# Information Diffusion on the Iterated Local Transitivity Model of Online Social Networks

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**Abstract**

We study a recently introduced deterministic model of competitive information diffusion on the Iterated Local Transitivity (ILT) model of Online Social Networks (OSNs). In particular, we show that, for 2 competing agents, an independent Nash Equilibrium (N.E.) on the initial graph remains a N.E. for all subsequent times. We also describe an example showing that this conclusion does not hold for general N.E. in the ILT process.

*Keywords:* Online Social Networks (OSNs); Information Diffusion; Nash Equilibria; Iterated Local Transitivity.

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## 1. Introduction and Motivation

Understanding how information and rumours spread is a key issue for modern society. Malicious or inaccurate rumours can lead to unnecessary panic and generate social and economic instability. From another perspective, understanding how information propagates through a population is a necessary first step in the design of viral marketing campaigns [1]. Recent advances in communications and the emergence of social networking sites such as Facebook and Twitter have greatly increased the power of individual agents to disseminate information. This provides strong motivation for analysing the mechanisms of information propagation and the role played by the individual in the process.
In this paper, we build on recent work in [5] in which a simple model of competitive information diffusion was introduced. The model in [5] considers the diffusion process as a competitive game taking place on a graph, which captures

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the underlying social structure. The model considers interested parties (agents) $\{1, \ldots, n\}$ who wish to propagate their idea or innovation through the network. The agents are initially assigned vertices $\mathbf{x} = (x_1, \ldots, x_n)$, which they "colour" at the first time-step in the diffusion process. At each following time-step, uncoloured vertices adjacent to vertices that are already coloured are coloured according to the following rules. If two vertices of different colour neighbour an uncoloured vertex, then in the next time-step this vertex is coloured grey. Grey nodes are treated differently to other colours and do not propagate; they represent individuals who choose to adopt neither idea and who do not pass on either idea. If an uncoloured vertex is adjacent to vertices of only one colour, then the uncoloured vertex takes this colour. All other uncoloured vertices remain uncoloured. This represents the spread of an idea through a social network. The diffusion process ends when no further vertices can be coloured. The utility $U_i(\mathbf{x})$ of agent $i$ is then the number of vertices coloured $i$ when the process ends.

A central theme in the study of games is the existence of Nash equilibria. A Nash Equilibrium (N.E.) occurs if no agent benefits from unilaterally changing its starting vertex. Formally, this means that $U_i((x_1, \ldots x_{i-1}, v, x_{i+1}, \ldots, x_n)) \leq U_i(\mathbf{x})$ for all $i$ and $v \neq x_i$. The authors of [5] considered the question of when N.E. exist for graphs of diameter 2. In [6], an example was presented to show that even in the case of a graph of diameter 2, with 2 agents, a N.E. need not exist. It is possible to ensure the existence of N.E. for graphs of diameter 2 under additional technical assumptions. In keeping with the wish of the authors of [5] that their results be extended to other models of social networks, we consider their model on a recently proposed model of online social networks (OSNs); namely, the Iterated Local Transitivity (ILT) model [2]. We present a result describing conditions in which a N.E. for a 2-agent game on the initial graph in the ILT process remains a N.E. throughout the process. The advantage of this is that it allows us to establish the existence of N.E. for quite complicated graphs, starting from a simple initial network.

The layout of the paper is as follows. In Section 2, we review related work on mathematical modeling of diffusion and social networks. We introduce the main notation used in the paper in Section 3 and present some preliminary technical results in Section 4. Then, in Section 5 we consider the diffusion process of [5] on the ILT model of social networks. We show that if an independent N.E. exists in the initial (seed) graph, then it remains a N.E. for all subsequent graphs in the ILT process. An example is described to illustrate that this conclusion will not necessarily hold for general, non-independent Nash equilibria. We also highlight differences between the diffusion process of [5] and that based on Voronoi games in this section. Finally, in Section 6 we present our conclusions and outline possible directions for future work.

