On the stability of switched positive linear systems

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Abstract

It was recently conjectured that the Hurwitz stability of the convex hull of a set of Metzler matrices is a necessary and sufficient condition for the asymptotic stability of the associated switched linear system under arbitrary switching. In this paper we show that: (i) this conjecture is true for systems constructed from a pair of second order Metzler matrices; (ii) the conjecture is true for systems constructed from an arbitrary finite number of second order Metzler matrices; and (iii) the conjecture is in general false for higher order systems. The implications of our results, both for the design of switched positive linear systems, and for research directions that arise as a result of our work, are discussed toward the end of the paper.

Key Words: Stability theory; Switched linear systems; Positive linear systems

1 Introduction

Positive dynamical systems are of fundamental importance to numerous applications in areas such as Economics, Biology, Sociology and Communications. Historically, the theory of positive linear time-invariant (LTI) systems has assumed a position of great importance in systems theory and has been applied in the study of a wide variety of dynamic systems [1, 2, 3, 4]. Recently, new studies in communication systems [5], formation flying [6], and other areas, have highlighted the importance of switched (hybrid) positive linear systems (PLS). In the last number of years, a considerable effort has been expended on gaining an understanding of the properties of general switched linear systems [7, 9]. As is the case for general switched systems, even though the main properties of positive LTI systems are well understood, many basic questions relating to switched PLS remain unanswered. The most important of these concerns their stability, and in this paper we present some initial results on the stability of switched PLS.

Recently, it was conjectured by the authors of [14], and independently by David Angeli, that the asymptotic stability of a positive switched linear system can be determined by testing the Hurwitz-stability of an associated convex cone of matrices. This conjecture was based on preliminary results on the stability of positive switched linear systems and is both appealing and plausible. Moreover, if it were true, it would have significant implications for the stability theory of positive switched linear systems. In this paper, we shall extend some earlier work and show that the above

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conjecture is true for some specific classes of positive systems. However, one of the the major contributions of the paper is to construct a counterexample which proves that, in general, the conjecture is false.

The layout of the paper is as follows. In Section 2 we present the mathematical background and notation necessary to state the main results of the paper. Then in Section 3, we present necessary and sufficient conditions for the uniform asymptotic stability of switched second order positive linear systems. In Section 4 we show by means of an abstract construction that the results derived in the preceding sections do not generalise to higher dimensional systems. Our conclusions are presented in Section 5.

2 Mathematical Preliminaries

In this section we present a number of preliminary results that shall be needed later and introduce the main notations used throughout the paper.

(i) Notation

Throughout, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^n$ stands for the vector space of all n-tuples of real numbers and $\mathbb{R}^{n\times n}$ is the space of $n \times n$ matrices with real entries. For x in $\mathbb{R}^n$, x_i denotes the i^{th} component of x, and the notation $x \succ 0$ ($x \succeq 0$) means that $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \succeq 0\}$ denotes the non-negative orthant in $\mathbb{R}^n$. Similarly, for a matrix A in $\mathbb{R}^{n\times n}$, a_{ij} denotes the element in the (i, j) position of A, and $A \succ 0$ ($A \succeq 0$) means that $a_{ij} > 0(a_{ij} \geq 0)$ for $1 \leq i, j \leq n$. $A \succ B$ ($A \succeq B$) means that $A - B \succ 0$ ($A - B \succeq 0$). We write A^T for the transpose of A and $exp(A)$ for the usual matrix exponential of $A \in \mathbb{R}^{n\times n}$.

For P in $\mathbb{R}^{n\times n}$ the notation $P > 0$ ($P \geq 0$) means that the matrix P is positive (semi-)definite, and $PSD(n)$ denotes the cone of positive semi-definite matrices in $\mathbb{R}^{n\times n}$. The spectral radius of a matrix A is the maximum modulus of the eigenvalues of A and is denoted by $\rho(A)$. Also we shall denote the maximal real part of any eigenvalue of A by $\mu(A)$. If $\mu(A) < 0$ (all the eigenvalues of A are in the open left half plane) A is said to be *Hurwitz* or *Hurwitz-stable*.

