conservative results if we only demand that $\dot{V}_i(x)$ is non-positive during time-intervals where the system actually is in mode i (then, V_i is often referred to as Lyapunov-like function). Further improvement can be achieved if we compare the values of V_i in (5.4) at consecutive *starting* points of mode i. For details see e.g. [30].

Remark 5.7. *The multiple Lyapunov function approach to stability analysis offers several advantages over other more conventional methods: (i) the underlying idea of the paradigm is easy to understand and is suitable for use in industry; (ii) the approach can be used for the stability analysis of heterogeneous systems; and (iii) the analysis can be based upon the existence of any type of Lyapunov function (not just quadratic functions).*

Remark 5.8. *There are several disadvantages associated with the approach: (i) no constructive procedure for choosing the best Lyapunov functions is currently known; (ii) a poor choice of Lyapunov functions V_i may lead to very conservative switching rules; (iii) in order to choose a Lyapunov function for each subsystem, the subsystems must be individually stable; (iv) the technique places conditions on all of the candidate*

Lyapunov functions (a condition that is clearly not necessary for exponential stability of the switched system).

Remark 5.9. *A number of interesting research questions in the multiple Lyapunov function framework remain unanswered. In our context, namely for switched linear systems, the most important of these pertains to developing a constructive method of choosing the candidate Lyapunov functions that minimise the dwell-time for each mode.*

5.3. State dependent switching and stability. In the previous subsection, we considered a variety of results concerned with switched systems for which the rate of switching is constrained in some way. An alternative type of constraint that arises in the study of switched systems is where the switching action is constrained by the state vector of the system. As we shall see in the next section, Internet congestion control is an example of one such system. If the rule for switching between the constituent systems of a switched system is determined by the state vector of the system, we say that the switching is *state dependent*. Loosely speaking, the stability problems associated with this type of switching regime can be divided into two classes. In the first of these, the state space is partitioned by a number of hyper-surfaces that determine the mode switches in the system dynamics, and the problem is to analyse the stability of the time-varying system defined in this way. On the other hand, in the second class of problem we are concerned with finding state-dependent rules for switching between a family of unstable systems that result in stability. Thus, in the former case, a partition of state space is specified and the problem is to determine the stability of the piecewise linear system defined by that partition, while in the latter case the aim is to find stabilizing state-dependent rules for switching between individually unstable systems.

5.4. Switched systems and state-dependent constraints. We shall now consider switched systems that are constrained in the sense that a mode switch occurs in the systems dynamics when its state vector crosses certain threshold surfaces in state space. We first turn our attention to the class of *piecewise linear systems.*

Piecewise linear systems:
A piecewise linear system is a dynamical system of the form

$$\dot{x} = A_i x \quad \text{for } x \in \Omega_i \tag{5.5}$$

where $\Omega_1, \ldots, \Omega_m$ are closed sets with pairwise disjoint interiors such that $\cup_i \Omega_i = \mathbb{R}^n$, and $A_1, \ldots, A_m$ are in $\mathbb{R}^{n \times n}$. For such systems, requiring the existence of a common Lyapunov function for the LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$ can be an unduly restrictive criterion for stability. For instance, in order for the quadratic form $V(x) = x^T P x$ to define a CQLF for $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$, for each $i \in \{1, \ldots, m\}$, $V(x)$ must decrease along all trajectories of Σ_{A_i} everywhere in the state space. However, it is clear that this may well lead to unnecessarily conservative stability conditions as it fails to take into account that the system Σ_{A_i} is only active within the region Ω_i. The so-called S-procedure [28, 77] seeks to exploit this fact in order to obtain less restrictive stability conditions for systems of the form (5.5).

The S-procedure:
The basic idea behind using the S-procedure to analyse the stability of systems of

the form (5.5) is the following. First of all, determine symmetric matrices $S_1, \ldots, S_m$ such that $x^T S_i x \geq 0$ for x in Ω_i and then search for a positive definite $P = P^T > 0$ satisfying the conditions

$$A_i^T P + PA_i + S_i < 0 \quad \text{for } 1 \leq i \leq m. \tag{5.6}$$

If P satisfies (5.6) then it follows that, for each $i \in \{1, \ldots, m\}$, $x^T(A_i^T P + PA_i)x < 0$ for $x \in \Omega_i$. Thus, the positive definite function, $V(x) = x^T Px$, decreases along all trajectories of the system (5.5), thereby establishing exponential stability. However, note that for some choices of the matrix S_i, the expression $x^T(A_i^T P + PA_i)x$ may be non-negative for x outside of Ω_i. For this reason, the conditions given by (5.6) may be less demanding than those imposed by requiring CQLF existence.

Piecewise quadratic Lyapunov functions:
Extending the ideas of the S-Procedure, a number of authors have studied piecewise quadratic Lyapunov functions [77, 129, 128, 63] in order to find less conservative stability criteria for piecewise linear systems. Here, rather than looking for a single quadratic Lyapunov function, $V(x) = x^T Px$, for the system (5.5), the idea is to search for a family of such functions satisfying certain local conditions, and then to piece these together appropriately to form a Lyapunov function for the overall system.

For convenience, and to illustrate the main ideas behind the use of piecewise quadratic Lyapunov functions, we shall focus mainly on the results of [77]. In this paper, under the assumption that the regions Ω_i are polyhedral, a numerical procedure is described for searching for a piecewise quadratic Lyapunov function of the form [8]

$$V(x) = x^T P_i x \quad \text{for } x \in \Omega_i, \tag{5.7}$$

where $P_i = P_i^T \in \mathbb{R}^{n \times n}$ for $i = 1, 2, \ldots, m$. Extending the basic idea of the S-procedure, the authors of [77] relax the conditions for stability given by CQLF existence in a number of ways.

(i) The use of different quadratic forms for the different operating regions Ω_i can lead to greater flexibility in the definition of the Lyapunov function V.

(ii) The matrices P_i are not required to be globally positive definite. In fact, by applying the S-procedure, the inequality $x^T P_i x > 0$ is only required to hold when $x \in \Omega_i$ for $1 \leq i \leq m$.

(iii) Similarly, the condition $x^T(A_i^T P_i + P_i A_i)x < 0$ is only required to hold for $x \in \Omega_i$.

A few specific points relating to the results described in [77] are worth noting.

(i) The matrices P_i are parameterized so as to ensure that the piecewise quadratic function $V(x)$ is continuous.

[8] The function takes a slightly different form in regions Ω_i that do not contain the origin. For details consult [77]

(ii) The conditions for (5.7) to define a piecewise quadratic Lyapunov function for the system (5.5) are expressed in the form of linear matrix inequalities (LMIs). Hence, modern convex optimization algorithms can be used to search for piecewise quadratic Lyapunov functions.

(iii) It is possible to use a partition other than that dictated by the system dynamics to define the piecewise quadratic function. Thus, if an initial search is unsuccessful, it may be possible to find a piecewise quadratic Lyapunov function defined with respect to an alternative, possibly finer, partition of the state space. However, the problem of selecting an initial partition, and of devising automatic methods of successively refining the partition to systematically search for piecewise quadratic Lyapunov functions is in general far from straightforward.

