Stability and Positivity of Equilibria for Subhomogeneous Cooperative Systems

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Abstract

Building on recent work on homogeneous cooperative systems, we extend results concerning stability of such systems to subhomogeneous systems. We also consider subhomogeneous cooperative systems with constant input, and relate the global asymptotic stability of the unforced system to the existence and stability of positive equilibria for the system with input.

Keywords: Nonlinear Systems; Cooperative Systems; Positive Systems; Subhomogeneity; Positivity of Equilibria.

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1. Introduction

Dynamical systems leaving the non-negative orthant invariant are of great practical importance due to their applications in Biology, Ecology, Economics, Communications and elsewhere. Systems of this type are known as positive systems and have been studied extensively in the literature. In particular, the theory of linear time-invariant (LTI) positive systems is now well developed and much recent work has been directed towards extending this theory to broader and more realistic system classes. For instance, the authors of [17] have considered positive systems defined by integro-differential equations, while the properties of switched positive systems have been studied in [5] [6] [10]. As many of the applications of positive systems give rise to nonlinear systems, it is natural to look for extensions of the theory of

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positive LTI systems to classes of nonlinear positive systems. In this paper, we consider a particular class of nonlinear positive systems, subhomogeneous cooperative systems, and derive results on their stability that echo the properties of positive LTI systems.

We first consider an extension of the following property of positive LTI systems. If the positive LTI system $\dot{x} = Ax$ is globally asymptotically stable, then so is the system $\dot{x} = DAx$ for any diagonal matrix D with positive diagonal entries. This property is commonly referred to as D-stability. It was shown in [15] [2] that a nonlinear analogue of D-stability property also holds for a significant class of nonlinear positive systems; namely cooperative systems that are homogeneous [2]. We shall show here that these results can be further extended to subhomogeneous cooperative systems (we define these formally in the following two sections). It should be pointed out that the methods used in [15] [2] relied heavily on an extension of the Perron-Frobenius theorem given in [1]. In contrast, our approach here makes use of the Knaster-Kuratowski-Mazurkiewicz (KKM) theorem [9] and is inspired by recent applications of this result to small-gain conditions and input-to-state stability in [3] [19].

While we formally define both homogeneous and subhomogeneous systems later, it is appropriate to point out some reasons why the class of subhomogeneous systems is of interest. First of all, subhomogeneous systems can include terms such as $\frac{x^\tau}{a+x^\tau}$ for $a > 0$, $\tau > 0$, which arise frequently in models of biochemical reaction networks [16]. From a more theoretical point of view, the class of homogeneous systems is not closed under the addition of positive constants. More formally, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous, then it is not true that $x \to f(x) + b$ is homogeneous for positive vectors b. However, if $f(.)$ is subhomogeneous, then so is $f(.) + b$ for any positive b. This property is of relevance to the second question we consider.

In [14] the stability properties of the homogeneous cooperative system $\dot{x} = f(x)$ were related to the existence and stability of positive equilibria for the associated system $\dot{x} = f(x) + b$ where b is a positive vector. Specifically, it was shown that if f is an irreducible vector field (defined in Section 2) and $\dot{x} = f(x)$ has a globally asymptotically stable (GAS) equilibrium at the origin (defined in Section 2), then for every positive vector b there exists a globally asymptotically stable equilibrium $\bar{x}$ of $\dot{x} = f(x) + b$. We show that this same result extends naturally to subhomogeneous systems. Again our analysis does not rely on the extension of the Perron-Frobenius theorem presented in [1] but uses the KKM theorem to establish stability.

The layout of the paper is as follows. In Section 2, we introduce notation as well as definitions and results that are needed throughout the paper. In Section 3, we establish some fundamental technical facts concerning subhomogeneous cooperative systems including an extension of Euler's formula to this setting. The main contributions of the paper are contained in Section 4 and Section 5. In Section 4, we provide two results extending the D-stability property to subhomogeneous cooperative systems; one covers systems with an equilibrium at the origin while the other is concerned with an equilibrium in the interior of the positive orthant. In Section 5, we extend the results of [14] on the existence of positive equilibria to subhomogeneous systems. Finally, in Section 6 we present our concluding remarks.

2. Preliminaries

Throughout the paper, $\mathbb{R}$ and $\mathbb{R}^n$ denote the field of real numbers and the vector space of all n-tuples of real numbers, respectively. $\mathbb{R}^{n\times n}$ denotes the space of $n \times n$ matrices with real entries. $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$. The interior of $\mathbb{R}^n_+$ is denoted by $\mathrm{int}\,(\mathbb{R}^n_+)$ and its boundary by $\mathrm{bd}\,(\mathbb{R}^n_+) := \mathbb{R}^n_+ \backslash \mathrm{int}\,(\mathbb{R}^n_+)$. For vectors $x, y \in \mathbb{R}^n$, we write: $x \geq y$ if $x_i \geq y_i$ for $1 \leq i \leq n$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x_i > y_i, 1 \leq i \leq n$. For $x \in \mathbb{R}^n$ and $i = 1, \dots, n$, x_i denotes the i^{th} coordinate of x. Similarly, for $A \in \mathbb{R}^{n\times n}$, a_{ij} denotes the $(i, j)^{th}$ entry of A. Also, for $x \in \mathbb{R}^n$, $\mathrm{diag}\,(x)$ is the $n \times n$ diagonal matrix in which $d_{ii} = x_i$.

For $A \in \mathbb{R}^{n\times n}$, we denote the *spectrum* of A by $\sigma(A)$. Also, the notation $\mu(A)$ denotes the *spectral abscissa* of A which is defined as follows:

$$\mu(A) := \max\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\}.$$

A real $n \times n$ matrix $A = (a_{ij})$ is *Metzler* if and only if its off-diagonal entries a_{ij}, $\forall i \neq j$ are nonnegative. The matrix A is *irreducible* if and only if for every nonempty proper subset K of $N := \{1, \cdots, n\}$, there exists an $i \in K$, $j \in N \backslash K$ such that $a_{ij} \neq 0$. When A is not irreducible, it is *reducible*.