Given a set of points, $\{x_1, \ldots, x_m\}$ in a finite-dimensional linear space V, we shall use the notations $CO(x_1, \ldots, x_m)$ and $Cone(x_1, \ldots, x_m)$ to denote the convex hull and the cone generated by $x_1, \ldots, x_m$ respectively. Formally:

$$CO(x_1, \ldots, x_m) = \{x = \sum_{i=1}^{m} \alpha_i x_i : \alpha_i \geq 0, 1 \leq i \leq m, \text{ and } \sum_{i=1}^{m} \alpha_i = 1\};$$

$$Cone(x_1, \ldots, x_m) = \{x = \sum_{i=1}^{m} \alpha_i x_i : \alpha_i \geq 0, 1 \leq i \leq m\}.$$

The properties of convex cones shall play a key role in the results described later in this paper. Formally, a closed *convex cone* in $\mathbb{R}^n$ is a set $\Omega \subseteq \mathbb{R}^n$ such that, for any $x, y \in \Omega$ and any $\alpha, \beta \geq 0$, $\alpha x + \beta y \in \Omega$. A convex cone is said to be:

(i) *Solid* if the interior of Ω, with respect to the usual norm topology on $\mathbb{R}^n$, is non-empty;

(ii) *Pointed* if $\Omega \cap (-\Omega) = \{0\}$;

(iii) *Polyhedral* if $\Omega = Cone(x_1, \ldots, x_m)$ for some finite set $\{x_1, \ldots, x_m\}$ of vectors in $\mathbb{R}^n$.

For the remainder of the paper, we shall call a closed convex cone that is both solid and pointed, a *proper convex cone.*

(ii) Positive LTI systems and Metzler matrices

The LTI system

$$\Sigma_A : \dot{x}(t) = Ax(t), \quad x(0) = x_0$$

is said to be positive if $x_0 \succeq 0$ implies that $x(t) \succeq 0$ for all $t \geq 0$. Basically, if the system starts in the non-negative orthant of $\mathbb{R}^n$, it remain there for all time. See [3] for a description of the basic theory and several applications of positive linear systems. It is well-known that the system Σ_A is positive if and only if the off-diagonal entries of the matrix A are non-negative. Matrices of this form are known as Metzler matrices. If A is Metzler we can write $A = N - \alpha I$ for some non-negative N and a scalar $\alpha \geq 0$. Note that if the eigenvalues of N are $\lambda_1, \ldots \lambda_n$, then the eigenvalues of $N - \alpha I$ are $\lambda_1 - \alpha, \ldots \lambda_n - \alpha$. Thus the Metzler matrix $N - \alpha I$ is Hurwitz if and only if $\alpha > \rho(N)$.

The next result concerning positive combinations of Metzler Hurwitz matrices was pointed out in [8].

Lemma 2.1 *Let A_1, A_2 be Metzler and Hurwitz. Then $A_1 + \gamma A_2$ is Hurwitz for all $\gamma > 0$ if and only if $A_1 + \gamma A_2$ is non-singular for all $\gamma > 0$.*

(iii) Common Quadratic Lyapunov Functions and Stability

It is well known that the existence of a common quadratic Lyapunov function (CQLF) for the family of stable LTI systems

$$\Sigma_{A_i} : \dot{x} = A_i x \quad i \in \{1, \ldots, k\}$$

is sufficient to guarantee that the associated switched system

$$\Sigma_S : \dot{x} = A(t)x \quad A(t) \in \{A_1, \ldots, A_k\} \tag{1}$$

is uniformly asymptotically stable under arbitrary switching. Throughout the paper, when we speak of the stability (uniform asymptotic stability) of a switched linear system, we mean stability (uniform asymptotic stability) under arbitrary switching.

Formally checking for the existence of a CQLF amounts to looking for a single positive definite matrix $P = P^T > 0$ in $\mathbb{R}^{n \times n}$ satisfying the k Lyapunov inequalities

$$A_i^T P + P A_i < 0 \quad i \in \{1, \ldots, k\}. \tag{2}$$

If such a P exists, then $V(x) = x^T Px$ defines a CQLF for the LTI systems Σ_{A_i}. While the existence of such a function is sufficient to assure the uniform asymptotic stability of the system (1), it is in general not necessary for stability [9]. Hence CQLF existence is in general a conservative way of establishing stability for switched linear systems. However, recent work has established a number of system classes for which this is not necessarily the case [10, 11]. The results in these papers relate the existence of an unbounded solution to a switched linear system to the Hurwitz-stability of the convex hull of a set of matrices and are based on the following theorem.