The results of [77] were actually derived for piecewise affine systems where the dynamics within each region Ω_i take the form $\dot{x} = A_i x + a_i$. The ideas and techniques of this paper were developed and extended in the subsequent papers [78, 142] and similar LMI conditions for the stability of piecewise linear systems based on piecewise quadratic Lyapunov functions have been presented in [63]. In this context, the work of [51] on applying piecewise quadratic methods to the problem of controller design for uncertain piecewise affine systems should also be noted.

The paper [185] describes closely related work on the stability of the class of *orthogonal* piecewise linear systems. For such systems, the hyper-planes that partition the state space, $\mathbb{R}^n$ take the general form $x_i = c_{i,j}$ for $i = 1, \ldots n$, $j = 1, \ldots n_i$, and divide the state space into a family of hyper-rectangles. The conditions for stability derived in [185] are based on the existence of piecewise *linear* Lyapunov functions as opposed to piecewise quadratic Lyapunov functions.

Switching rules specified by switching surfaces:
Similar ideas to those used to analyse piecewise linear systems have also been used to study more general state dependent switching rules. Typically, these rules are defined by specifying a set of surfaces S_{ij}, $1 \leq i, j \leq m$, in the state space such that the system switches from mode i to mode j if the i^{th} subsystem is currently active and the state vector crosses the surface S_{ij} [129, 77, 173]. For instance, in [77] piecewise quadratic Lyapunov functions are employed to obtain LMI-based stability conditions for such systems under the assumption that the switching surfaces are given by hyper-planes. The Lyapunov function used to derive the conditions in this paper is not required to be continuous provided that the value of the function decreases whenever the system switches from one mode to another. Similar results based on piecewise quadratic Lyapunov functions have also been presented in [129]. The results of this last paper are based on a Lyapunov function that need not be continuous and, moreover, is not required to decrease at each switching instant. In fact, all that is needed for stability is that the values of the function are bounded by some continuous function of its initial value. The conditions in [129] are again formulated as LMIs, and can also be applied to systems with non-linear constituent systems.

Piecewise linear systems and the Popov criterion:
The classical Popov criterion for the absolute stability of non-linear systems [133, 168]

can also be used in the analysis of certain piecewise linear systems. An example illustrating this is described in [77], where the stability of a two-dimensional piecewise linear system is established using the Popov criterion. For the system considered in this example, the Popov criterion ensures the existence of a Lyapunov function of Lur'e Postnikov form, meaning that it is the sum of an integral of a non-linear function and a quadratic form. For the system considered in [77], this function actually reduces to a piecewise quadratic Lyapunov function of the form considered above. It should be noted that the connection between Lyapunov functions of Lur'e Postnikov form and piecewise quadratic Lyapunov functions was earlier pointed out by Weissenberger in [171], and that the idea of using piecewise quadratic Lyapunov functions for stability analysis had been suggested by Power and Tsoi in [134].

Positive switched linear systems:
The classes of switched system considered above are constrained in the sense that the system dynamics must undergo a mode switch when the state vector crosses some surface in state space. The nature of these systems has led naturally to the consideration of both piecewise quadratic and piecewise linear Lyapunov functions in their stability analysis. A different type of constraint arises in the study of positive switched linear systems, where any trajectory starting from non-negative initial conditions must remain within the non-negative orthant for all subsequent times [50, 98]. In view of the considerable restriction that this imposes on the possible trajectories of a positive system, it is natural to consider so-called co-positive Lyapunov functions when analysing the stability of such systems [18]. These functions are only required to satisfy the requirements of a traditional Lyapunov function within the non-negative orthant and may lead to less conservative stability conditions for positive switched linear systems than can be obtained using traditional Lyapunov functions. Some initial results on common copositive Lyapunov function existence can be found in [61, 108, 106]

5.5. Stabilizing switching rules. The results discussed in the last subsection were concerned with establishing the stability of systems subject to some specified state dependent switching rule. Another problem of interest in this context is that of determining stabilizing switching rules for systems with unstable constituent systems.

In [173], the following problem was addressed. Given two LTI systems Σ_{A_1}, Σ_{A_2} where both A_1 and A_2 have some eigenvalues in the right half plane, determine if there exists some rule for switching between these systems that results in stability. It has been established, [173], that if some convex combination of the matrices A_1 and A_2 is Hurwitz, then such a stabilizing switching rule does indeed exist. Formally, this amounts to testing for the existence of some α with $0 < \alpha < 1$ such that the matrix

$$A(\alpha) = \alpha A_1 + (1 - \alpha)A_2$$

is Hurwitz. Moreover, the authors of [173] describe how to construct a state-dependent stabilizing switching rule when such a stable convex combination exists.

The basic idea behind this construction is the following. As the matrix $A(\alpha)$ is Hurwitz, there exists some positive definite matrix $P = P^T > 0$ such that

$$A(\alpha)^T P + PA(\alpha) < 0. \tag{5.8}$$

It follows that the two cones Ω_1, Ω_2 defined by

$$\Omega_i = \{x \in \mathbb{R}^n : x^T(A_i^T P + PA_i)x < 0\} \tag{5.9}$$

cover the space (meaning that $\Omega_1 \cup \Omega_2 = \mathbb{R}^n$). Using this fact, it is possible to define two switching surfaces close to the boundaries of these cones such that the associated switching rule asymptotically stabilizes the overall system. In fact, the quadratic function, $V(x) = x^T Px$, where P is a solution of (5.8), is a Lyapunov function for the system defined by this switching rule. For this reason, we say that the switching rule defined in the above manner quadratically stabilizes the system.

The switching rule described in the previous paragraph is constructed so as to ensure that only a finite number of switches occur in any finite time interval. Two other switching rules for stabilizing the system are also described in [173], but both of these allow for the possibility of infinitely many switches occurring in a finite time interval and for practical reasons this may be undesirable.

The result that a stabilizing switching rule exists if some convex combination of the matrices A_1 and A_2 is Hurwitz extends to the case of an arbitrary finite family of matrices. Formally, given a family of unstable LTI systems $\Sigma_{A_1}, \ldots, \Sigma_{A_m}$, there is some rule for switching between them that results in quadratic stability if there are non-negative real numbers $\alpha_1, \ldots, \alpha_m$ with $\alpha_1 + \cdots + \alpha_m = 1$ such that the matrix $\alpha_1 A_1 + \cdots + \alpha_m A_m$ is Hurwitz. In general this condition is not known to be necessary for the existence of such a quadratically stabilizing switching rule. However, for the case of switching between two unstable systems, the existence of a Hurwitz convex combination of the system matrices is known to be *equivalent* to the existence of a quadratically stabilizing switching rule [52].