The next result concerning Metzler matrices is standard [8] and follows from the Perron-Frobenius theorem.

Theorem 2.1. *Let $A \in \mathbb{R}^{n\times n}$ be Metzler. Then $\mu(A) \in \sigma(A)$. In addition, if A is also irreducible then there exist vectors $v \gg 0$, $w \gg 0$ such that $v^T A = \mu(A)v^T$, $Aw = \mu(A)w$.*

The following notation and assumptions are adopted throughout the paper. $\mathcal{W}$ is a neighbourhood of $\mathbb{R}^n_+$ and $f : \mathcal{W} \to \mathbb{R}^n$ is a C^1 vector field on $\mathcal{W}$.

We are concerned with the system:

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \tag{1}$$

The *forward solution* of (1) with initial condition $x_0 \in \mathcal{W}$ at $t = 0$ is denoted as $x(t, x_0)$ and is defined on the maximal forward interval of existence $\mathcal{I}_{x_0} := [0, T_{max}(x_0))$. A set $D \subset \mathbb{R}^n$ is called *forward invariant* if and only if for all $x_0 \in D$, $x(t, x_0) \in D$ for all $t \in \mathcal{I}_{x_0}$.

We shall be exclusively concerned with *positive systems*. The system (1) is positive if $\mathbb{R}^n_+$ is forward invariant. Formally, it means that $x_0 \geq 0$ implies $x(t, x_0) \geq 0$ for all $t \geq 0$. It is intuitively clear and shown in [13] that given the uniqueness of solutions of the system (1) the following property is necessary and sufficient for positivity of the system:

$$\textbf{P:}\ \forall x \in \mathrm{bd}\,(\mathbb{R}^n_+) : x_i = 0 \Rightarrow f_i(x) \geq 0$$

As is standard, we say that the C^1 vector field $f : \mathcal{W} \to \mathbb{R}^n$ is *cooperative* on $\mathcal{U} \subseteq \mathcal{W}$ if the Jacobian matrix $\frac{\partial f}{\partial x}(a)$ is Metzler for all $a \in \mathcal{U}$. When we say that f is cooperative without specifying the set $\mathcal{U}$, we understand that it is cooperative on $\mathbb{R}^n_+$. It is well known that cooperative systems are monotone [20]. Formally, this means that if $f : \mathcal{W} \to \mathbb{R}^n$ is cooperative on $\mathbb{R}^n_+$ then $x_0 \leq y_0$, $x_0, y_0 \in \mathbb{R}^n_+$ implies $x(t, x_0) \leq x(t, y_0)$ for all $t \geq 0$.

Our results here extend previous work on homogeneous systems. In the interest of completeness, we recall now the definitions of dilation map and homogeneity [1].

Given an n-tuple $r = (r_1, \dots, r_n)$ of positive real numbers and $\lambda > 0$, the *dilation map* $\delta^r_\lambda(x) : \mathbb{R}^n \to \mathbb{R}^n$ is given by $\delta^r_\lambda(x) = (\lambda^{r_1} x_1, ..., \lambda^{r_n} x_n)$. For an $\alpha \geq 0$, the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *homogeneous* of degree α with respect to $\delta^r_\lambda(x)$ if

$$\forall x \in \mathbb{R}^n, \lambda \geq 0, \quad f(\delta^r_\lambda(x)) = \lambda^\alpha \delta^r_\lambda(f(x))$$

If $r = (1, \cdots, 1)$, then $\delta^r_\lambda(x)$ is the standard dilation map.

We next recall various fundamental stability concepts. As we are dealing with positive systems throughout the paper, all definitions are with respect to the state space $X = \mathbb{R}^n_+$.

Definition 2.1. *Let the system (1) have an equilibrium at* $p \geq 0$*. Then we say that the equilibrium point* p *is*

- *stable, if for each* $\epsilon > 0$*, there is* $\delta = \delta(\epsilon) > 0$ *such that*

$$\|x_0 - p\| < \delta \Rightarrow \|x(t, x_0) - p\| < \epsilon, \forall t > 0.$$

- *unstable, if it is not stable;*

- *asymptotically stable if it is stable and there exists a neighbourhood* N *of* p *such that*

$$x_0 \in N \Rightarrow \lim_{t \to \infty} x(t, x_0) = p.$$

The set

$$A(p) := \{x_0 \in \mathbb{R}^n_+ : x(t, x_0) \to p, \ \text{as } t \to \infty\}$$

is the domain of attraction of p*. If* $A(p) = \mathbb{R}^n_+$*, then we say that* p *is globally asymptotically stable (GAS).*

In this manuscript, we will often use the following lemma, which is Proposition 3.2.1 in [20].

Lemma 2.1. *Let* $f : \mathcal{W} \to \mathbb{R}^n$ *be a cooperative vector field and assume there exists a vector* w *such that* $f(w) \ll 0$ $(f(w) \gg 0)$*. Then the trajectory* $x(t, w)$ *of system (1) is decreasing (increasing) for* $t \geq 0$ *with respect to the order on* $\mathbb{R}^n_+$*. In the case of* $f(w) \leq 0$ $(f(w) \geq 0)$*, the trajectory will be non-increasing (non-decreasing).*

One immediate consequence of Lemma 2.1 is the following result, which we shall use often in the sequel.