## 2. Related Work

We now give a very brief review of some recent models of OSNs; for more details consult [14]. In [3] a simple deterministic model for network growth based on the matrix theoretic Kronecker product was introduced. This model generates graphs whose diameter decreases over time and also obeys a *densification power law*; both of these properties have been observed in data on real OSNs. Moreover, as highlighted in [3], the simplicity of the model renders it more amenable to rigorous analysis than, typically more complex, stochastic models for network growth. The so-called forest fire model also possess a densification power law and shrinking diameter; however its definition is more complicated than the Kronecker model and involves more parameters. The ILT model we adopt here is based on two fundamental observations concerning the nature of social interactions: transitivity of connections and local growth rules (the update rule for Kronecker graphs is essentially global in nature). The ILT model also reflects the community structure of social networks, as reflected in its spectral and expansion properties. As argued in [2], apart from its theoretical interest, it provides a simple mathematical metaphor that can be used to obtain insights into complex processes taking place on social networks. Given the complexity of the processes behind competitive diffusion, having a graph model of the network that is sufficiently simple to make analysis tractable is a distinct advantage. The same arguments can be advanced for the Kronecker model, and while we do not study these models here, it is hoped that similar analysis can be done for this model class in future work.

While several authors have considered the problem of innovation and information diffusion through networks, relatively little has been done on competitive diffusion. The *threshold model* considered in [13] considers a single innovation and the question of how best to select an initial set of seed nodes to propagate the innovation. The work of [10] is more closely related to our results as it is game-theoretic in nature. However, the models considered in these papers differ fundamentally from the one analysed here. In particular, they are concerned with a single innovation and treat each node in the network as an *agent* in the game, who must choose between adopting or not adopting the innovation. The payout depends on the choices made by an agent's neighbours. In contrast, the model we study views the agents as existing outside the network and competing against each other. Each agent selects a seed node in the network with the aim of maximising the number of nodes adopting their innovation or idea.

As highlighted by the work of [10, 4, 12], the analysis of network games is far from straightforward. For the related Voronoi game on graphs, the computation or identification of N.E. for a fixed number of agents involves exhaustively enumerating all possible strategy profiles. While this provides a polynomial algorithm, it is nonetheless a computational bottleneck, particularly as network size increases. Motivated by this, the authors of [12] sought graphical properties that guarantee the existence of a N.E. for the Voronoi game. Even for the simple case of 2 agents, their results are restrictive, and apply only to a subclass of *strongly transitive* graphs. For the simple model considered here, providing conditions for the existence of Nash equilibria has also proven troublesome. In [6], an example is given of a graph of diameter 2 for which no N.E. exists even for the case of 2 agents. Our approach is to take advantage of the iterative nature of many graph models for social networks. If it is possible to show that the existence of a N.E. on a graph at any time in the process implies the existence of one at the next time step, then a simple inductive argument can be used to establish the existence of N.E.s for quite complex graphs, starting from simple initial networks. We hope that this simple paradigm may prove useful in deriving results for more realistic and complex models than those we consider.

## 3. Notation and Background

Our graph theoretical terminology and notation is standard [7]. The graphs we consider are finite and undirected; formally, a graph $G$ consists of a set of nodes $V(G)$ and a set of edges $E(G)$ of the form $\{v, w\}$ for $v, w \in V(G)$, $v \neq w$. For notational simplicity, we denote the edge $\{v, w\}$ by $vw$ (or $wv$). The neighbourhood $N(v)$ of $v \in V(G)$ is defined as

$$N(v) = \{u \in V(G) : uv \in E(G)\}.$$

For a set $X$, we denote the cardinality of $X$ by $|X|$.

### *3.1. The Diffusion Process* $\boldsymbol{D}$

[5]

Consider a graph $G$ with vertex set $V(G)$, $|V(G)| = N$ and a set of agents indexed as $[1, n] = \{1, \dots, n\}$. At time 0, each agent $i \in [1, n]$ selects a seed node, $x_i$, in $V(G)$, which is labelled (or coloured) $i$. The $n$-tuple $\mathbf{x} = (x_1, \dots, x_n)$ is known as a *strategy profile*. Throughout the paper, we shall only consider strategy profiles in which all of the $x_i$ are distinct. All other nodes at time 0 are labelled 0 (corresponding to white nodes in [5]). In addition to the labels $0, 1, \dots, n$ we also use the label $-1$ to denote grey nodes. In keeping with the original model of [5], grey nodes *do not propagate*. For $v \in V(G)$ and $t \geq 0$, we use $l^t(v)$ to denote the label of $v$ at time $t$.