Theorem 2.1 *[12, 13] Let $A_i \in \mathbb{R}^{n\times n}$, $i = \{1, 2\}$, be Hurwitz matrices. A sufficient condition for the existence of an unstable switching signal for the system*

$$\dot{x} = A(t)x,\ A(t) \in \{A_1, A_2\},$$

is that $A_1 + \gamma A_2$ has an eigenvalue with a positive real part for some positive γ.

The relationship between the existence of a CQLF, the existence of an unbounded solution to a switched linear system and the Hurwitz-stability of convex sets of matrices will play a crucial role in this paper.

Finally, we note that a defining characteristic of switched positive linear systems is that any trajectory originating in the positive orthant will remain there as time evolves. Consequently, to demonstrate the stability of such systems, one need not search for a common quadratic Lyapunov function, but rather the existence of a copositive Lyapunov function. Formally checking for the existence of a copositive CQLF amounts to looking for a single symmetric matrix P such that $x^T Px > 0$ for $x \in \mathbb{R}^n, x \succeq 0$, $x \neq 0$, satisfying the k Lyapunov inequalities

$$x^T(A_i^T P + PA_i)x^T < 0 \quad i \in \{1, \dots, k\},\ \forall x\ \succeq 0. \tag{3}$$

3 Second Order Positive Linear Systems

In this section, we shall show that the conjecture in [14] is true for second order positive switched linear systems. To begin with, we recall the result of [10] which described necessary and sufficient conditions for the existence of a CQLF for a pair of general second order LTI systems.

Theorem 3.1 *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz. Then a necessary and sufficient condition for Σ_{A_1}, Σ_{A_2} to have a CQLF is that the matrix products A_1A_2 and $A_1A_2^{-1}$ have no negative eigenvalues.*

We next show that it is only necessary to check one of the products in the above theorem if the individual systems $\Sigma_{A_1}, \Sigma_{A_2}$ are positive systems.

Lemma 3.1 *Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz and Metzler. Then the product A_1A_2 has no negative eigenvalue.*

Proof: First of all, as A_1, A_2 are both Hurwitz, the determinant of A_1A_2 must be positive. Secondly, a straightforward calculation shows that all of the diagonal entries of A_1A_2 must be positive. Hence the trace of A_1A_2 is also positive. It now follows easily that the product A_1A_2 cannot have any negative eigenvalues as claimed.

A straightforward combination of Theorem 3.1 and Lemma 3.1 yields the following result.

Theorem 3.2 *Let $\Sigma_{A_1}, \Sigma_{A_2}$ be stable positive LTI systems with $A_1, A_2 \in \mathbb{R}^{2\times 2}$. Then the following statements are equivalent:*

(a) Σ_{A_1} and Σ_{A_2} have a CQLF;

(b) Σ_{A_1} and Σ_{A_2} have a common copositive quadratic Lyapunov function;

(c) The switched system

$$\dot{x} = A(t)x, \quad A(t) \in \{A_1, A_2\}$$

is uniformly asymptotically stable;

(d) The matrix product $A_1A_2^{-1}$ has no negative real eigenvalues.

Proof :

(a) $\Leftrightarrow$ (d)

From Lemma 3.1 it follows that the matrix product A_1A_2 cannot have a negative eigenvalue. Hence, from Theorem 3.1, a necessary and sufficient condition for a CQLF for Σ_{A_1} and Σ_{A_2} is that $A_1A_2^{-1}$ has no negative eigenvalue.

(b) $\Leftrightarrow$ (d)

Since the fact that $A_1A_2^{-1}$ has no negative eigenvalue is necessary and sufficient for a CQLF for Σ_{A_1} and Σ_{A_2}, it follows that this condition is certainly sufficient for the existence of a copositive common quadratic Lyapunov function for this pair of positive LTI systems. Suppose now that $A_1A_2^{-1}$ has a negative eigenvalue. It follows that $A_1 + \gamma A_2$ has a real non-negative eigenvalue for some $\gamma_0 > 0$. Since, $A_1 + \gamma_0 A_2 = N - \alpha_0 I$, where $N \succeq 0$, it follows that the eigenvector corresponding to this eigenvalue is the Perron eigenvector of N and consequently lies in the positive orthant [2]. It follows that a copositive Lyapunov function cannot exist and the condition that $A_1A_2^{-1}$ has no negative eigenvalue is necessary and sufficient for the existence of a copositive common quadratic Lyapunov function for Σ_{A_1} and Σ_{A_2}.