Lemma 2.2. *Let* $f : \mathcal{W} \to \mathbb{R}^n$ *be cooperative and satisfy* ***P****. Suppose that the system (1) has an equilibrium point at* $p \in \mathbb{R}^n_+$*. The following statements hold.*

(i) If there exists $w > p$ *with* $f(w) \geq 0$*, then* $x(t, w) \geq w$ *for all* $t \geq 0$*.*

(ii) If there exists $0 \leq w < p$ *with* $f(w) \leq 0$*, then* $x(t, w) \leq w$ *for all* $t \geq 0$*.*

In particular, in both case (i) and case (ii), w *cannot lie in the domain of attraction* $A(p)$ *of* p*.*

Lemma 2.3. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be cooperative with a unique equilibrium at the origin. Then the system (1) is positive.*

Proof: Since f is cooperative, the system (1) is monotone, which means for every initial conditions x_1 and x_2 we have:

$$x_1 \leq x_2 \Rightarrow x(t, x_1) \leq x(t, x_2) \text{ for all } t \geq 0$$

Since $x(t, 0) = 0$ for all $t \geq 0$, then for all initial conditions $x_0 \geq 0$ we have

$$x_0 \geq 0 \Rightarrow x(t, x_0) \geq x(t, 0) = 0 \text{ for all } t \geq 0$$

which means the positive orthant is an invariant set for the system (1) and this concludes the proof. □

D-STABILITY

One well-known fact about positive LTI systems is that they are *D-stable* [4]. Formally, if the positive LTI system

$$\dot{x} = Ax$$

has a GAS equilibrium at the origin, then so does the system

$$\dot{x} = DAx$$

for all diagonal D with positive diagonal entries.

Extending the notion of D-stability to nonlinear positive systems is a central theme for this paper. A restricted form of this concept was considered in [15] for a specific class of homogeneous systems. A more general definition was then introduced in [2] wherein D-stability results for arbitrary homogeneous cooperative systems were established. Such systems are automatically positive and moreover, they always possess an equilibrium at the origin. In this earlier paper, it was shown that, analogous to positive LTI systems, any homogeneous cooperative system with a globally asymptotically stable equilibrium at the origin is in fact D-stable.

To study nonlinear extensions of D-stability to systems such as (1), we consider the system

$$\dot{x}(t) = \text{diag}\,(d(x(t)))f(x(t)) \tag{2}$$

where $d : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 mapping satisfying the following condition.

Condition **D**:

(i) $d(x_1, x_2, \ldots, x_n) = (d_1(x_1), d_2(x_2), \ldots, d_n(x_n))$ for C^1 mappings $d_i : \mathbb{R} \to \mathbb{R}$, $1 \leq i \leq n$;

(ii) for $1 \leq i \leq n$, $d_i(x_i) > 0$, for $x_i > 0$.

In Section 4 we present results that relate the stability properties of the system (2) to those of the system (1). First, we make the following simple observation that the properties of cooperativity and positivity are preserved under pre-multiplication by $\mathrm{diag}\,(d(x))$.

Proposition 2.1. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be cooperative and satisfy condition **P**. Further, let $d : \mathbb{R}^n \to \mathbb{R}^n$ satisfy condition **D**. Then the vector field $g : \mathcal{W} \to \mathbb{R}^n$ given by $g(x) = \mathrm{diag}\,(d(x))f(x)$ is cooperative and satisfies condition **P**.*

Proof: If $x_i = 0$, then $f_i(x) \geq 0$ as f satisfies condition **P**. It is now immediate that $g_i(x) = d_i(x_i)f_i(x) \geq 0$ as $d_i(0) \geq 0$ by continuity. Hence g satisfies condition **P**. Direct calculation shows that for $i \neq j$

$$\frac{\partial g_i}{\partial x_j}(a) = d_i(a_i)\frac{\partial f_i}{\partial x_j}(a) \geq 0$$

for all $a \in \mathbb{R}^n_+$ as f is cooperative. Hence g is cooperative as claimed.

KKM Lemma

The arguments presented later in the paper will make considerable use of the Knaster, Kuratowski, Mazurkiewicz (KKM) Lemma [18][12][9]. The lemma as originally stated was concerned with coverings of a simplex by closed sets, but it is a later version of the result concerning open coverings that we make use of here. Before stating the lemma we first need to recall some definitions.

A set $\{a_0, a_1, \cdots, a_r\} \in \mathbb{R}^n$ is affinely independent if the system of vectors

$$(a_1 - a_0), \cdots, (a_r - a_0)$$

is linearly independent. Given a set of affinely independent vectors, $a_0, a_1, \cdots, a_r$, the set of all vectors of the form

$$x = \lambda_0 a_0 + \lambda_1 a_1 + \cdots + \lambda_r a_r$$

where $\lambda_i \geq 0$, $0 \leq i \leq r$, $\lambda_0 + \lambda_1 + \cdots + \lambda_r = 1$ is called an r-dimensional *simplex*, or briefly an r-simplex. The points $a_0, a_1, \cdots, a_r$ are the *vertices* of the simplex. For simplicity, we denote the simplex by $S(a_0, \ldots, a_r)$.

The simplex whose vertices are the standard basis vectors $e_1, \dots, e_n$ of $\mathbb{R}^n$ is referred to as the *standard simplex* and denoted by Δ_n.

Given a simplex $S(a_0, \dots, a_r)$ and indices $0 \le i_0 < i_1 < \dots < i_p \le r$, the simplex $S(a_{i_0}, \dots, a_{i_p})$ is a *face* of $S(a_0, \dots, a_r)$. We shall need the following open version of the KKM Lemma [18].

Theorem 2.2 (KKM Lemma). *Let $\Delta := S(a_0, a_1, \dots, a_r)$ be an r-simplex and let $F_0, F_1, \dots, F_r$ be (relatively) open subsets of Δ. If*

$$S(a_{i_0}, \dots, a_{i_p}) \subset F_{i_0} \cup F_{i_1} \cup \dots \cup F_{i_p}$$

holds for all faces $S(a_{i_0}, \dots, a_{i_p})$, $1 \le p \le r$, $0 \le i_0 < i_1 < \dots < i_p \le r$, then

$$F_0 \cap F_1 \dots \cap F_r \neq \emptyset.$$

3. Subhomogeneous Systems

In this section, we introduce the class of subhomogeneous cooperative systems and present some basic properties of such systems that shall prove useful later.