The labelling map $l^0$ is given by

$$l^0(v) = \begin{cases} i & \text{if } v = x_i \text{ where } 1 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

At each subsequent time $t \geq 1$, we define $l^t$ as follows. If $l^{t-1}(v) \neq 0$, then $l^t(v) = l^{t-1}(v)$ (so only vertices labelled 0 can change their label). For nodes $v$ with $l^{t-1}(v) = 0$, let $\mathcal{L}(v) = \{l^{t-1}(w) : w \in N(v)\}$ denote the set of labels of the neighbours of $v$. Then:

- if $\mathcal{L}(v) \cap [1, n] = \{i\}$, then $l^t(v) = i$;
- if $|\mathcal{L}(v) \cap [1, n]| \geq 2$ then $l^t(v) = -1$;
- $l^t(v) = l^{t-1}(v)$ otherwise.

For $i \in [1, n]$, we denote by $L_i(t)$ those nodes labelled $i$ at exactly time $t$. Formally $L_i(t) = \{v \in V(G) : l^{t-1}(v) = 0, l^t(v) = i\}$. The process *terminates* at some time $t$ if $L_i(t+1)$ is empty for all $1 \in [1, n]$. Thus no new vertices are labelled $i \in \{1, \ldots n\}$ in the time step $t \to t+1$. Note however, that it is possible for nodes to be labelled $-1$ in the step $t \to t+1$. As we are only interested in vertices ultimately labelled $i \in [1, n]$, this is unimportant. Clearly, the process must terminate before time $t = N - n$.

### *3.2. Iterated Local Transitivity (ILT) Graphs*

In [2], the *Iterated Local Transitivity (ILT)* model for online social networks was introduced. Let an initial, connected graph $G_0$ be given. For each time $t \geq 1$, $G_t$ is formed by adding a *clone* $v'$ of every node $v$ in $V(G_{t-1})$, an edge $vv'$ and an edge $v'w$ between $v'$ and every neighbour $w$ of $v$ in $G_t$. Several basic properties of this model were derived in [2] and a stochastic extension was also introduced.

## 4. Preliminary Results

We are interested in the following question for the ILT model discussed in Section 3. When does the existence of a N.E. in the initial graph $G_0$ imply the existence of a N.E. in $G_t$ for all $t \geq 0$?

We adopt the following notation. For a connected graph $G$, $\hat{G}$ denotes the graph obtained through applying one step of the ILT process to $G$. For $1 \leq i \leq n$, $t \geq 0$, $\hat{L}_i(t)$ denotes the set of nodes in $\hat{G}$ labelled $i$ at *exactly* time $t$. We also use $W(t)$ ($\hat{W}(t)$) to denote the *white* nodes in $G$ ($\hat{G}$) at time $t$. Formally, $W(t) = \{v \in V(G) : l^t(v) = 0\}$ ($\hat{W}(t) = \{v \in V(\hat{G}) : l^t(v) = 0\}$).

For a set $U \subseteq V(G)$ ($U \subseteq V(\hat{G})$), $N(U)$ ($\hat{N}(U)$) denotes the neighbours of $U$ in $G$ ($\hat{G}$).

If a strategy profile $\mathbf{x} = (x_1, \ldots, x_n)$ consists entirely of vertices from $V(G)$, we say that it is a strategy profile in $G$. If the set $\{x_1, \ldots, x_n\}$ is an independent set, we say that the strategy profile $\mathbf{x}$ is independent.

**Definition 4.1.** *Let a connected graph $G$, a set of agents $[1, n] = \{1, \ldots, n\}$ and a strategy profile $\mathbf{x}$ be given. If the diffusion process $\boldsymbol{D}$ terminates at time $T$ on $G$, the Utility $U_i(\mathbf{x})$ of agent $i \in [1, n]$ is given by*

$$U_i(\mathbf{x}) = |\{v \in V(G) : l_T(v) = i\}|$$

Informally, the utility of agent $i$ is the total number of nodes in $G$ labelled $i$ when the process terminates. We use $\hat{U}_i$ to denote utilities in $\hat{G}$.

Given a strategy profile $\mathbf{x}$, a node $v \notin \{x_1, \ldots, x_n\}$ and $i \in [1, n]$, we denote by $\mathbf{x}_{-i}(v)$ the profile given by

$$\mathbf{x}_{-i}(v) = (x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots x_n).$$

We next recall the definition of N.E..