The work of [173] has been extended in the paper [172] where conditions for the existence of stabilizing switching rules were derived using piecewise quadratic Lyapunov functions as opposed to quadratic Lyapunov functions. Note also the related work on piecewise quadratic Lyapunov functions described in [180], and the recent paper [8] where it is shown that, under an additional assumption, there exists a stabilizing rule for switching between a pair of unstable LTI systems Σ_{A_1}, Σ_{A_2}, provided that there is some convex combination of the system matrices A_1, A_2 with the property that all of its eigenvalues have non-positive real parts and any eigenvalues on the imaginary axis are simple.

6. The Lure' problem. The study of stability reached a peak in the 1960's and early 1970's in the engineering community. During this period a vast number of researchers studied the Lure' problem. Consequently, it is impossible to write a review on this topic without mentioning this work even briefly.

The SISO version of the Lure' problem is depicted in Figure 6.1 (although our comments are equally valid for MIMO versions of the Lure' problem).

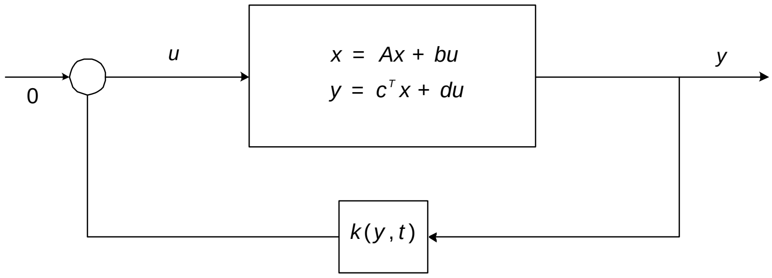

Fig. 6.1: Graphical depiction of the Lure' system. For the present discussion we consider SISO systems: A is a stable $n \times n$ matrix, b, c are appropriately dimensioned vectors, and d is a scalar.

As can be seen the Lure' system is composed of a feedback connection of a stable LTI system and a non-linear and possibly time-varying gain that is constrained to lie in some interval, say $k(y,t) \in [0,1]$.

REMARK 6.1. *Note that the Lure' system reduces to an example of a classical switched system when* $k(y,t) \in \{0,1\}$. *Evidently, tools developed for the study of Lure' systems can therefore be applied to the analysis of special classes of switched systems.* Three problems were widely studied in the context of the Lure' system.

(i) When is the equilibrium solution of the Lure' system globally uniformly exponentially stable for arbitrary time-varying gains $k(t)$.

(ii) Can one impose constraints on $k(y,t)$ such that the equilibrium solution of the Lure' system is globally uniformly exponentially stable?

(iii) Can one impose conditions on the rate of change of $k(y,t)$ so that the equilibrium solution of the Lure' system is globally uniformly exponentially stable?

Problems (i), (ii) and (iii) are clearly analogs of the *Arbitrary switching*, *Restricted switching* and the *Dwell-time* problems respectively. Their study has led to many classical stability criteria: the Kalman-Yacubovich-Popov-Meyer lemma; the Circle Criterion; the off-axis Circle criterion; the Passivity theorem; and the Popov Criterion [1]. The main tool used in developing these results was the Lure'-Postnikov Lyapunov function. That is, one sought conditions on $A, b, c, d, k(y,t)$ to ensure that a Lyapunov function of the form:

$$V(x) = x^T P x + \lambda \int_0^y k(\delta, t) d\delta \tag{6.1}$$

where $\lambda \in \mathbb{R}^+$ and where $P = P^T > 0$. Note that $V(x)$ is rarely quadratic. Due mainly to the remarkable results of Popov, and later of Kalman and Meyer, it was found that the existence of a function $V(x)$ for the Lure' system could be deduced by testing whether or not

$$1 + Re(H(j\omega)G(j\omega)) > 0, \ \ \forall \ \omega \in \mathbb{R} \tag{6.2}$$

where $G(j\omega) = d + c^T(j\omega I - A)^{-1}b$ and $H(j\omega)$ is some rational function of ω that is referred to as a multiplier. Recently, a number of results have been obtained that relate results derived in the context of the Lure' problem to more general results derived in the context of switched linear systems. In particular, the following result has proved to be a useful bridge between these areas.

THEOREM 6.2. *Let $G(j\omega) = \frac{N(j\omega)}{D(j\omega)}$ be a proper real rational transfer function and $K \in \mathbb{R}^+$. Let $\{A, b, c, d\}$ be a realisation of $G(j\omega)$ so that $G(j\omega) = c^T(j\omega I - A)^{-1}b + d$. Assume that A and $\left(A - \frac{1}{K+d}bc^T\right)$ are Hurwitz. Then, a necessary and sufficient condition for*

$$K + Re\{G(j\omega)\} > 0, \ \forall\, \omega \in \mathbb{R} \cup \{\infty\}, \tag{6.3}$$

is that the matrix-product $A\left(A - \frac{bc^T}{K+d}\right)$ has no negative real eigenvalues.

Theorem 6.2 has a number of implications for classical frequency domain stability criteria.

(i) *The Kalman-Yacubovich-Popov (KYP) Lemma :* The single-input single-output (SISO) version of the Kalman-Yacubovich-Popov lemma [80] is expressed in the form of a strictly positive real (SPR) condition: namely, A and $A - \frac{1}{\gamma}bc^T$ are Hurwitz and

$$\gamma + Re\left\{c^T(j\omega I_n - A)^{-1}b\right\} > 0 \ \forall\, \omega \in \mathbb{R},$$

for some $\gamma \in \mathbb{R}^+$. Hence it follows from Theorem 6.2 that a time-domain version of the SPR condition for SISO systems is that the matrices A and $\left(A - \frac{1}{\gamma}bc^T\right)$ are Hurwitz and $A(A - \frac{1}{\gamma}bc^T)$ does not have any negative real eigenvalues.

(ii) *The Circle Criterion [158]:* The SISO version of the Circle Criterion is derived directly from the SISO KYP lemma. Here, conditions are derived for the existence of a Lyapunov function $V(x) = x^TPx, P = P^T \in \mathbb{R}^{n\times n}$ for the non-linear Lur'e type system. In this case a necessary and sufficient condition for the existence of a quadratic Lyapunov function $V(x)$ is that [119] A and $A - bc^T$ are Hurwitz and that

$$1 + Re\left\{c^T(j\omega I_n - A)^{-1}b\right\} > 0 \ \forall\, \omega \in \mathbb{R}.$$

It follows from Theorem 6.2 that a time-domain version of the Circle Criterion with $0 \le k(y,t) \le 1$ is that matrices A and $A - bc^T$ are Hurwitz and that matrix $A(A - bc^T)$ does not have any negative real eigenvalues.

(iii) *The Popov Criterion* : The SISO Popov criterion [163] considers the stability of the Lure' system where the nonlinearity k is non-linear but time-invariant. A sufficient condition for the absolute stability of this system is that A and $A - \frac{1}{\gamma}bc^T$ are Hurwitz and there exists a strictly positive $\alpha \in \mathbb{R}$ such that

$$\frac{1}{k} + Re\left\{(1 + j\alpha\omega)c^T(j\omega I_n - A)^{-1}b\right\} > 0 \ \forall\, \omega \in \mathbb{R}.$$

It follows from Theorem 6.2 that a time-domain version of the Popov criterion is that A and $A - \frac{1}{\gamma}bc^T$ are Hurwitz and there exists a positive $\alpha \in \mathbb{R}$ such that matrix $\bar{A}(\bar{A} - \frac{1}{\bar{d}+\frac{1}{k}}\bar{b}\bar{c})$ does not have any negative real eigenvalues where $\{\bar{A}, \bar{b}, \bar{c}, \bar{d}\}$ is a realisation of $(1+\alpha s)c^T(sI_n - A)^{-1}b$.