(c) $\Leftrightarrow$ (d)

Suppose that $A_1A_2^{-1}$ has a negative eigenvalue; namely, $A_1 + \gamma A_2$ is non-Hurwitz for some $\gamma > 0$. It now follows from Theorem 2.1 that there exists some switching signal for which the switched system

$$\Sigma_S : \dot{x} = A(t)x \quad A(t) \in \{A_1, A_2\} \tag{4}$$

is not uniformly asymptotically stable. This proves that (c) implies (d). Conversely, if $A_1A_2^{-1}$ has no negative eigenvalues, then Σ_{A_1}, Σ_{A_2} have a CQLF and (4) is uniformly asymptotically stable. This completes the proof.

The equivalence of (c) and (d) in the previous theorem naturally gives rise to the following question. Given a finite set $\{A_1, \ldots, A_k\}$ of Metzler, Hurwitz matrices in $\mathbb{R}^{2\times 2}$, does the Hurwitz stability of $CO(A_1, ..., A_k)$ imply the uniform asymptotic stability of the associated switched system? This is indeed the case and follows from the following theorem, which can be thought of as an edge theorem for positive systems. Formally, we can state this result as follows.

Theorem 3.3 *Let $A_1, \ldots, A_k$ be Hurwitz, Metzler matrices in $\mathbb{R}^{2\times 2}$. Then the positive switched linear system,*

$$\dot{x} = A(t)x \quad A(t) \in \{A_1, \ldots, A_k\}, \tag{5}$$

is uniformly asymptotically stable (GUAS) if and only if each of the switched linear systems,

$$\dot{x} = A(t)x \quad A(t) \in \{A_i, A_j\}, \tag{6}$$

for $1 \leq i < j \leq k$ is uniformly asymptotically stable.

Outline of Proof

In the interests of clarity, we now outline the main steps involved in proving that the uniform asymptotic stability of each of the systems (6) implies that the overall system (5) is uniformly asymptotically stable. The proof of the converse is straightforward.

The basic idea is to use a piecewise quadratic Lyapunov function to prove uniform asymptotic stability of the system (5). We show that such a function may always be constructed provided that each of the systems (6) is uniformly asymptotically stable.

(a) The first step is to show that the non-negative orthant, $\mathbb{R}^2_+$, can be partitioned into a finite collection of cones or wedges, Sec_j, $1 \leq j \leq m$ with the following property: there are integers $L(j)$, $U(j)$ in $\{1, \ldots, k\}$ for each j, $1 \leq j \leq m$, with

$$Cone(A_1 x, \ldots, A_k x) = Cone(A_{L(j)}x, A_{U(j)}x).$$

Effectively, this fact means that we need only consider the two systems corresponding to $A_{L(j)}$, $A_{U(j)}$ within the region Sec_j.

(b) We next construct quadratic forms, $x^T P_j x$ for $1 \leq j \leq m$ which are non-increasing along each trajectory of (5) within Sec_j. Formally, for $x \in Sec_j$ and $1 \leq i \leq k$,

$$x^T(A_i^T P_j + P_j A_i)x \leq 0.$$

(c) The forms in (b) are next used to show that the system (5) has uniformly bounded trajectories.

(d) Finally, a simple perturbation argument shows that, by choosing a sufficiently small $\epsilon > 0$ the same conclusion will hold if we replace each system matrix A_i with $A_i + \epsilon I$. This then establishes the uniform asymptotic stability of the system (5).

Proof : It is immediate that if the system (5) is uniformly asymptotically stable (for arbitrary switching), then each of the systems (6) is also.