**Definition 4.2.** *Let a connected graph $G$ and a set of agents $[1, n] = \{1, \ldots, n\}$ be given. A strategy profile $\mathbf{x}$ is a N.E. for the process $\boldsymbol{D}$ on $G$ if $U_i(\mathbf{x}) \geq U_i(\mathbf{x}_{-i}(v))$ for all $i \in [1, n]$ and all $v \in V(G) \backslash \{x_1, \ldots, x_n\}$.*

Note the following simple facts, which follow immediately from the definition of the process $\mathbf{D}$.

**Lemma 4.1.** *A vertex $v$ is in $L_i(t+1)$ if and only if: (i) $v \in W(t)$; (ii) $v \in N(L_i(t))$; (iii) $v \notin N(L_j(t))$ for all $j \in [1, n] - i$.*

**Lemma 4.2.** *A vertex $v$ is in $W(t+1)$ if and only if: (i)$v \in W(t)$; (ii) $v \notin N(L_j(t))$ for all $j \in [1, n]$.*

**Proposition 4.1.** *Let $(x_1, \ldots, x_n)$ be a strategy profile in a connected graph $G$. Then for $1 \leq i \leq n$*

$$\hat{L}_i(1) = \begin{cases} L_i(1) \cup (L_i(1))' \cup \{x_i'\} & \textit{if } \mathbf{x} \textit{ is independent} \\ L_i(1) \cup (L_i(1))' & \textit{otherwise} \end{cases}.$$

$$\hat{W}(1) = W(1) \cup W(1)'.$$

*Proof.* Note that $\hat{W}(0) = W(0) \cup (W(0))' \cup \{x_1', \ldots, x_n'\}$ and that $\hat{N}(x_j) = N(x_j) \cup (N(x_j))' \cup \{x_j'\}$ for $1 \leq j \leq n$. As $x_i' \in \hat{N}(x_j)$ if and only if $x_i \in N(x_j)$, the result follows readily from Lemma 4.1. □

The following proposition clarifies the relationship between $\hat{L}_i(t)$ and $L_i(t)$ for $t = 2, 3, \ldots$.

**Proposition 4.2.** *Let $(x_1, \ldots, x_n)$ be a strategy profile in a connected graph $G$. Then for $1 \leq i \leq n$, $t \geq 2$,*

$$\hat{L}_i(t) = L_i(t) \cup (L_i(t))'.$$

$$\hat{W}(t) = W(t) \cup W(t)'.$$

*Proof.* We prove the result by induction on $t$. First, we use Proposition 4.1 to establish the result for the case $t = 2$. Lemma 4.1 implies that a vertex $w \in V(\hat{G})$ is in $\hat{L}_i(2)$ if and only if: $w \in \hat{W}(1)$; $w \in \hat{N}(\hat{L}_i(1))$; $w \notin \hat{N}(\hat{L}_j(1))$ for $j \in [1, n] - i$. Using Proposition 4.1, we can see that $\hat{N}(x_i') \cap \hat{W}(1)$ is empty (this follows from Lemma 4.2 as $\hat{N}(x_i')$ is given by $\{x_i\} \cup N(x_i)$) . Furthermore, for $1 \leq k \leq n$

$$\hat{N}(L_k(1) \cup L_k(1)') = \hat{N}(L_k(1)) = N(L_k(1)) \cup N(L_k(1))'.$$

It follows from Lemma 4.1 that $\hat{L}_i(2) = L_i(2) \cup L_i(2)'$. The conclusion $\hat{W}(2) = W(2) \cup W(2)'$ follows from Proposition 4.1 and Lemma 4.2.
Now assume that the result is true for some $t \geq 2$. Lemma 4.1 implies that $w \in \hat{L}_i(t+1)$ if and only if: $w \in \hat{W}(t)$; $w \in \hat{N}(\hat{L}_i(t))$; $w \notin \hat{N}(\hat{L}_j(t))$ for $j \in [1, n] - i$. Using the induction hypothesis we see that for $k \in [1, n]$

$$\hat{N}(\hat{L}_k(t)) = \hat{N}(L_k(t) \cup L_k(t)') = \hat{N}(L_k(t)) = N(L_k(t)) \cup N(L_k(t))'.$$

