Stability and D-stability for Switched Positive Systems*

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Abstract. We consider a number of questions pertaining to the stability of positive switched linear systems. Recent results on common quadratic, diagonal, and copositive Lyapunov function existence are reviewed and their connection to the stability properties of switched positive linear systems is highlighted. We also generalise the concept of D-stability to positive switched linear systems and present some preliminary results on this topic.

1 Introduction

While the stability properties of positive linear time-invariant (LTI) systems have been thoroughly investigated and are now completely understood, the theory for nonlinear, uncertain and time-varying positive systems is considerably less well-developed. In fact, many natural and fundamental questions on the stability of such systems remain unanswered. It is clear that for many practical applications there is a need to extend the theory for positive LTI systems to broader and more realistic system classes incorporating nonlinearities and time-varying parameters.

Our principal focus in the present paper is on the stability properties of switched positive linear systems [2]. In particular, we review recent work on the stability of these systems, highlighting the connection between various notions of stability and the existence of corresponding types of common Lyapunov function. We also consider an extension of the concept of D-stability to positive switched linear systems, present some preliminary results for this question and highlight some directions for future research.

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2 Notation and Background

Throughout, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^n$ stands for the vector space of all n-tuples of real numbers and $\mathbb{R}^{m\times n}$ is the space of $m \times n$ matrices with real entries. For x in $\mathbb{R}^n$, x_i denotes the i^{th} component of x, and the notation $x \succ 0$ ($x \succeq 0$) means that $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. The notations $x \prec 0$ and $x \preceq 0$ are defined in the obvious manner.

We write A^T for the transpose of $A \in \mathbb{R}^{n\times n}$ and for a symmetric P in $\mathbb{R}^{n\times n}$ the notation $P > 0$ means that the matrix P is positive definite.

Throughout the paper, in an abuse of notation, for LTI systems we shall use the term stability to denote asymptotic stability. Also, when referring to switched linear systems, stability shall be used to denote asymptotic stability under arbitrary switching [2].

For a positive LTI system

$$\dot{x}(t) = Ax(t) \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix (meaning that the off-diagonal entries of A are non-negative), the equivalences we collect in the following result are well known.

Proposition 1. *[4] Let $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix. The following statements are equivalent:*

(i) The LTI system (1) is stable;
(ii) A is Hurwitz, meaning that its eigenvalues lie in the open left half plane;
(iii) There exists $P > 0$ such that $A^TP + PA < 0$;
(iv) There exists a diagonal matrix $D > 0$ such that $A^TD + DA < 0$;
(v) There exists a vector $v \succ 0$ in $\mathbb{R}^n$ with $Av \prec 0$;
(vi) For any diagonal matrix $D > 0$, the system $\dot{x}(t) = DAx(t)$ is stable.

While the equivalence of (i), (ii) and (iii) in the previous result also holds for any LTI system, properties (iv), (v) and (vi) are specific to positive LTI systems.

The property described in (vi) is known as D-stability and establishes that stability of positive LTI systems is robust with respect to parametric uncertainties given by diagonal scaling. Later in the paper, we shall be concerned with investigating the connection between concepts similar to those in (v) and (vi) for switched positive linear systems. Before this, in the following section, we shall review some recent work on the stability of switched positive linear systems.

3 Lyapunov Functions and Stability for Switched Positive Linear Systems

It is well known that a switched positive linear system of the form

$$\dot{x}(t) = A(t)x(t) \quad A(t) \in \{A_1, A_2\} \tag{2}$$

can be unstable for certain choices of switching sequence even when the individual system matrices A_1, A_2 are asymptotically stable [2]. This observation has led to great interest in the stability of such systems under arbitrary switching regimes. A key result in this connection is that stability of (2) is equivalent to the existence of a common Lyapunov function for the individual component LTI systems [2]. In the light of Proposition 1, three classes of Lyapunov function naturally suggest themselves for positive switched linear systems:

(i) *Common Quadratic Lyapunov Functions (CQLFs)*: $V(x) = x^T P x$ where $P = P^T > 0$ and $A_i^T P + P A_i < 0$ for $i = 1, 2$;
(ii) *Common Diagonal Lyapunov Functions (CDLFs)*: $V(x) = x^T D x$ where $D = \text{diag}(d_1, \ldots, d_n)$, $D > 0$ and $A_i^T D + D A_i < 0$ for $i = 1, 2$;
(iii) *Common Linear Copositive Lyapunov Functions (CLLFs)*: $V(x) = v^T x$ where $v \succ 0$ and $A_i^T v \prec 0$ for $i = 1, 2$;

In the interests of brevity, we shall abuse notation slightly and say that the matrices A_1, A_2 have a CQLF, CDLF or CLLF rather than always referring to the associated LTI systems.