REMARK 6.3. *The most interesting observation arising from this theorem is that all stability problems have been reduced to a CQLF existence problem. For example, the Popov criterion, which searches for the existence of a non-quadratic Lyapunov function for the original non-linear system, has been reduced to a CQLF existence problem for $\dot{x} = \bar{A}x$, and $\dot{x} = (\bar{A} - \frac{1}{\bar{d}+\frac{1}{k}}\bar{b}\bar{c})x$. This appears to be an unexplored (and perhaps important) observation.*

7. Passivity. Apart from the Circle Criterion, one of the big successes of this period, was the derivation of the Passivity theorem [168]. The Passivity Theorem gives a sufficient condition for stability of the following time-varying, nonlinear system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ u(t) &= -\Phi(t, y(t)) \end{aligned} \tag{7.1}$$

where $x \in \mathbb{R}^n, u, y, z \in \mathbb{R}^m, m < n$, $\Phi : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, and A, B, C, D are matrices with the appropriate dimensions. These equations describe the dynamics of a control system where $x(t)$ is the state variable, $y(t)$ is the output variable and $u(t)$ is the feedback. The dynamics of $x(t)$ is separated into a linear part Ax and a nonlinear part Bu, and it is assumed that the matrix A is Hurwitz so that the linear part is stable. The question of finding conditions on the nonlinearity Φ which are sufficient to guarantee stability of the system is known as the Lur'e problem. The Passivity Theorem provides the following solution: the system (7.1) is gobally exponentially stable if $\Phi(t, 0) = 0$, $y^T\Phi(t, y) \geq 0$ for all $t \in \mathbb{R}$, all $y \in \mathbb{R}^m$, and if $H(j\omega) + H^*(j\omega)$ is positive definite for every real ω, where the transfer matrix $H(s)$ is defined by

$$H(s) = C(sI - A)^{-1}B + D, \quad s \in \mathbb{C}. \tag{7.2}$$

When these conditions are satisfied the Kalman-Yacubovitch Lemma guarantees the existence of a quadratic Lyapunov function x^TPx for the system (7.1) [168].

This problem is relevant to the study of switching systems because in many cases the dynamics of a switching system has the following form:

$$\dot{x}(t) = Ax(t) - z(t, x(t)) \tag{7.3}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is Hurwitz and $z : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a time-varying, nonlinear vector field. The Lur'e system (7.1) can be rewritten in the form (7.3) with $z = \Phi(t, y(t))$, where $y(t)$ depends on the state $x(t)$ through the implicit relation $y(t) = Cx(t) - D\Phi(t, y(t))$. Thus (7.3) is more general than the Lur'e system, because z is allowed to depend on x in an arbitrary way.

One example of such a system arises in state-dependent switching where $\mathbb{R}^n$ is partitioned into disjoint sets $\{\Omega_i\}$, and the dynamics switches between different linear systems as the state crosses from one region to another. So the dynamics is given by

$$\dot{x}(t) = A(x)\, x(t), \quad A(x) = A_i \ \forall\ x \in \Omega_i \tag{7.4}$$

This system can be presented in the form (7.3) by defining $z(t, x) = (A - A(x))x(t)$ for some Hurwitz matrix A.

The Passivity Theorem provides a method to analyse the stability of systems of the form (7.3). The crucial ingredient is the existence of a quadratic Lyapunov function $x^T Px$ for the system. Instead of using the Kalman-Yacubovitch Lemma to provide the matrix P, one can instead postulate the existence of such a function and use it to derive conditions which must be satisfied by the nonlinearity Φ. This idea leads to the following result [84].

THEOREM 7.1. *Let* $\{A_1, \ldots, A_m\}$ *be a collection of real matrices such that the convex combination* $\tilde{A} = \sum_{i=1}^m \alpha_i A_i$ *is Hurwitz for some* $\alpha_1, \ldots \alpha_m$ *satisfying* $\alpha_i \geq 0$, $\sum \alpha_i = 1$. *Choose* $P = P^T > 0$ *such that* $\tilde{A}^T P + P\tilde{A} < 0$ *and define* $K_i = \{x : x^T P(\tilde{A} - A_i)x \geq 0\}$. *Then*

$$\bigcup_{i=1}^m K_i = \mathbb{R}^n.$$

For $i = 1, \ldots, m$ *let* $\Omega_i \subset K_i$ *such that the* Ω_i *are disjoint and their union is* $\mathbb{R}^n$. *Then the system (7.4) is globally exponentially stable.*

The result is proved by choosing $z(t, x) = (\tilde{A} - A(x))x(t)$ in (7.4) where $A(x) = A_i$ for all $x \in \Omega_i$. Then the definition of K_i implies that $x^T Pz(t, x) \geq 0$ for all $x \in \mathbb{R}^n$, and this is sufficient to guarantee that $x^T Px$ is a Lyapunov function for (7.4).

The results of Theorem 7.1 can be used to design stable switching systems of the form (7.4). In particular, given a collection of real matrices $\{A_1, \ldots, A_m\}$ for which some convex combination is Hurwitz, one would like to determine all matrices $P = P^T > 0$ which satisfy $\tilde{A}^T P + P\tilde{A} < 0$, as these would describe possible stable state-dependent switching rules for this collection. In general it is difficult to find a compact parametrisation of these matrices. However the procedure simplifies when the matrices A_i have the special form $A_i = \tilde{A} - BD_i^T$, where $\tilde{A}$ is Hurwitz and B is some fixed matrix. In this case, if there is a matrix C such that $H(j\omega) + H^*(j\omega)$ is positive definite for all real ω, where now $H(s) = C^T(sI - A)^{-1}B$, then the cones are K_i are given as $\{x : x^T CD_i^T x \geq 0\}$. Therefore the search for matrices P can be replaced by a simpler search for matrices C that satisfy this positivity condition. In the case $m = 1$ where B, C are vectors in $\mathbb{R}^n$, the set of all possible vectors C can be described in the following compact and useful way [84]: a vector C satisfies the positivity condition above if and only if $C^T AB < 0$ and $C^T(A^2 + \omega^2)^{-1}AB < 0$ for all $\omega \in \mathbb{R}$. This allows a constructive procedure to find the vectors [84].

8. Case study: Internet congestion control. To conclude our paper we present an example of a class of switched system that is currently finding application in many different areas of computer science and engineering. It has already been discussed that switched systems in which the states remain positive are ubiquitous [106]. A subclass of such systems that is currently arising in many current applications is the class of discrete time switched positive system models in which the matrices in the switching set are not only positive, but also Markov (stochastic). Such models can readily be found in the study of communication networks [151, 75]; in the study of concensus algorithms [75]; and in other applications from computer science such as Google [33]. This system class is particularly interesting since it involved switching between subsystems that are themselves marginally stable. We focus here on the study of TCP based communication networks as it illustrates some of the interesting features of such systems.