Definition 3.1. *A vector field $f : \mathcal{W} \to \mathbb{R}^n$ is subhomogeneous of degree $\tau > 0$ if $f(\lambda v) \le \lambda^\tau f(v)$, for all $v \in \mathbb{R}^n_+$, $\lambda \in \mathbb{R}$ with $\lambda \ge 1$.*

The class of subhomogeneous vector fields given above includes concave vector fields [11]. Furthermore,it includes vector fields which are homogeneous with respect to the standard dilation map (given by $x \to \lambda x$ for $\lambda > 0$).

Comment: We are assuming that the vector field f is C^1 on a neighbourhood $\mathcal{W}$ of $\mathbb{R}^n_+$. In [14], vector fields were not required to be C^1 at the equilibrium at the origin. As shown by the authors of this paper, for homogeneous systems this is sufficient to guarantee uniqueness of solutions. However, for subhomogeneous systems, this is not the case as can be seen from the simple 1-dimensional example $\dot{x} = 3x^{2/3}$ which has multiple solutions satisfying $x(0) = 0$.

The following result establishes an inequality for subhomogeneous vector fields that is reminiscent of Euler's formula for homogeneous functions [14].

Lemma 3.1. *The vector field $f : \mathcal{W} \to \mathbb{R}^n$ is subhomogeneous of degree $\tau > 0$ if and only if:*

$$\frac{\partial f}{\partial x}(x)x \le \tau f(x) \quad \text{for all } x \ge 0. \tag{3}$$

Proof: We first show that f is subhomogeneous of degree τ if and only if for any $x \geq 0$, the mapping

$$\lambda \to \lambda^{-\tau} f(\lambda x)$$

is a non-increasing function for $\lambda > 0$.

Let $x \geq 0$ be given. If f is subhomogeneous, then for any $\mu \geq \lambda > 0$ we have

$$\begin{aligned} f(\mu x) = f\left(\frac{\mu}{\lambda}\lambda x\right) &\leq \left(\frac{\mu}{\lambda}\right)^{\tau} f(\lambda x) \\ \Rightarrow \mu^{-\tau} f(\mu x) &\leq \lambda^{-\tau} f(\lambda x) \end{aligned}$$

Thus, we can conclude that $\lambda^{-\tau} f(\lambda x)$ is a non-increasing function with respect to λ for all $\lambda > 0$. Conversely, if this function is non-increasing for $\lambda > 0$, then by choosing $\mu \geq \lambda = 1$, we see immediately that $f(\mu x) \leq \mu^{\tau} f(x)$.

Differentiating with respect to λ, we see that f is subhomogeneous if and only if for all $\lambda > 0$

$$\frac{d}{d\lambda}\left(\lambda^{-\tau} f(\lambda x)\right) \leq 0$$

$$\Leftrightarrow -\tau\lambda^{-\tau-1} f(\lambda x) + \lambda^{-\tau}\frac{\partial f}{\partial x}(\lambda x)\, x \leq 0$$

Rearranging this inequality, we see that f is subhomogeneous if and only if

$$\frac{\partial f}{\partial x}(\lambda x)(\lambda x) \leq \tau f(\lambda x) \quad \forall x \geq 0; \forall \lambda > 0$$

This last statement is equivalent to

$$\frac{\partial f}{\partial x}(x)(x) \leq \tau f(x) \ \forall x \geq 0$$

This concludes the proof. □

In the following corollary, some of the basic properties of subhomogeneous systems are stated.

Corollary 3.1. *(i) The set of subhomogeneous vector fields of degree τ on $\mathbb{R}^n_+$ is a convex cone.*

(ii) A non-negative constant vector field $f(x) \equiv c$ is subhomogeneous of any degree $\tau > 0$.

(iii) Any affine map $f(x) = Ax + b$ where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n_+$ is subhomogeneous of degree 1*.*

Proof:

(i) The claim follows as the condition (3) has to be satisfied pointwise and is clearly convex in f and invariant under positive scaling of f.

(ii) Immediate from (3) as $f(x) \geq 0 = \partial f/\partial x(x)$ for all $x \geq 0$.

(iii) The claim follows from (i), (ii) and (3) as for linear maps we have $f(x) = Ax = \partial f/\partial x(x) \cdot x$. □

In the following result, we show that subhomogeneous cooperative systems are positive.

Theorem 3.1. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be subhomogeneous of degree $\tau > 0$ and cooperative. Then the system (1) is positive.*

Proof: It follows from Lemma 3.1 and the fact that the Jacobian matrix is Metzler for all $x \in \mathbb{R}^n_+$ that $f_i(x) \geq 0$ for all $x \in \mathbb{R}^n_+$ with $x_i = 0$. Therefore, condition **P** is satisfied and this immediately implies that (1) is positive. □

4. D-Stability for Subhomogeneous Cooperative Systems

In this section, we are concerned with extending results on D-stability to subhomogeneous cooperative systems. We shall consider two distinct cases: systems with a GAS equilibrium at the origin and systems with an asymptotically stable equilibrium in $\text{int}\,(\mathbb{R}^n_+)$ whose region of attraction includes $\text{int}\,(\mathbb{R}^n_+)$.

4.1. Equilibrium at the origin

The main result of this subsection states that if a subhomogeneous cooperative system (1) has a GAS equilibrium at the origin then the system (2) also has a GAS equilibrium at the origin. Before stating this theorem, we establish some preliminary results.

Lemma 4.1. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be a cooperative vector field such that system (1) has a GAS equilibrium at the origin. Let $d : \mathbb{R}^n \to \mathbb{R}^n$ satisfy condition **D**. Then the system (2) has a unique equilibrium at the origin.*

Proof: Clearly, (2) has an equilibrium at the origin. It remains to show that it is unique.

Based on the definition of $d(x)$, we know that (2) cannot have any equilibrium points in $\mathrm{int}\,(\mathbb{R}^n_+)$. Now, by way of contradiction, suppose $\mathrm{diag}\,(d(p))f(p) = 0$ for some $p \neq 0$ in $\mathrm{bd}\,(\mathbb{R}^n_+)$.