Recall the following well-known necessary condition for the stability of positive switched linear systems (in fact this is a necessary condition for stability for general switched linear systems)[2].

Lemma 1. *Let $A_1, A_2 \in \mathbb{R}^{n \times n}$ be Metzler and Hurwitz. Suppose that the associated switched positive linear system (2) is stable. Then for any real $\gamma \geq 0$, $A_1 + \gamma A_2$ is Hurwitz.*

(i) Common Quadratic Lyapunov Functions (CQLFs)

In [1], the relationship between the existence of CQLFs, the stability of all matrices of the form $A_1 + \gamma A_2$ with $\gamma \geq 0$, and the stability of the system (2) was considered. For 2-dimensional systems, the following result was established.

Theorem 1. *Let $A_1, A_2 \in \mathbb{R}^{2 \times 2}$ be Hurwitz and Metzler. Then the following statements are equivalent:*

(a) A_1, A_2 have a CQLF;
(b) The switched system (2) is stable;
(c) $A_1 + \gamma A_2$ is Hurwitz for all real $\gamma \geq 0$.

Further, the equivalence of (b) and (c) can be extended to the case of an arbitrary finite number of positive LTI systems. Formally, it was shown in [1] that given Metzler, Hurwitz matrices $A_1, \ldots, A_k$ in $\mathbb{R}^{2 \times 2}$, the switched system $\dot{x}(t) = A(t)x(t)$, $A(t) \in \{A_1, \ldots, A_k\}$ is stable if and only if $A_1 + \gamma_2 A_2 + \cdots + \gamma_k A_k$ is Hurwitz for all real $\gamma_2 \geq 0, \ldots \gamma_k \geq 0$.

The equivalence of (a), (b) and (c) fails immediately for 3-dimensional systems. Moreover, the equivalence of (b) and (c) is not true for arbitrary dimensions[1]. In fact, in a very recent paper [3], a 3-dimensional example of an unstable switched system for which $A_1 + \gamma A_2$ was Hurwitz for all $\gamma \geq 0$ was explicitly described. In connection with CQLF existence and the stability of positive switched linear systems, it has been shown in [11] for 2 and 3 dimensional systems that if $\text{rank}(A_2 - A_1) = 1$, and A_2, A_1 are both Hurwitz, then the associated LTI systems always possess a CQLF and the switched linear system (2) is stable.

(ii) Common Diagonal Lyapunov Functions (CDLFs)

As stable positive LTI systems have diagonal Lyapunov functions, it is natural to ask under what conditions families of such systems will possess a common diagonal Lyapunov function. In the paper [9], the following result was derived for systems with irreducible system matrices (for the definition of irreducible matrices, see [5]).

Theorem 2. *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be irreducible, Metzler and Hurwitz. A_1, A_2 have a CDLF if and only if $A_1 + DA_2D$ is Hurwitz for all diagonal matrices $D > 0$.*

The above result allows us to establish a connection between the existence of a CDLF and a form of robust stability for switched positive linear systems. First of all, note that for A_1, A_2 irreducible, Metzler and Hurwitz, Theorem 2 shows that if A_1, A_2 have a CDLF, then so do $D_1A_1D_1$, $D_2A_2D_2$ for any choice of diagonal matrices $D_1 > 0$, $D_2 > 0$. Hence the existence of a CDLF guarantees the stability of the positive switched linear system

$$\dot{x}(t) = A(t)x(t) \quad A(t) \in \{D_1A_1D_1, D_2A_2D_2\} \tag{3}$$

for any diagonal matrices $D_1 > 0$, $D_2 > 0$.

Conversely, if A_1, A_2 do not have a CDLF, then it follows from Theorem 2 that there is some diagonal matrix $D > 0$ such that $A_1 + DA_2D$ is not Hurwitz. This then immediately implies from Lemma 1 that the switched system (3) is not stable with $D_1 = I$, and $D_2 = D$. This discussion establishes the following result.