A communication network consists of a number of sources and sinks connected together via links and routers. We assume that these links can be modelled as a constant propagation delay together with a queue, that the queue is operating according to a drop-tail discipline, and that all of the sources are operating a TCP-like congestion control algorithm. TCP (transmission control protocol) operates a window based congestion control algorithm. The TCP standard defines a variable *cwnd* called the congestion window. Each source uses this variable to track the number of sent unacknowledged packets that can be in transit at any time. When the window size is exhausted, the source must wait for an acknowledgement before sending a new packet. Congestion control is achieved by dynamically adapting the window size according to an additive-increase multiplicative-decrease (AIMD) law. Roughly speaking, the basic idea is for a source to gently probe the network for spare capacity by increasing the rate at which packets are inserted into the network, and to rapidly back-off the number of packets transmitted through the network when congestion is detected. We shall see that the *AIMD* paradigm with drop-tail queuing gives rise to networks whose dynamics can be accurately modelled as a positive linear system. While we are ultimately interested in general communication networks, for reasons of exposition it is useful to begin our discussion with a description of networks in which packet drops are synchronised (i.e. every source sees a drop at each congestion event). We show that many of the properties of communication networks that are of interest to network designers can be characterised by properties of a square matrix whose dimension is equal to the number of sources in the network. The approach is then extended to a model of unsynchronised networks. Even though the mathematical details are more involved, many of the qualitative characteristics of synchronised networks carry over to the non-synchronised case if interpreted in a stochastic fashion.

8.1. Synchronised communication networks. We begin our discussion by considering communication networks for which the following assumptions are valid: (i) at congestion every source experiences a packet drop; (ii) each source has the same round-trip-time (RTT)[9]; and (iii) each source competes through a single bottleneck link. In this case an exact model of the network dynamics may be found as follows [150]. Let $w_i(k)$ denote the congestion window size of source i immediately before

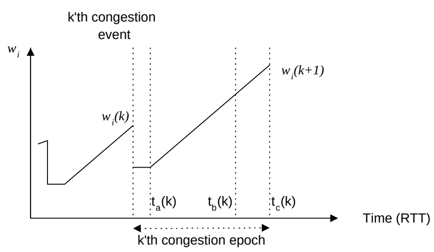

Fig. 8.1: Evolution of window size

the k'th network congestion event is detected by the source. Over the k'th congestion epoch three important events can be discerned: $t_a(k), t_b(k)$ and $t_c(k)$; as depicted in

[9]One RTT is the time between sending a packet and receiving the corresponding acknowledgement when there are no packet drops.

Figure 8.1. The time $t_a(k)$ denotes the instant at which the number of unacknowledged packets in flight equals $\beta_i w_i(k)$; $t_b(k)$ is the time at which the bottleneck queue is full; and $t_c(k)$ is the time at which packet drop is detected by the sources, where time is measured in units of RTT[10]. It follows from the definition of the *AIMD* algorithm that the window evolution is completely defined over all time instants by knowledge of the $w_i(k)$ and the event times $t_a(k), t_b(k)$ and $t_c(k)$ of each congestion epoch. We therefore only need to investigate the behaviour of these quantities.

We assume that each source is informed of congestion one RTT after the queue at the bottleneck link becomes full; that is $t_c(k) - t_b(k) = 1$. Also,

$$w_i(k) \geq 0, \sum_{i=1}^{n} w_i(k) = P + \sum_{i=1}^{n} \alpha_i, \ \forall k > 0, \tag{8.1}$$

where P is the maximum number of packets which can be in transit in the network at any time; P is usually equal to $q_{max} + BT_d$ where q_{max} is the maximum queue length of the congested link, B is the service rate of the congested link in packets per second and T_d is the round-trip time when the queue is empty. At the $(k+1)$'th congestion event

$$w_i(k+1) = \beta_i w_i(k) + \alpha_i[t_c(k) - t_a(k)]. \tag{8.2}$$

and

$$t_c(k) - t_a(k) = \frac{1}{\sum_{i=1}^{n} \alpha_i}[P - \sum_{i=1}^{n} \beta_i w_i(k)] + 1. \tag{8.3}$$

Hence, it follows that

$$w_i(k+1) = \beta_i w_i(k) + \frac{\alpha_i}{\sum_{j=1}^{n} \alpha_j}[\sum_{i=1}^{n}(1-\beta_i)w_i(k)] \tag{8.4}$$

and that the dynamics of an entire network of such sources is given by

$$W(k+1) = AW(k), \tag{8.5}$$

where $W^T(k) = [w_1(k), \cdots, w_n(k)]$, and

$$A = \begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \beta_n \end{bmatrix} + \frac{1}{\sum_{j=1}^{n} \alpha_j} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{bmatrix} \begin{bmatrix} 1-\beta_1 & 1-\beta_2 & \cdots & 1-\beta_n \end{bmatrix} \tag{8.6}$$

The matrix A is a positive matrix (all the entries are positive real numbers) and it follows that the synchronised network (8.5) is a positive linear system [15]. Many results are known for positive matrices and we exploit some of these to characterise the properties of synchronised communication networks. In particular, from the viewpoint of designing communication networks the following properties are important:

[10]Note that measuring time in units of RTT results in a linear rate of increase for each of the congestion window variables between congestion events.

(i) network fairness; (ii) network convergence and responsiveness; and (iii) network throughput. While there are many interpretations of network fairness, in this paper we concentrate on window fairness. Roughly speaking, window or pipe fairness refers to a steady state situation where n sources operating $AIMD$ algorithms have an equal number of packets P/n in flight at each congestion event; convergence refers to the existence of a unique fixed point to which the network dynamics converge; responsiveness refers to the rate at which the network converges to the fixed point; and throughput efficiency refers to the objective that the network operates at close to the bottleneck-link capacity. It is shown in [149, 16] that these properties can be deduced from the network matrix A.

THEOREM 8.1. *[150, 16] Let A be defined as in Equation (8.6). Then A is a column stochastic matrix with Perron eigenvector $x_p^T = [\frac{\alpha_1}{1-\beta_1}, ..., \frac{\alpha_n}{1-\beta_n}]$ and whose eigenvalues are real and positive. Further, the network converges to a unique stationary point $W_{ss} = \Theta x_p$, where Θ is a positive constant such that the constraint (8.1) is satisfied; $\lim_{k\to\infty} W(k) = W_{ss}$; and the rate of convergence of the network to W_{ss} is bounded by the second largest eigenvalue of A.*

Comment: Networks of synchronised sources and drop-tail queues have already been the subject of several studies [72, 7, 35, 4, 66]. The novelty of our approach is that we use facts from the theory of positive matrices to analyse not only the network steady-state behaviour but also the network dynamics, directly relating the qualitative properties of synchronised networks to source and network parameters.