As in the previous paragraph, it follows that

$$\hat{L}_i(t+1) = L_i(t+1) \cup (L_i(t+1))'.$$

Moreover, combining the induction hypothesis with Lemma 4.2 yields $\hat{W}(t+1) = W(t+1) \cup W(t+1)'$. □

## 5. Nash Equilibria and the ILT Model

In [5], the existence of Nash equilibria for the diffusion process was investigated on graphs of low diameter. We shall provide conditions under which such equilibria are guaranteed to exist for $G_t$ in the ILT model for all $t$.
The following lemma shows that in the 2-agent case, neither agent can improve their utility by unilaterally changing from $x_i$ to its clone $x_i'$.

**Lemma 5.1.** *Let $(x_1, x_2)$ be a strategy profile in $G$. Then*

$$\hat{U}_1(x_1', x_2) \leq \hat{U}_1(x_1, x_2).$$

*Proof.* We use $\bar{l}^t$ to denote the labelling map for the profile $(x_1', x_2)$, and $l^t$ for the labelling map for $(x_1, x_2)$. It is clear that $\bar{l}^1(v) = 1$ implies $l^1(v) = 1$, and that $l^1(v) = 2$ implies $\bar{l}^1(v) = 2$. Suppose that $\hat{U}_1(x_1', x_2) > \hat{U}_1(x_1, x_2)$. Then there must exist some $t > 1$ and $v$ such that $\bar{l}^t(v) = 1$, $l^t(v) \neq 1$. Let $t_0$ be the minimal $t > 1$ for which this occurs. It is immediate that $l^{t_0-1}(v) \neq 1$. If $t_0 - 1 = 1$, then this implies that $\bar{l}^{t_0-1}(v) \neq 1$. If $t_0 - 1 > 1$, then as $t_0$ is minimal, it also follows that $\bar{l}^{t_0-1}(v) \neq 1$. We can thus conclude that $\bar{l}^{t_0-1}(v) = 0$.
As $\bar{l}^{t_0}(v) = 1$, there is some $w_1 \in \hat{N}(v)$ with $\bar{l}^{t_0-1}(w_1) = 1$ and there exists no $w \in N(v)$ with $\bar{l}^{t_0-1}(w) = 2$. We know that $l^{t_0}(v) \neq 1$ and $l^{t_0-1}(w_1) = 1$. It follows from this that there must exist some $w_2 \in \hat{N}(v)$ with $l^{t_0-1}(w_2) = 2$. Moreover, we know that $\bar{l}^{t_0-1}(w_2) \neq 2$. This implies that $t_0 - 1 > 1$. Thus if we define $t_1$ to be the minimum $t > 1$ for which there exists $u$ with $\bar{l}^t(u) \neq 2$, $l^t(u) = 2$, we can see that $1 < t_1 < t_0$. A similar argument to that used above will show that there must exist some $w_3$ in $\hat{N}(u)$ such that $l^{t_1-1}(w_3) \neq 1$, $\bar{l}^{t_1-1}(w_3) = 1$. As $1 < t_1 < t_0$ (and this cannot happen for $t = 1$ so that $t_1 - 1 > 1$), this contradicts the minimality of $t_0$. This shows that $\hat{U}_1(x_1', x_2) \leq \hat{U}_1(x_1, x_2)$ as claimed. □

The example in Figure 1 below shows that the previous result need not hold for 3 or more agents. If $\mathbf{x} = (v_1, v_2, v_3)$ and $\mathbf{x}_1 = (v_1, v_2', v_3)$, then $\hat{U}_2(x) < \hat{U}_2(x')$.
The next lemma, which follows from Proposition 4.2, shows how the utility $U_i(\mathbf{x})$ of an agent on $G$ relates to its utility $\hat{U}_i(\mathbf{x})$ on $\hat{G}$ for a strategy profile $\mathbf{x}$ in $G$.

**Lemma 5.2.** *Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a strategy profile in $G$. Then*

$$\hat{U}_i(\mathbf{x}) = \begin{cases} 2U_i(\mathbf{x}) - 1 & \text{if } x_i \text{ is neighboured by some } x_j \\ 2U_i(\mathbf{x}) & \text{if } x_i \text{ is not neighboured by some } x_j \end{cases}$$