Proposition 2. *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be irreducible, Metzler and Hurwitz. The switched system (3) is stable for any diagonal matrices $D_1 > 0$, $D_2 > 0$ if and only if A_1, A_2 have a CDLF.*

(iii) Common Linear Copositive Lyapunov Functions (CLLFs)

It is also possible to establish the stability of positive switched linear systems using copositive linear Lyapunov functions. As noted in [6], traditional Lyapunov functions may give conservative stability conditions for positive switched systems as they fail to take into account that trajectories are naturally constrained to the positive orthant. The existence of a CLLF for a pair of Metzler, Hurwitz matrices A_1, A_2 is equivalent to the feasibility of the linear inequalities $v \succ 0$, $A_1^T v \prec 0$, $A_2^T v \prec 0$. In the following sections, we shall investigate closely the connection between the related but distinct question of the feasibility of $v \succ 0$, $A_1 v \prec 0$, $A_2 v \prec 0$ and an extension of the concept of D-stability for switched positive linear systems.

Note that an algebraic condition for CLLF existence was derived in [10]. In the interests of brevity, we shall not explicitly state this result here but rather state the following technical result which follows from Theorem 3.1 in that paper. This fact shall prove useful in our later discussion.

Lemma 2. *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz. Suppose that there is no non-zero $v \succeq 0$ in $\mathbb{R}^n$ with $A_1 v \preceq 0$, $A_2 \preceq 0$. Then there is some diagonal $D > 0$ such that $A_1 + DA_2$ is singular.*

4 Switched Positive Linear Systems and D-Stability: The 2-d Case

In this and the following section, we shall investigate the following generalisation of the notion of D-stability to positive switched linear systems.

Definition 1. *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz. The associated switched positive linear system (2) is said to be D-stable if for any diagonal matrices $D_1, D_2 \in \mathbb{R}^{n\times n}$ with $D_1 > 0$, $D_2 > 0$, the system*

$$\dot{x}(t) = A(t)x(t) \qquad A(t) \in \{D_1 A_1, D_2 A_2\} \tag{4}$$

is stable.

For positive LTI systems, Proposition 1 shows that stability and D-stability are equivalent. Our first observation, in Example 1, is to note that this equivalence is not true in the switched case. First of all, we note the following simple necessary condition for D-stability, which follows immediately from Lemma 1.

Lemma 3. *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz. Suppose that the associated switched positive linear system (2) is D-stable. Then for any diagonal matrix $D > 0$, $A_1 + DA_2$ is Hurwitz.*

Example 1. Consider the Metzler, Hurwitz matrices in $\mathbb{R}^{2\times 2}$

$$A_1 = \begin{pmatrix} -2 & 0 \\ 1 & -4 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix}$$

It is straightforward to verify that $A_1 + \gamma A_2$ is Hurwitz for all $\gamma \geq 0$. Hence by Theorem 1, the associated switched system is stable. On the other hand, choosing

$$D = \begin{pmatrix} 20 & 0 \\ 0 & 0.5 \end{pmatrix}$$

it is easily verified that $A_1 + DA_2$ is not Hurwitz. Hence by Lemma 3 the associated switched system is not D-stable.

The above example illustrates that for switched positive linear systems, the concepts of stability and D-stability are not equivalent, in contrast to the LTI system case. In the following result, we show that the necessary condition given in Lemma 3 is also sufficient for D-stability for 2-dimensional systems.

Theorem 3. *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Metzler and Hurwitz. The positive switched linear system (2) is D-stable if and only if A_1+DA_2 is Hurwitz for all diagonal matrices $D > 0$.*

Proof: Lemma 3 has already established the necessity of this condition. For sufficiency let $D_1 > 0$, $D_2 > 0$ be diagonal matrices and let $\gamma \geq 0$ be any non-negative real number. By hypothesis, $A_1 + \gamma D_1^{-1} D_2 A_2$ is Hurwitz for $\gamma > 0$ and it is trivially true for $\gamma = 0$. However, this matrix is also Metzler and hence by point (vi) of Proposition 1, $D_1A_1 + \gamma D_2 A_2 = D_1(A_1 + \gamma D_1^{-1} D_2 A_2)$ is also Hurwitz. It now follows immediately from Theorem 1 that the switched system (4) associated with D_1A_1, D_2A_2 is stable. As this is true for any diagonal $D_1 > 0$, $D_2 > 0$, the system (2) is D-stable.