8.2. Models of unsynchronised network. The preceding discussion illustrates the relationship between important network properties and the eigensystem of a positive matrix. Unfortunately, the assumptions under which these results are derived, namely of source synchronisation and uniform RTT, are quite restrictive (although they may, for example, be valid in many long-distance networks [181] and are useful in the consideration of networks with multiple bottleneck links). It is therefore of great interest to extend our approach to more general network conditions. As we will see the model that we obtain shares many structural and qualitative properties of the synchronized model described above. To distinguish variables, we will from now on denote the nominal parameters of the sources used in the previous section by α_i^s, β_i^s, $i = 1, \ldots, n$. Here the index s may remind the reader that these parameters describe the *synchronized case*, as well as that these are the parameters that are chosen by each *source*.

Consider the general case of a number of sources competing for shared bandwidth in a generic dumbbell topology (where sources may have different round-trip times and drops need not be synchronised). The evolution of the *cwnd* of a typical source as a function of time, over the $k'th$ congestion epoch, is depicted in Figure 8.2. As before a number of important events may be discerned, where we now measure time in seconds, rather than units of RTT. Denote by $t_{ai}(k)$ the time at which the number of packets in flight belonging to source i is equal to $\beta_i^s w_i(k)$; $t_q(k)$ is the time at which the bottleneck queue begins to fill; $t_b(k)$ is the time at which the bottleneck queue is full; and $t_{ci}(k)$ is the time at which the i'th source is informed of congestion. In this case the evolution of the i'th congestion window variable does not evolve linearly with time after t_q seconds due to the effect of the bottleneck queue filling and the resulting variation in RTT; namely, the RTT of the i'th source increases according

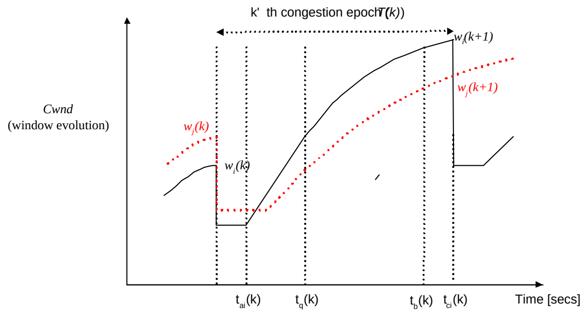

Fig. 8.2: Evolution of window size over a congestion epoch. $T(k)$ is the length of the congestion epoch in seconds.

to $RTT_i(t) = T_{d_i} + q(t)/B$ after t_q, where T_{d_i} is the RTT of source i when the bottleneck queue is empty and $0 \leq q(t) \leq q_{\max}$ denotes the number of packets in the queue. Note also that we do not assume that every source experiences a drop when congestion occurs. For example, a situation is depicted in Figure 8.2 where the i'th source experiences congestion at the end of the epoch whereas the j'th source does not.

Given these general features it is clear that the modelling task is more involved than in the synchronised case. Nonetheless, it is possible to relate $w_i(k)$ and $w_i(k+1)$ using a similar approach to the synchronised case by accounting for the effect of non-uniform RTT's and unsynchronised packet drops as follows.

(i) **Unsynchronised source drops :** Consider again the situation depicted in Figure 8.2. Here, the i'th source experiences congestion at the end of the epoch whereas the j'th source does not. This corresponds to the i'th source reducing its congestion window by the factor β_i^s after the $k+1$'th congestion event, and the j'th source not adjusting its window size at the congestion event. We therefore allow the back-off factor of the i'th source to take one of two values at the k'th congestion event.

$$\beta_i(k) \in \{\beta_i^s, 1\}, \tag{8.7}$$

corresponding to whether the source experiences a packet loss or not.

(ii) **Non-uniform RTT :** Due to the variation in round trip time, the congestion window of a flow does not evolve linearly with time over a congestion epoch. Nevertheless, we may relate $w_i(k)$ and $w_i(k+1)$ linearly by defining an average rate $\alpha_i(k)$ depending on the k'th congestion epoch:

$$\alpha_i(k) := \frac{w_i(k+1) - \beta_i(k)w(k)}{T(k)}, \tag{8.8}$$

where $T(k)$ is the duration of the k'th epoch measured in seconds. Equivalently we have

$$w_i(k+1) = \beta_i(k)w_i(k) + \alpha_i(k)T(k)\,. \tag{8.9}$$

In the case when $q_{max} << BT_{d_i}$, $i = 1, \ldots, n$, the average α_i are (almost) independent of k and given by $\alpha_i(k) \approx \alpha_i^s / T_{d_i}$ for all $k \in \mathbb{N}, i = 1, \ldots, n$. The situation when

$$\alpha_i \approx \frac{\alpha_i^s}{T_{d_i}}, \quad i = 1, \ldots, n \tag{8.10}$$

is of considerable practical importance and such networks are the principal concern of this paper. This corresponds to the case of a network whose bottleneck buffer is small compared with the delay-bandwidth product for all sources utilising the congested link. Such conditions prevail on a variety of networks; for example networks with large delay-bandwidth products, and networks where large jitter and/or latency cannot be tolerated. In view of (8.7) and (8.9) a convenient representation of the network dynamics is obtained as follows. At congestion the bottleneck link is operating at its capacity B, i.e.,

$$\sum_{i=1}^{n} \frac{w_i(k) - \alpha_i}{RTT_{i,max}} = B, \tag{8.11}$$

where $RTT_{i,max}$ is the RTT experienced by the i'th flow when the bottleneck queue is full. Note, that $RTT_{i,max}$ is independent of k. Setting $\gamma_i := (RTT_{i,max})^{-1}$ we have that

$$\sum_{i=1}^{n} \gamma_i w_i(k) = B + \sum_{i=1}^{n} \gamma_i \alpha_i \, . \tag{8.12}$$

By interpreting (8.12) at $k+1$ and inserting (8.9) for $w_i(k+1)$ it follows furthermore that

$$\sum_{i=1}^{n} \gamma_i \beta_i(k) w_i(k) + \gamma_i \alpha_i T(k) = B + \sum_{i=1}^{n} \gamma_i \alpha_i \, . \tag{8.13}$$

Using (8.12) again it follows that

$$T(k) = \frac{1}{\sum_{i=1}^{n} \gamma_i \alpha_i} \left(\sum_{i=1}^{n} \gamma_i (1 - \beta_i(k)) w_i(k) \right) . \tag{8.14}$$

Inserting this expression into (8.9) we finally obtain

$$\begin{aligned} w_i(k+1) &= \beta_i(k) w_i(k) \\ &\quad + \frac{\alpha_i}{\sum_{j=1}^{n} \gamma_j \alpha_j} \left(\sum_{j=1}^{n} \gamma_j (1 - \beta_j(k)) w_j(k) \right) \end{aligned} \tag{8.15}$$

and the dynamics of the entire network of sources at the k-th congestion event are described by

$$W(k+1) = A(k) W(k), \; A(k) \in \{A_1, ..., A_m\}. \tag{8.16}$$

where

$$A(k) = \begin{bmatrix} \beta_1(k) & 0 & \cdots & 0 \\ 0 & \beta_2(k) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \beta_n(k) \end{bmatrix} + \frac{1}{\sum_{j=1}^{n} \gamma_j \alpha_j} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{bmatrix} [\gamma_1(1-\beta_1(k)), \ldots, \gamma_n(1-$$