We define $Z := \{i : p_i = 0\}$ and $NZ := \{i : p_i \neq 0\}$. As $d_i(p_i) > 0$ for all $i \in NZ$ by assumption, we must have $f_i(p) = 0$ for all $i \in NZ$. As the origin is a GAS equilibrium of (1), it follows from Lemma 2.2 that we cannot have $f(p) \geq 0$. Hence, there must be some $i_0 \in Z$ such that $f_{i_0}(p) < 0$.

On the other hand

$$\frac{\partial f_{i_0}}{\partial x_j}(s) \geq 0$$

for all $j \neq i_0$ and for all $s \in \mathbb{R}^n_+$. Furthermore, $p_{i_0} = 0$ as $i_0 \in Z$. Thus

$$f_{i_0}(p) = f_{i_0}(0) + \int_0^1 \sum_{j=1}^n \frac{\partial f_{i_0}}{\partial x_j}(sp)p_j ds \geq 0.$$

This is a contradiction and we can conclude that the origin is the only equilibrium of (2). □

The following proposition plays a key role in proving later results. The argument presented here is essentially the same as was used in [3][19] albeit for a different class of systems.

Proposition 4.1. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be a cooperative vector field such that (1) has a GAS equilibrium at the origin. Then there exists $v \gg 0$ such that $f(v) \ll 0$.*

Proof: Firstly, Lemma 2.3 implies that the system (1) is positive. Secondly, consider the standard simplex Δ_n. We define $C_i = \{x \in \Delta_n : f_i(x) < 0\}$ for $i = 1, \cdots, n$. As f is continuous, C_i is a relatively open set in Δ_n for $i = 1, \cdots, n$. On the other hand, since the system (1) has a GAS equilibrium at 0, there is no $w > 0$ in the simplex, such that $f(w) \geq 0$ by Lemma 2.2. Therefore, $\cup_{i=1}^n C_i = \Delta_n$.

Let $S(e_{i_0}, e_{i_1}, \cdots, e_{i_s})$ be an arbitrary face of the simplex and let $x \in S(e_{i_0}, e_{i_1}, \cdots, e_{i_s})$. Then $x_j = 0$ for $j \notin \{i_0, \cdots, i_s\}$. Since the positive orthant is an invariant set for (1), it follows that $f_j(x) \geq 0$ for $j \notin \{i_0, \cdots, i_s\}$. Therefore as (1) has a GAS equilibrium at the origin, Lemma 2.2 implies that $f_j(x) < 0$ for some $j \in \{i_0, \cdots, i_s\}$. This means that

$$x \in C_{i_0} \cup C_{i_1} \cup \cdots \cup C_{i_s}.$$

As x was arbitrary, we conclude that for any face of the simplex, we have

$$S(e_{i_0}, e_{i_1}, \cdots, e_{i_s}) \subset C_{i_0} \cup C_{i_1} \cup \cdots \cup C_{i_s}.$$

It follows from Theorem 2.2 that $\cap_{i=1}^{n} C_i \neq \varnothing$. As f is continuous, this means there exists a $v \gg 0$ in Δ_n such that $f(v) \ll 0$. □

Now we are ready to state and prove the following theorem.

Theorem 4.1. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be a cooperative vector field that is subhomogeneous of degree $\tau > 0$. Let $d : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 mapping satisfying condition **D**. Assume that (1) has a GAS equilibrium at the origin. Then (2) also has a GAS equilibrium at the origin.*

Proof: It follows from Proposition 2.1 that the system (2) is positive and monotone.

As the origin is a GAS equilibrium of (1), Proposition 4.1 implies that there exists a $v \gg 0$ such that $f(v) \ll 0$.

Let $x_0 \in \mathbb{R}^n_+$ be given. We can find a $\lambda \geq 1$ such that $w = \lambda v > x_0$. From subhomogeneity, it follows that

$$f(w) = f(\lambda v) \leq \lambda^{\tau} f(v) \ll 0.$$

Further, it follows from Property **D**, that

$$\operatorname{diag}(d(w)) f(w) \ll 0.$$

Lemma 2.1 implies that the trajectory $x(t, w)$ of (2) starting from $x(0) = w$ is decreasing. In addition $\mathbb{R}^n_+$ is invariant under (2). It now follows from Theorem 1.2.1 of [20] that $x(t, w)$ converges to an equilibrium of (2) as $t \to \infty$. Lemma 4.1 implies that the origin is the only equilibrium of (2). It follows immediately that $x(t, w) \to 0$ as $t \to \infty$.

As (2) is positive and monotone and as $x_0 < w$, it follows that

$$0 \leq x(t, x_0) \leq x(t, w) \leq w$$

for all $t \geq 0$. This implies that $x(t, x_0) \to 0$ as $t \to \infty$. This concludes the proof. □

Example 4.1. Consider the system

$$\dot{x} = f(x) = \begin{pmatrix} -x_1 + \dfrac{x_2}{a + x_2} \\ \\ -x_2 + \dfrac{x_1}{b + x_1} \end{pmatrix} \tag{4}$$

where $a > 1$, $b > 1$. It can be easily checked that f is C^1 on $\mathbb{R}^n \setminus \{(-b, -a)\}$, cooperative and subhomogeneous. Hence (4) is positive and monotone.

Also $f(x) = 0$ has two solutions, one is $x = 0$ and the other is

$$x = \left(\frac{1-ab}{1+a}, \frac{1-ab}{1+b}\right)$$

Since $a, b > 1$, the second solution is outside the positive orthant. Hence the system (4) has a unique equilibrium in $\mathbb{R}^n_+$. As $f(1,1) \ll 0$, the argument in the previous proof can be readily adapted to show that the origin is a GAS equilibrium of (4).

If we define

$$d(x) = \begin{pmatrix} \dfrac{x_1^2}{x_1^2+1} \\ \\ 1+\sin^2(x_2) \end{pmatrix}$$

then it satisfies condition **D**. Now based on Theorem 4.1 we can say that the system

$$\dot{x} = \operatorname{diag}(d(x)) f(x)$$

has a GAS equilibrium at the origin. Note that this new system is cooperative but not subhomogeneous.