Now suppose that for each i, j with $1 \le i < j \le k$, the system (6) is uniformly asymptotically stable. We can assume without loss of generality that for all $a > 0$ and $1 \le i < j \le k$, the matrix $A_i - aA_j$ is not zero. Let $\mathbb{R}^2_+$ be the nonnegative orthant in $\mathbb{R}^2$. For any vector $x \in \mathbb{R}^2$,

$$Cone(A_1x, \ldots, A_kx) = \cup_{1 \le i < j \le k} Cone(A_ix, A_jx).$$

Moreover, as the switched system (6) is uniformly asymptotically stable for $1 \le i < j \le k$ and the system matrices are Metzler, $Cone(A_ix, A_jx) \cap \mathbb{R}^2_+ = \{0\}$ for all $1 \le i < j \le k$ and nonzero $x \in \mathbb{R}^2_+$. Therefore $Cone(A_1x, \ldots, A_kx) \cap \mathbb{R}^2_+ = \{0\}$.

For a nonzero vector $x \in \mathbb{R}^2$ define $arg(x)$, the argument of x in the usual way, viewing x as a complex number. Let $(l(x), u(x)), 1 \le l(x), u(x) \le k$ be a pair of integers such that $arg(A_{l(x)}x) \le arg(A_ix) \le arg(A_{u(x)}x)$. Then clearly

$$Cone(A_1x, \ldots, A_kx) = Cone(A_{l(x)}, A_{u(x)}).$$

For $1 \le i, j \le k$ define

$$D_{(i,j)} \quad = \quad \{y \in \mathbb{R}^2_+, y \ne 0 : Cone(A_1y, \ldots, A_ky) = Cone(A_iy, A_jy)\}$$

Here (i, j) is a pair of integers, not necessarily ordered and possibly equal and $arg(A_iy) \le arg(A_jy)$. It now follows that $\mathbb{R}^2_+ - \{0\} = \cup_{1 \le i,j \le k} D_{(i,j)}$. Note that $D_{(i,j)} \cup \{0\}$ is a closed cone, not necessarily convex and that if $x \in D_{(i,j)}$ and $arg(A_ix) < arg(A_mx) < arg(A_jx)$ for $m \ne i, j$ then x belongs to the interior of $D_{(i,j)}$.

Consider

$$Symp = \{\hat{a} =: (a, 1-a)^T : 0 \le a \le 1\},$$

and define

$$d_{(i,j)} = Symp \cap D_{(i,j)}.$$

We shall write $\hat{a} < \hat{b}$ if and only if $a < b$. The sets $d_{(i,j)}$ are closed and their (finite) union is equal to *Symp*. Moreover, the only way for $x \in Symp$ not to lie in the interior of some $d_{(i,j)}$ is if there exists $b > 0, 1 \le l \ne m \le k$ such that $A_lx = bA_mx$. As we assumed that for all $a > 0, 1 \le i < j \le k$ the matrix $A_i - aA_j$ is not zero, it follows that there exists a finite subset $Sing =: \{0 \le \hat{a_1} < ... < \hat{a_q} \le 1\}$ such that all vectors in $Symp - Sing$ belong to the interior of some $d_{(i,j)}$.

It now follows that *symp* can be partitioned into a finite family of closed intervals, each of them contained in some $d_{(i,j)}$. This in turn defines a partition of $\mathbb{R}^2_+ - \{0\}$ into finitely many closed cones/wedges Sec_j, $1 \le j \le m$, each of which is contained in some $D_{(L(j),U(j))}$. We shall label the rays which define this partition $r_1, \ldots, r_{m+1}$ where r_1 is the y-axis, r_{m+1} is the x-axis and the rays are enumerated in the clockwise direction.

Now, by assumption, the switched system

$$\dot{x} = A(t)x \quad A(t) \in \{A_{L(j)}, A_{U(j)}\}$$

is uniformly asymptotically stable for all $1 \le j \le m$. Thus, it follows from Theorem 3.2 that, for $1 \le j \le m$, there exist quadratic forms x^TP_jx, $P_j = P_j^T > 0$, such that

$$A^T_{L(j)}P_j + P_jA_{L(j)} < 0, \quad A^T_{U(j)}P_j + P_jA_{U(j)} < 0.$$

As $Sec_j \subset D_{(L(j),U(j))}$, it follows that $x^T P_j A_i x \leq 0$ for all $x \in Sec_j$ and all i with $1 \leq i \leq k$.