The next result establishes a connection between the existence of a common solution to the inequalities $v \succ 0$, $A_i v \prec 0$ for $i = 1, 2$ and D-stability for (2).

Corollary 1. *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Metzler and Hurwitz. Then:*

(i) If there is some $v \succ 0$ with $A_1 v \prec 0$, $A_2 v \prec 0$ then the system (2) is D-stable;
(ii) If (2) is D-stable then there exists some non-zero $v \succeq 0$ with $A_1 v \preceq 0$, $A_2 v \preceq 0$.

Proof: (i) Suppose there is some $v \succ 0$ with $A_i v \prec 0$ for $i = 1, 2$. Then for any diagonal $D > 0$, $DA_2 v \prec 0$ and $(A_1 + DA_2)v \prec 0$. Moreover, $A_1 + DA_2$ is Metzler. Hence, from point (v) of Proposition 1, it follows that $A_1 + DA_2$ is Hurwitz. Theorem 3 now implies that the switched system (2) is D-stable.

(ii) If (2) is D-stable, then Theorem 3 implies that $A_1 + DA_2$ is Hurwitz for all diagonal $D > 0$. It now follows from Lemma 2 that there must exist some non-zero $v \succeq 0$ with $A_1 v \preceq 0$, $A_2 v \preceq 0$.

Note that the sufficient condition for D-stability presented in Corollary 1 is not necessary as demonstrated by the following example.

Example 2. Consider the Metzler, Hurwitz matrices A_1, A_2 given by:

$$A_1 = \begin{pmatrix} -2 & 1 \\ 2 & -2 \end{pmatrix}, A_2 = \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix}$$

Using Theorem 4.1 of [10] it is straightforward to show that there is no vector $v \succ 0$ with $A_1 v \prec 0$, $A_2 v \prec 0$. On the other hand, it can be verified algebraically that for any diagonal $D > 0$, $A_1 + DA_2$ is Hurwitz and hence the switched system (2) is D-stable by Theorem 3.

5 D-Stability in Higher Dimensions

In this section, we highlight a result extending Corollary 1 to higher dimensional positive switched systems. We only provide an outline of the proof here due to space limitations.

Theorem 4. *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz. Then:*

(i) If there is some $v \succ 0$ with $A_1 v \prec 0$, $A_2 v \prec 0$ then the system (2) is D-stable;
(ii) If (2) is D-stable then there exists some non-zero $v \succeq 0$ with $A_1 v \preceq 0$, $A_2 v \preceq 0$.

Proof Outline: The key to proving (i) is to show that the existence of such a v is sufficient for the stability of the switched system (2). Once this is established, the result follows immediately as $D_1 A_1 v \prec 0$, $D_2 A_2 v \prec 0$ for any diagonal $D_1 > 0$, $D_2 > 0$.

To show that the existence of $v \succ 0$ satisfying $A_1 v \prec 0$, $A_2 v \prec 0$ implies the system (2) is stable, we first show that any trajectory starting from the initial condition given by v converges asymptotically to the origin. We can then combine this fact with the monotonicity properties of positive LTI systems [4] to conclude that (2) is stable. The result given by (ii) follows immediately from Lemma 3 and Lemma 2.

Note that the result given by (i) provides a condition for stability of (2) that is distinct although related to the condition given by CLLF existence.

6 Concluding remarks

In this paper, we have discussed a number of problems relating to the stability properties of switched positive linear systems. In particular, we have reviewed recent work on common quadratic, copositive and diagonal Lyapunov functions for these systems and on the relationship between the existence of such functions and various notions of stability for switched positive systems. We have also discussed the notion of D-stability for positive switched systems and presented separate necessary and sufficient conditions for D-stability for n-dimensional systems. More detailed and complete results have also been given for 2-dimensional systems.

A number of interesting directions for future research emerge from the work described here. For instance, it would be interesting to investigate the possibility of Theorem 3 extending to dimensions higher than 2, even for some restricted system class. Also, the question of whether stability and D-stability are equivalent for any subclass of positive switched linear systems arises naturally. It is straightforward to show that this is true for upper (or lower) triangular positive systems, for example, but are there any more interesting such classes?

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