We next note that the result of Theorem 4.1 is not true for general cooperative systems (not necessarily subhomogeneous) with an equilibrium at the origin.

Example 4.2. Consider the system on $\mathbb{R}^2_+$ given by

$$\dot{x} = f(x) = \begin{pmatrix} -\dfrac{x_1}{1+x_1^3} + x_2 \\ \\ -x_2 \end{pmatrix} \tag{5}$$

It is easy to verify that f is cooperative and that the origin is the only equilibrium of this system. Also, for $v = (1, 0.25)^T$, $f(v) = (-0.25, -0.25)^T \ll 0$. This system satisfies all the conditions of Theorem 4.1 except subhomogeneity. We will prove that (5) has a GAS equilibrium at the origin but is not D-stable.

First note that

$$\dot{x}_1 + \dot{x}_2 = -\frac{x_1}{1+x_1^3} + x_2 - x_2 = -\frac{x_1}{1+x_1^3} \leq 0$$

for all $x_1, x_2 \in \mathbb{R}_+$. This implies that for every $K > 0$, the bounded set

$$\{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 \leq K\}$$

is invariant under (5). In particular, the trajectories of (5) have compact closure in $\mathbb{R}_+^2$. Using Theorem 3.2.2 in [20], we can conclude that the single equilibrium of this system, which is the origin, is globally asymptotically stable.

In order for (5) to be D-stable, the associated system (2) should have a GAS equilibrium at the origin for all choices of $d(x)$ satisfying Condition **D**. Choosing $d(x) = (1, x_2^3)$, (2) takes the form:

$$\dot{x} = \text{diag}\,(d(x))f(x) = \begin{pmatrix} -\dfrac{x_1}{1 + x_1^3} + x_2 \\ \\ -x_2^4 \end{pmatrix} \tag{6}$$

As stated in Example 3.11 of [19], the origin is *not* a GAS equilibrium of (6). In fact, for the initial condition $(x_1(0), x_2(0)) = (1, 1)$, the x_1 component of the associated solution grows without bound. This shows that the system (5) is not D-stable.

4.2. Equilibrium at $p \gg 0$

We next derive a version of Theorem 4.1 for the case where (1) has a unique equilibrium at $p \gg 0$. For this scenario, rather than showing that GAS of p for (1) implies GAS of p for (2), we shall show that if the domain of attraction of p under (1) contains $\text{int}\,(\mathbb{R}_+^n)$, then the domain of attraction of p under (2) contains $\text{int}\,(\mathbb{R}_+^n)$.

To see why this is necessary, consider a Mutualistic Lotka-Volterra system [7]:

$$\dot{x} = f(x) = \text{diag}\,(x)(Ax + b) \tag{7}$$

where A is Metzler and $b > 0$.

Based on Corollary 3.1, we know $f(x) = Ax + b$ is subhomogeneous of order 1. Also since A is Metzler, f is also cooperative. However, even if this system has a GAS equilibrium at some point $p \in \text{int}\,(\mathbb{R}_+^n)$, each axis, $x_i = 0$ is an invariant set for the system (7). Hence, p cannot be GAS for (7) in this case.

To prove the main result of this section, we will need the following variant of Proposition 4.1. In the statement of the proposition we use the following notation for $p \in \text{int}(\mathbb{R}^n_+)$: $R_1(p) = \{x : x \gg p\}$; $R_2(p) = \{x \in \text{int}(\mathbb{R}^n_+) : x \ll p\}$.

Proposition 4.2. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be a cooperative vector field. Assume that (1) has an asymptotically stable equilibrium at $p \gg 0$ and that the domain of attraction of p contains* $\text{int}(\mathbb{R}^n_+)$. *Then there exists $v_1 \in R_1(p)$ such that $f(v_1) \ll 0$ and $v_2 \in R_2(p)$ such that $f(v_2) \gg 0$.*

Proof: Firstly, we prove there exists a $v_1 \in R_1(p)$ such that $f(v_1) \ll 0$. Let Δ_n be the standard simplex. We consider $p + \Delta_n$, the standard simplex shifted to point p and define $C_i = \{x \in p + \Delta_N : f_i(x) < 0\}$ for $i = 1, \cdots, n$. Note the following facts.

- (i) The set $\{x \in \mathbb{R}^n_+ : x \geq p\}$ is forward invariant under (1). This is because p is an equilibrium and (1) is monotone.
- (ii) There is no $x > p$ with $f(x) \geq 0$. This follows from Lemma 2.2 as the domain of attraction of p contains $\text{int}(\mathbb{R}^n_+)$.

Using (i) and (ii), we can apply Theorem 2.2 in the same way as in Proposition 4.1 to conclude that there exists $v_1 \gg p$ such that $f(v_1) \ll 0$.

We next show that there exists a $v_2 \in R_2(p)$ with $f(v_2) \gg 0$. First, choose $r > 0$ small enough to ensure that the shifted simplex $p - r\Delta_n$ is wholly contained in $\text{int}(\mathbb{R}^n_+)$. As above, it follows that $\{x \in \mathbb{R}^n_+ : x \leq p\}$ is forward invariant under (1) and that there can be no $x < p$ with $f(x) \leq 0$. Again applying Theorem 2.2, we conclude that there exists a $v_2 \ll p$, such that $f(v_2) \gg 0$. □

With the above proposition, we can now prove the following.

Theorem 4.2. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be subhomogeneous of degree τ and cooperative. Let $d : \mathbb{R}^n \to \mathbb{R}^n$ satisfy condition **D**. Assume that (1) has an asymptotically stable equilibrium at $p \gg 0$ and that the domain of attraction of p under (1) contains* $\text{int}(\mathbb{R}^n_+)$. *Then the system (2) has an asymptotically stable equilibrium at $p \gg 0$ and the domain of attraction of p under (2) contains* $\text{int}(\mathbb{R}^n_+)$.