$\beta_n(k))]$

and where $\beta_i(k)$ is either 1 or β_i^s. The non-negative matrices $A_2, .., A_m$ are constructed by taking the matrix A_1,

$$A_1 = \begin{bmatrix} \beta_1^s & 0 & \cdots & 0 \\ 0 & \beta_2^s & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \beta_n^s \end{bmatrix} + \frac{1}{\sum_{j=1}^n \gamma_j \alpha_j} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{bmatrix} \begin{bmatrix} \gamma_1(1-\beta_1^s), & \ldots, & \gamma_n(1-\beta_n^s) \end{bmatrix}$$

and setting some, but not all, of the β_i to 1. This gives rise to $m = 2^n - 1$ matrices associated with the system (8.16) that correspond to the different combinations of source drops that are possible. We denote the set of these matrices by $\mathcal{A}$.

Finally we note that another, sometimes very useful, representation of the network dynamics can be obtained by considering the evolution of scaled window sizes at congestion; namely, by considering the evolution of $W_\gamma^T(k) = [\gamma_1 w_1(k) \ , \gamma_2 w_2(k), ..., \gamma_n w_n(k)]$. Here one obtains the following description of the network dynamics:

$$W_\gamma(k+1) = \bar{A}(k) W_\gamma(k) \tag{8.17}$$

with $\bar{A}(k) \in \bar{\mathcal{A}} = \{\bar{A}_1, ..., \bar{A}_m\}$, $m = 2^n - 1$ and where the $\bar{A}_i$ are obtained by the similarity transformation associated with the change of variables; in particular,

$$\bar{A}_1 = \begin{bmatrix} \beta_1^s & 0 & \cdots & 0 \\ 0 & \beta_2^s & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \beta_n^s \end{bmatrix} + \frac{1}{\sum_{j=1}^n \gamma_j \alpha_j} \begin{bmatrix} \gamma_1 \alpha_1 \\ \gamma_2 \alpha_2 \\ \cdots \\ \gamma_n \alpha_n \end{bmatrix} \begin{bmatrix} 1-\beta_1^s & 1-\beta_2^s & \cdots & 1-\beta_n^s \end{bmatrix}.$$

As before the non-negative matrices $\bar{A}_2, .., \bar{A}_m$ are constructed by taking the matrix $\bar{A}_1$ and setting some, but not all, of the β_i^s to 1. All of the matrices in the set $\bar{\mathcal{A}}$ are now column stochastic; for convenience we use this representation of the network dynamics to prove the main mathematical results presented in this paper.

Comment : Before proceeding we note that networks of unsynchronised sources have also been the subject of wide study in the TCP community: see [82, 97, 104, 88, 109, 69, 170, 79, 36, 70, 71, 70] and the accompanying references for further details. While most of this work has concentrated on developing and analysing TCP models that are based upon fluid analogies, several authors have recently developed hybrid systems models of networks with a single bottleneck link which employ AIMD congestion control mechanisms: most notably by Hespanha [65] and Baccelli and Hong [6].

8.3. Main results. The unsynchronised dynamics introduced in the previous section are rather more complex than those of synchronised networks. In particular, their dynamics is described by products of stochastic matrices. However, by working in terms of average properties, it is still possible to gain considerable insight into the dynamics in the unsynchronised case.

It follows from (8.16) that $W(k) = \Pi(k)W(0)$, where $\Pi(k) = A(k)A(k-1)....A(0)$. Consequently, the behaviour of $W(k)$, as well as the network fairness and convergence properties, are governed by the properties of the matrix product $\Pi(k)$. The objective of this section is to analyse the average behaviour of $\Pi(k)$ with a view to making concrete statements about these network properties. To facilitate analytical tractability we will make two mild simplifying assumptions.

ASSUMPTION 8.2. *The probability that $A(k) = A_i$ in (8.16) is independent of k and equals ρ_i.*

In other words Assumption 8.2 says that the probability that the network dynamics are described by $W(k+1) = A(k)W(k), A(k) = A_i$ over the k'th congestion epoch is ρ_i and that the random variables $A(k), k \in \mathbb{N}$ are independent and identically distributed (i.i.d.).

Given the probabilities ρ_i for $i \in \{1, ..., 2^n - 1\}$, one may then define the probability Λ_j that source j experiences a backoff at the k'th congestion event as follows:

$$\Lambda_j = \sum \rho_i \, ,$$

where the summation is taken over those i which correspond to a matrix in which the j'th source sees a drop. Or to put it another way, the summation is over those indices i for which the matrix A_i is defined with a value of $\beta_j \neq 1$.

ASSUMPTION 8.3. *We assume that $\Lambda_j > 0$ for all $j \in \{1, ..., n\}$.*

Simply stated, Assumption 8.3 states that almost surely all flows must see a drop at some time (provided that they persist for a long enough time).

Comment: A consequence of the above assumptions is that the probability that source j experiences a drop at the k'th congestion event is not independent of the other sources. For example, if the first $n-1$ sources do not see a drop then this implies that source n must see a drop (in accordance with the usual notion of a congestion event, we require at least one flow to see a drop at each congestion event). Hence, the events cannot be independent.

Under the foregoing assumptions we have the following key result.

THEOREM 8.4. *Consider the stochastic system defined in the above preamble. Let $\Pi(k)$ be the random matrix product arising from the evolution of the first k steps of this system:*

$$\Pi(k) = A(k)A(k-1)....A(0).$$

Then, the expectation of $\Pi(k)$ is given by

$$E(\Pi(k)) = (\sum_{i=1}^{m} \rho_i A_i)^k; \tag{8.18}$$

and the asymptotic behaviour of $E(\Pi(k))$ satisfies

$$\lim_{k\to\infty} E(\Pi(k)) = x_p y_p^T, \tag{8.19}$$

where the vector x_p *is given by* $x_p^T = \Theta(\frac{\alpha_1}{\Lambda_1(1-\beta_1)}, \frac{\alpha_2}{\Lambda_2(1-\beta_2)}, ..., \frac{\alpha_n}{\Lambda_n(1-\beta_n)})$, $y_p^T = (\gamma_1, ..., \gamma_n)$. *Here* $\Theta \in \mathbb{R}$ *is chosen such that equation (8.12) is satisfied if* w_i *is replaced by* $x_{pi} = \Theta\alpha_i/(\Lambda_i(1-\beta_i))$.

Theorem 8.4 characterises the ensemble average behaviour of the congestion variable vector $W(k)$. The congestion variable vector of a network of flows starting from initial condition $W(0)$ and evolving for k congestion epochs is given by $W(k) = \Pi(k)W(0)$. The average window vector over many repetitions is given by $E(\Pi(k))W(0)$. Theorem 8.4 provides an expression for calculating this average in terms of the network parameters and the probabilities ρ_i. The following facts follow immediately from Theorem 8.4.