Using the above quadratic forms, we shall now show that the trajectories of the system (5) are uniformly bounded. First, choose a point $T_1 = (0, y)^T$, $y > 0$ and consider the level curve of $x^T P_1 x$ which passes through T_1. This curve must intersect the second ray r_2 at some point T_2; now consider the level curve of $x^T P_2 x$ going through T_2, it will intersect the third ray r_3 at some point T_3; continue this process until we reach some point T_{m+1} on the x-axis. This gives us a domain bounded by the y-axis, the chain of ellipsoidal arcs defined above and the x-axis.

The domain constructed above is an invariant set for the switched system (5) and, hence, this system has uniformly bounded trajectories. By a simple perturbation argument the same conclusion will hold if we replace the system matrices $A_1, \ldots, A_k$ with $\{A_1 + \epsilon I, .., A_k + \epsilon I\}$ for some small enough positive ϵ. This implies that the original system (5) is in fact uniformly asymptotically stable and completes the proof of the theorem.

4 Higher Dimensional Systems

In view of the results obtained above for second order systems, and keeping in mind that the trajectories of a positive switched linear system are constrained to lie in the positive orthant for all time, it may seem reasonable to hope that analogous results could be obtained for higher dimensional systems. Recently, such considerations have led a number of authors to the following conjecture.

> **Conjecture 1**
>
> Let $A_1, \ldots, A_k$ be a finite family of Hurwitz, Metzler matrices in $\mathbb{R}^{n \times n}$. Then the following statements are equivalent:
>
> (i) All matrices in the convex hull, $CO(A_1, \ldots, A_k)$, are Hurwitz;
>
> (ii) The switched linear system
>
> $$\dot{x} = A(t)x \quad A(t) \in \{A_1, \ldots, A_k\}$$
>
> is uniformly asymptotically stable.

Unfortunately, while this conjecture is both appealing and plausible, it is untrue in general for higher dimensional systems. In the remainder of this section, we shall present a counterexample to Conjecture 1, based on arguments first developed by Gurvits in [15](which extended the results in [16, 17]).

Systems with Invariant Cones

The result of the following lemma is fundamental to our construction of a counterexample to Conjecture 1.

Lemma 4.1 *Let $A_1, \ldots, A_k$ be a finite family of matrices in $\mathbb{R}^{n \times n}$. Assume that there exists a proper polyhedral convex cone Ω in $\mathbb{R}^n$ such that $exp(A_i t)(\Omega) \subseteq \Omega$ for all $t \geq 0$ and $1 \leq i \leq k$.*

Then there is some integer $N \geq n$ and a family of Metzler matrices $A_1^M, \ldots A_k^M$ in $\mathbb{R}^{N \times N}$ such that:

(i) All matrices in $CO(A_1, \ldots, A_k)$ are Hurwitz if and only if $CO(A_1^M, \ldots, A_k^M)$ consists entirely of Hurwitz matrices;

(ii) The switched linear system $\dot{x} = A(t)x$, $A(t) \in \{A_1, \ldots, A_k\}$ is uniformly asymptotically stable if and only if the positive switched linear system $\dot{x} = A(t)x$, $A(t) \in \{A_1^M, \ldots, A_k^M\}$ is uniformly asymptotically stable.

Proof:

As Ω is polyhedral, solid and pointed, we can assume without loss of generality that there exist vectors $z_1, \ldots, z_N$ in $\mathbb{R}^n$, with $N \geq n$, such that:

(i) $\Omega = Cone(z_1, \ldots, z_N)$;

(ii) There is some $h \in \mathbb{R}^n$ with $h^T z_i = 1$ for $1 \leq i \leq N$.

From this, it follows that, for $1 \leq i \leq k$, $exp(A_i t)(\Omega) \subseteq \Omega$ for all $t \geq 0$ if and only if there is some $\tau > 0$ such that $I + \tau A_i(\Omega) \subseteq \Omega$.

Define a linear operator $\Phi : R^N \to R^n$ by $\Phi(e_i) = z_i$ for $1 \leq i \leq N$ where $e_1, \ldots, e_N$ is the standard basis of $\mathbb{R}^N$. We shall now show how to construct Metzler matrices $A_i^M \in \mathbb{R}^{N \times N}$ satisfying the requirements of the lemma.