Proof: Proposition 4.2 implies that there exists a $v_1 \gg p$ such that $f(v_1) \ll 0$ and there exists a v_2 with $0 \ll v_2 \ll p$ such that $f(v_2) \gg 0$. It follows from the subhomogeneity of f that for any $\lambda \geq 1$,

$$f(\lambda v_1) \leq \lambda^\tau f(v_1) \ll 0$$

Similarly, for any $0 < \mu \leq 1$,

$$f(\mu v_2) \geq \mu^\tau f(v_2) \gg 0$$

Let $x_0 \in \text{int}\,(\mathbb{R}^n_+)$ be an arbitrary initial condition. Then we can choose $\lambda > 1$ and $\mu < 1$ such that

$$\mu v_2 \leq x_0 \leq \lambda v_1 \tag{8}$$

From the properties of d, it follows immediately that $d(\lambda v_1)f(\lambda v_1) \ll 0$ and $d(\mu v_2)f(\mu v_2) \gg 0$. Hence the trajectory $x(t, \lambda v_1)$ of (2) is decreasing while the trajectory $x(t, \mu v_2)$ is increasing. Further, as p is an equilibrium of (2) and (2) is monotone, the trajectories $x(t, \lambda v_1)$, $x(t, \mu v_2)$ of (2) satisfy $p \leq x(t, \lambda v_1) \leq \lambda v_1$, $\mu v_2 \leq x(t, \mu v_2) \leq p$ for all $t \geq 0$.

Taken together this implies that the trajectories $x(t, \lambda v_1)$, $x(t, \mu v_2)$ of (2) converge monotonically to p. It now follows from the monotonicity of the system (2) and (8) that

$$x(t, \mu v_2) \leq x(t, x_0) \leq x(t, \lambda v_1)$$

for all $t \geq 0$. Hence, $x(t, x_0)$ must also converge to p. This completes the proof. □

5. Stability and Positivity of Equilibria

In [14], results relating the stability properties of a homogeneous cooperative irreducible system $\dot{x} = f(x)$ to the existence of positive equilibria of the associated system $\dot{x} = f(x) + b$ were presented. The arguments of this earlier paper relied on an extension of the Perron-Frobenius theorem to homogeneous cooperative systems [1]. In this section, we extend some of these results to subhomogeneous systems. Specifically, we consider the system (1), where f is assumed to be cooperative, subhomogeneous and irreducible, and relate its stability properties to the existence and stability of positive equilibria of the associated system

$$\dot{x} = f(x) + b, \quad b > 0. \tag{9}$$

We first recall the definition of irreducibility from [14].

Definition 5.1. *The vector field $f : \mathcal{W} \to \mathbb{R}^n$ is irreducible, if:*

(i) for all $a \in \text{int}\,(\mathbb{R}^n_+)$, the Jacobian matrix $\frac{\partial f}{\partial x}(a)$ is irreducible;

(ii) for $x \in \mathrm{bd}\,(\mathbb{R}^n_+) \setminus \{0\}$ with $x_i = 0$, we must have $f_i(x) > 0$.

The following proposition establishes a sufficient condition for the system (9) to be positive.

Proposition 5.1. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be subhomogeneous and cooperative and let $b \geq 0$. Then the system (9) is positive.*

Proof: Note that the vector field $f(x)+b$ will again be subhomogeneous and cooperative under these hypotheses. The result now follows from Theorem 3.1. □

Proposition 5.2. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be subhomogeneous of degree τ, cooperative and irreducible and let $b > 0$ be given. Assume that the system (1) has a GAS equilibrium at the origin. Then the system (9) has at least one equilibrium in $\mathrm{int}\,(\mathbb{R}^n_+)$.*

Proof: From Proposition 4.1, we know there exists a $v \gg 0$ such that $f(v) \ll 0$. The subhomogeneity of f implies that $f(\lambda v) \leq \lambda^\tau f(v)$ for all $\lambda \geq 1$. By choosing λ large enough we can ensure that $f(\lambda v) + b \ll 0$. Since $g(x) = f(x)+b$ is also cooperative, it follows from Lemma 2.1 that the trajectory $x(t, \lambda v)$ of (9) starting from λv will be decreasing.

Given any $x_0 \in \mathbb{R}^n_+$, we can find $\lambda > 1$ with $\lambda v \geq x_0$ and $f(\lambda v) + b \ll 0$. Further, as (9) is positive, this implies that

$$0 \leq x(t, x_0) \leq x(t, \lambda v) \leq \lambda v$$

for all $t \geq 0$. Hence, the forward orbit $\{x(t, x_0) : t \geq 0\}$ is relatively compact for any $x_0 \in \mathbb{R}^n_+$.

It follows immediately from Theorem 1.2.1 of [20] that $x(t, \lambda v)$ converges to an equilibrium point $p \in \mathbb{R}^n_+$.

We have now shown that there exists an equilibrium in $\mathbb{R}^n_+$. To complete the proof, we show that every equilibrium of (9) is in $\mathrm{int}\,(\mathbb{R}^n_+)$. (Our argument is the same as presented in [14] but we include it here for completeness.) Since $b > 0$ and $f(0) = 0$, $z = 0$ cannot be an equilibrium of the system (9). Next consider $z \in \mathrm{bd}\,(\mathbb{R}^n_+) \setminus \{0\}$ with $z_i = 0$. Since $b > 0$, and $f_i(z) > 0$, $f(z) + b$ cannot be zero. This concludes the proof. □

We next show that when f is subhomogeneous, cooperative and irreducible with a GAS equilibrium at the origin, then the system (9) has a *unique* equilibrium in $\mathrm{int}\,(\mathbb{R}^n_+)$. We will need the following proposition, which extends Proposition 4 of [14] to subhomogeneous vector fields.

Proposition 5.3. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be subhomogeneous, cooperative and irreducible and let $b > 0$ be given. Then the Jacobian matrix of $f(x) + b$ evaluated at an equilibrium point of the system (9) is a Hurwitz matrix.*

Proof: We prove the proposition by contradiction. Choose an arbitrary equilibrium point p. Based on Proposition 5.2 we know that $p \in \text{int}\,(\mathbb{R}^n_+)$.