(i) **Convergence:** The congestion window vector $W(k)$ converges, on average, to the unique value $\bar{W}_{ss} = \Theta x_p$ where Θ is a positive constant such that the constraint (8.12) is satisfied. When the $\Lambda_i, i = 1, ..., n$ are equal, x_p is identical to the Perron eigenvector obtained in the case of synchronised networks; that is, the ensemble average in the unsynchronised case is identical to the fixed point in the deterministic situation where packet drops are synchronised.

(ii) **Fairness:** Window fairness is achieved, on average, when the vector x_p is a scalar multiple of the vector $[1, ..., 1]$; that is, when the ratio $\frac{\alpha_i}{\Lambda_i(1-\beta_i)}$ does not depend on i. Observe that unlike in the synchronised case, fairness now depends upon on the relative drop probability of each flow. When the flows have equal drop probability Λ_i then the foregoing fairness condition is identical to that in the synchronised case.

(iii) **Network responsiveness:** The magnitude of the second largest eigenvalue λ_2 of the matrix $\sum_{i=1}^{m} \rho_i A_i$ bounds the convergence properties of the network. The network rise-time when measured in number of congestion epochs is, on average, bounded by $n_r = \log(0.95)/\log(\lambda_2)$.

(iv) **Network throughput :** Theorem 8.4 is concerned with the expected behaviour of the source congestion windows at the k'th congestion epoch. For k sufficiently large the expected throughput before backoff can be approximated as $\sum_{i=1}^{k} \frac{\alpha_i}{\Lambda_i(1-\beta_i)RTT_{i,max}}$. The worst case throughput after backoff (which occurs when the queue is on average empty after backoff) is approximately $\sum_{i=1}^{k} \frac{\alpha_i}{\Lambda_i(1-\beta_i)RTT_{i,min}}$. An immediate consequence of this observation is that the bottleneck link is guaranteed to be operating at capacity (on average) for k large enough if $\beta_i = \frac{RTT_{i,min}}{RTT_{i,max}}$.

8.4. The reachable set. The evolution of $W(k)$ is governed by the stochastic equation:

$$W(k+1) = A(k)W(k), \; A(k) \in \mathcal{A}; \tag{8.20}$$

where $A(k)$ is a non-negative column stochastic matrix. The asymptotic behaviour of $W(k)$ is therefore governed by infinite products of matrices drawn stochastically from the set $\mathcal{A}$. Although it is beyond the scope of this present paper we note that, under the assumptions of Section 3, all such products converge *almost surely* to a subsect of the rank-1 idempotent non-negative column stochastic matrices [151]. The image of these matrices, projected onto the simplex $\sum_{i=1}^{n} \gamma_i w_i(k) = B$, defines the set of reachable states for the asymptotic process. It is known that such sets can be very complicated; even Fractal (see Chapter 11 in [62]). For infinite products associated with AIMD networks this appears to be the case; an example of such an image set projected onto the simplex $\sum_{i=1}^{n} \gamma_i w_i(k) = B$ is depicted in Figure 8.4.

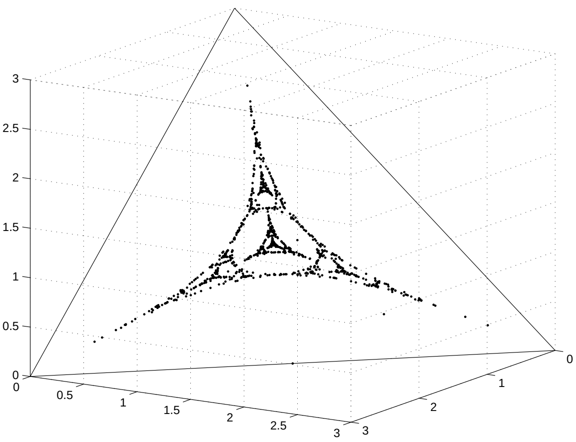

Fig. 8.3: Approximation of asymptotic image set for network of three flows. $\alpha_i = 1, \beta_i = 0.25$ for all i.

9. Some open problems and future directions. In this section, we shall briefly review and summarise some of the open questions arising from the results and issues discussed throughout the paper. Broadly speaking, the major open problems in the stability theory of switched linear systems can be divided into three categories, corresponding to the first three problems discussed at the end of Section 1; namely, the problem of stability under arbitrary switching, the dwell-time problem, and the problem of determining stabilising switching signals.

(i) In the context of stability under arbitrary switching, Theorem 4.3 and Theorem 4.4 provide simple conditions for CQLF existence that are related to the dynamics of switched linear systems via Theorem 4.18. While Theorem 4.7 gives necessary and sufficient conditions for a general family of stable LTI systems to have a CQLF, the conditions described by this result are extremely complicated and difficult to check, even for the case of a pair of third order systems. Hence, an open problem of some interest is to determine system classes, such as the class of second order systems or pairs of systems with system matrices differing by a rank one matrix, for which simple conditions for CQLF existence can be given. In this context, the work reported in [153] should be noted. In this paper, it was shown that the system classes covered by Theorem 4.3 and Theorem 4.4 can be treated within a common framework provided by the main result of [159]. This may provide some insights as to how to obtain further system classes for which similarly simple conditions for CQLF existence can be derived.

(ii) A closely related problem to that described above is that of determining classes of switched linear systems for which CQLF existence is equivalent to exponential stability under arbitrary switching. Two examples of such system classes have been described in Section 4 above, and for such systems, the problem of determining stability under arbitrary switching is simplified considerably.

(iii) Theorem 4.22 gives a simple spectral condition for the existence of a common piecewise linear Lyapunov function (PLLF) for a pair of stable second order LTI systems, and to date there are very few results of this kind available in the literature. This gives rise to the question of whether or not it is possible to extend this result to higher dimensional systems.

(iv) For the class of positive switched linear systems, as mentioned in Section 5, it is natural to consider co-positive Lyapunov functions. In particular, given that the trajectories of positive systems are constrained to remain within the non-negative orthant, such Lyapunov functions may lead to less conservative stability criteria than those obtained through requiring CQLF existence. This raises the problem of determining verifiable conditions for common co-positive Lyapunov function existence for families of positive LTI systems.

(v) Apart from the above problems on stability for arbitrary switching signals, the important issue of determining non-conservative estimates of the *dwell-time* for constrained switching regimes is still unresolved.

(vi) On the question of determining stabilising switching signals for unstable constituent systems, the work of Feron, Peleties, de Carlo and others discussed in Section 5 provides sufficient conditions for quadratic stabilisation laws. Also, it is known that for the case of two constituent systems, the conditions discussed above are both necessary and sufficient for the existence of a quadratically stabilising switching rule. To the best of the authors' knowledge, necessary and sufficient conditions for the existence of a general (not necessarily quadratic) stabilising switching law are not known. Some results related to this topic were previously mentioned in Section 5, where appropriate references were also given.

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