First, we note the following readily verifiable facts:

(i) For any trajectory,

$$x(t) = \sum_{1 \leq i \leq N} \alpha_i(t) z_i, \;\; \alpha_i(t) \geq 0, \; t \geq 0$$

in Ω, $\lim_{t \to \infty} x(t) = 0$ if and only if $\lim_{t \to \infty} \alpha_i(t) = 0$ for $1 \leq i \leq N$;

(ii) For each $i \in \{1, \ldots, k\}$ and $q \in \{1, \ldots, N\}$, we can write (non-uniquely)

$$A_i(z_q) = \sum_{p=1}^{N} a_{pq} z_p \qquad \text{where } a_{pq} \geq 0 \text{ if } p \neq q.$$

In this way, we can associate a Metzler matrix, $A_i^M = (a_{pq} : 1 \leq p, q \leq N)$ in $\mathbb{R}^{N \times N}$ with each of the system matrices A_i in $\mathbb{R}^{n \times n}$.

(iii) By construction, $\Phi A_i^M = A_i \Phi$ and $\Phi(exp(A_i^M t)) = (exp(A_i t))\Phi$ for all $t \geq 0$. Hence, $A_i^{(M)}$ is Hurwitz if and only if A_i is Hurwitz for $1 \leq i \leq k$.

From points (i) and (iii) above we can conclude that all matrices in the convex hull $CO(A_1, ..., A_k)$ are Hurwitz if and only if all matrices in the convex hull $CO(A_1^{(M)}, ..., A_k^{(M)})$ are Hurwitz. Moreover, the switched linear system

$$\dot{x} = A(t)x, \;\; A(t) \in \{A_1, \ldots, A_k\}$$

is uniformly asymptotically stable if and only if the positive switched linear system

$$\dot{x} = A(t)x, \;\; A(t) \in \{A_1^M, \ldots, A_k^M\}$$

is uniformly asymptotically stable. This proves the lemma.

Comments

(i) It follows from Lemma 4.1 that if Conjecture 1 was true then the same statement would also hold for switched linear systems having an invariant proper polyhedral convex cone.

(ii) In the course of the proof, it was shown that, for $A \in \mathbb{R}^{n\times n}$ and a proper polyhedral cone $\Omega \in \mathbb{R}^n$,

$$exp(At)(\Omega) \subseteq \Omega \text{ for all } t \geq 0$$

if and only if

$$(I + \tau A)(\Omega) \subseteq \Omega$$

for some $\tau > 0$.

Lyapunov Operators

Given a matrix $A \in \mathbb{R}^{n\times n}$, define the linear operator $\hat{A}$, on the space of $n \times n$ real symmetric matrices, by

$$\hat{A}(X) = A^T X + XA. \tag{7}$$

Consider the linear dynamical system on the space of symmetric matrices in $\mathbb{R}^{n\times n}$ given by

$$\dot{X} = \hat{A}(X). \tag{8}$$

It is a straightforward exercise to verify that if $x_1(t)$ and $x_2(t)$ are solutions of the system $\dot{x} = A^T x$ with initial conditions $x_1(0) = x_1$, $x_2(0) = x_2$, then $x_1(t)x_2(t)^T + x_2(t)x_1(t)^T$ is a solution of the linear system (8) with initial conditions $x_1x_2^T + x_2x_1^T$. The following result follows easily by combining this observation with standard facts about the existence and uniqueness of solutions to linear systems.

Lemma 4.2 *Consider a family, $\{A_1, \ldots, A_k\}$, of matrices in $\mathbb{R}^{n\times n}$. Then:*

(i) $CO(A_1, \ldots, A_k)$ consists entirely of Hurwitz stable matrices if and only if all of the operators in $CO(\hat{A}_1, \ldots, \hat{A}_k)$ are Hurwitz stable;

(ii) The cone, $PSD(n)$, of positive semi-definite matrices in $\mathbb{R}^{n\times n}$ is an invariant cone for the switched system

$$\dot{X} = \hat{A}(t)X \quad \hat{A}(t) \in \{\hat{A}_1, \ldots, \hat{A}_k\}; \tag{9}$$

(iii) The system (9) is uniformly asymptotically stable if and only if the system

$$\dot{x} = A(t)x \quad A(t) \in \{A_1, \ldots, A_k\}$$

is uniformly asymptotically stable.