As f is irreducible and cooperative and $p \in \text{int}\,(\mathbb{R}^n_+)$, $\frac{\partial f}{\partial x}(p)$ is an irreducible Metzler matrix. By way of contradiction, suppose that $\frac{\partial f}{\partial x}(p)$ is not a Hurwitz matrix. Writing μ for the maximal real part of the eigenvalues of $\frac{\partial f}{\partial x}(p)$, we have $\mu \geq 0$. Theorem 2.1 then implies that there exists a vector $v \in \text{int}\,(\mathbb{R}^n_+)$ with

$$v^T \frac{\partial f}{\partial x}(p) = \mu v^T \tag{10}$$

On the other hand based on Lemma 3.1, we know that

$$\frac{\partial f}{\partial x}(p)p \leq \tau f(p). \tag{11}$$

Multiplying (11) by v^T on the left and invoking (10) we have

$$\mu v^T p \leq \tau v^T f(p) \tag{12}$$

We know that $f(p) = -b < 0$. Therefore there exists at least one j such that $f_j(p) < 0$. This implies that the right hand side of (12) is strictly negative, while the left hand side is nonnegative. We have therefore reached a contradiction and we can conclude that $\frac{\partial f}{\partial x}(p)$ is a Hurwitz matrix. □

We next prove that the system (9) has a unique GAS stable equilibrium in $\text{int}\,(\mathbb{R}^n_+)$ for each $b > 0$ provided that f is subhomogeneous, cooperative and irreducible and that (1) has a GAS equilibrium at 0. This was established in [14] for homogeneous systems using degree theoretic arguments. Our argument does not involve degree theory but relies directly on Proposition 5.3.

Theorem 5.1. *Let $f : \mathcal{W} \to \mathbb{R}^n$ be subhomogeneous of degree τ, cooperative and irreducible such that the system (1) has a GAS equilibrium at the origin. Then for any $b > 0$, the system (9) has a unique equilibrium in $\text{int}\,(\mathbb{R}^n_+)$, and this equilibrium is GAS.*

Proof: We know from Proposition 5.2 that (9) has an equilibrium in $\text{int}\,(\mathbb{R}^n_+)$. We first prove that this equilibrium is unique.

To this end, suppose that there are two distinct equilibria $p \gg 0, q \gg 0$.

Proposition 5.3 implies that that Jacobian of $g(x) = f(x) + b$ evaluated at each equilibrium point is Hurwitz. Further, as g is cooperative and irreducible, the Jacobian evaluated at each equilibrium point is irreducible and Metzler. Theorem 2.1 implies that there exist vectors x^p, x^q with $\|x^p\| = 1$, $\|x^q\| = 1$ and

$$\frac{\partial g}{\partial x}(p)x^p \ll 0$$

$$\frac{\partial g}{\partial x}(q)x^q \ll 0.$$

Without loss of generality, we can assume that

$$\max_i \frac{q_i}{p_i} > 1 \quad \forall i = 1, \cdots, n$$

As g is C^1, it follows from Taylor's theorem that by choosing $t > 0$ sufficiently small, we can ensure that $g(p+tx^p) \ll 0$, $g(q-tx^q) \gg 0$. Define $v = p+tx^p$, $w = q - tx^q$. Then $g(v) \ll 0$, $g(w) \gg 0$. Also, choosing a smaller t if neccessary, we can ensure that

$$\lambda := \max_i \frac{w_i}{v_i} = \frac{w_k}{v_k} > 1$$

Now note the following facts:

(i) $\lambda v \geq w$ and $\lambda v_k = w_k$;

(ii) $g(\lambda v) \leq \lambda^\tau g(v)$ (as $b > 0$, g will also be subhomogeneous).

As g is cooperative, it follows from (i) that

$$g_k(\lambda v) \geq g_k(w) > 0$$

On the other hand, it follows from (ii) that

$$g_k(\lambda v) \leq \lambda^\tau g_k(v) < 0.$$

This is a contradiction, which shows that there can only be one equilibrium of (9) in $\operatorname{int}(\mathbb{R}^n_+)$ as claimed.

To complete the proof, we show that this unique equilibrium point is GAS. Let $p \gg 0$ be the equilibrium point of (9). As the Jacobian of g evaluated at

p is Hurwitz, Metzler and irreducible, it follows from Taylor's theorem (as in the previous paragraph) that there is some $v \geq p$ with $g(v) \ll 0$. Further, as $f(0) = 0$, we have $g(0) = b > 0$. Hence from Lemma 2.1 the trajectory $x(t, 0)$ of (9) is non-decreasing and satisfies

$$0 \leq x(t, 0) \leq p$$

for all $t \geq 0$. As p is the only equilibrium of (9) it follows that $x(t, 0) \to p$ as $t \to \infty$.

Let $x_0 \in \mathbb{R}^n_+$ be given. As g is subhomogeneous, we can find a $\lambda > 1$ such that $w = \lambda v \gg x_0$, and $g(w) \ll 0$. Lemma 2.1 implies that the trajectory $x(t, w)$, starting from w is decreasing and satisfies

$$w \geq x(t, w) \geq p$$

for all $t \geq 0$. Thus $x(t, w) \to p$ as $t \to \infty$.

As $0 \leq x_0 \leq w$ and (9) is monotone, it follows that

$$x(t, 0) \leq x(t, x_0) \leq x(t, w)$$

for all $t \geq 0$. It is now immediate that $x(t, x_0) \to p$ as $t \to \infty$. □

6. Conclusions

We have extended recent results on nonlinear versions of the concept of D-stability to subhomogeneous cooperative systems. Specifically, we have presented two separate results relating to D-stability: one for the case of a GAS equilibrium at the origin and one for the case of an asymptotically stable equilibrium in the interior of $\mathbb{R}^n_+$, whose domain of attraction includes int $(\mathbb{R}^n_+)$. We have also extended a result of [14] from homogeneous systems to subhomogeneous systems.

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