The Counterexample

Using Lemmas 4.1 and 4.2, we can now present a counterexample to Conjecture 1 above. To begin, consider the following two matrices in $\mathbb{R}^{2\times 2}$

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$$

where $a > b \geq 0$. Then, for some $t_1, t_2 > 0$ the spectral radius $\rho((exp(A_1 t_1)(exp(A_2 t_2)) > 1$. In fact, if we take $a = 2, b = 1$ then this is true with $t_1 = 1, t_2 = 3/2$.

By continuity of eigenvalues, if we choose $\epsilon > 0$ sufficiently small, we can ensure that

$$\rho((exp((A_1 - \epsilon I)t_1))(exp((A_2 - \epsilon I)t_2))) > 1.$$

Hence, the switched linear system associated with the system matrices $A_1 - \epsilon I$, $A_2 - \epsilon I$ is unstable and moreover, all matrices in the convex hull $CO(\{A_1 - \epsilon I, A_2 - \epsilon I\})$ are Hurwitz.

The above remarks establish the existence of Hurwitz matrices B_1, B_2 in $\mathbb{R}^{2\times 2}$ such that:

(i) All matrices in $CO(B_1, B_2)$ are Hurwitz;

(ii) The switched linear system $\dot{x} = A(t)x$, $A(t) \in \{B_1, B_2\}$ is unstable.

Next, consider the Lyapunov operators, $\hat{B}_1$, $\hat{B}_2$ on the symmetric 2×2 real matrices. It follows from Lemma 4.2 that $CO(\hat{B}_1, \hat{B}_2)$ consists entirely of Hurwitz stable operators and that the switched linear system associated with $\hat{B}_1$, $\hat{B}_2$ is unstable, and leaves the proper (not polyhedral) cone $PSD(2)$ invariant. Formally, $exp(\hat{B}_i t)(PSD(2)) \subseteq PSD(2)$ for $i = 1, 2$, and all $t \geq 0$.

From examining the power series expansion of $exp(\hat{B}_i t)$, it follows that for any $\epsilon > 0$, there exists $\tau > 0$ and two linear operators $\Delta_i, i = 1, 2$ such that

$$(\tau I + \hat{B}_i + \Delta_i)(PSD(2)) \subseteq PSD(2) \tag{10}$$

with $||\Delta_i|| < \epsilon$ for $i = 1, 2$.

Combining the previous fact with standard results on the existence of polyhedral approximations of arbitrary proper cones in finite dimensions, we can conclude that for any $\epsilon > 0$, there exists a proper polyhedral cone $PH_\epsilon \subset PSD(2)$, and two linear operators δ_i, $i = 1, 2$ such that

$$(\tau I + \hat{B}_i + \Delta_i + \delta_i)(PH_\epsilon) \subseteq PH_\epsilon \tag{11}$$

with $||\Delta_i||, ||\delta_i|| < \epsilon$ for $i = 1, 2$.

Recall that $CO(\hat{B}_1, \hat{B}_2)$ consists entirely of Hurwitz-stable operators and that the switched linear system associated with $\hat{B}_1$, $\hat{B}_2$ is unstable. For $\epsilon > 0$, define the linear operators $B_{i,\epsilon} = \hat{A}_i + \Delta_i + \delta_i$ for $i = 1, 2$. By choosing $\epsilon > 0$ sufficiently small, we can ensure that:

(i) All operators in $CO(B_{1,\epsilon}, B_{2,\epsilon})$ are Hurwitz-stable;

(ii) The switched linear system

$$\dot{x} = A(t)x \quad A(t) \in \{B_{1,\epsilon}, B_{2,\epsilon}\}$$

is unstable. Moreover, from (11) this switched linear system leaves the proper, polyhedral cone PH_ϵ invariant.

Thus, the statement of Conjecture 1 is not true for switched linear systems with an invariant proper, polyhedral cone and hence, it follows from Lemma 4.1 that Conjecture 1 itself is also false.

5 Conclusions

In this paper we have presented a counterexample to a recent conjecture presented in [14], and formulated independently by David Angeli, concerning the uniform asymptotic stability of switched positive linear systems. In particular, we have shown that the stability of a positive switched linear system is not in general equivalent to the Hurwitz stability of the convex hull of its system matrices. While this conjecture is now known to be false, counterexamples are rare, and the construction presented here suggests that such examples may only exist in very high dimensions. This gives some hope that it may be possible to extend some of the work reported here and to derive related, straightforward conditions for CQLF existence for subclasses of positive systems of practical interest.

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