On the stability of switched positive linear systems

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Abstract

It was recently conjectured that the Hurwitz stability of the convex hull of a set of Metzler matrices is a necessary and sufficient condition for the asymptotic stability of the associated switched linear system under arbitrary switching. In this paper we show that: (i) this conjecture is true for systems constructed from a pair of second order Metzler matrices; (ii) the conjecture is true for systems constructed from an arbitrary finite number of second order Metzler matrices; and (iii) the conjecture is in general false for higher order systems. The implications of our results, both for the design of switched positive linear systems, and for research directions that arise as a result of our work, are discussed toward the end of the paper.

Key Words: Stability theory; Switched linear systems; Positive linear systems

I. Introduction

Positive dynamical systems are of fundamental importance to numerous applications in areas such as Economics, Biology, Sociology and Communications. Historically, the theory of positive linear time-invariant (LTI) systems has assumed a position of great importance in systems theory and has been applied in the study of a wide variety of dynamic systems [1], [2], [3], [4]. Recently, new studies in communication systems [5], formation flying [6], and other areas, have highlighted the importance of switched (hybrid) positive linear systems (PLS). In the

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last number of years, a considerable effort has been expended on gaining an understanding of the properties of general switched linear systems [7], [8]. As is the case for general switched systems, even though the main properties of positive LTI systems are well understood, many basic questions relating to switched PLS remain unanswered. The most important of these concerns their stability, and in this paper we present some initial results on the stability of switched PLS.

Recently, it was conjectured by the authors of [9], and independently by David Angeli, that the asymptotic stability of a positive switched linear system can be determined by testing the Hurwitz-stability of an associated convex set of matrices. This conjecture was based on preliminary results on the stability of positive switched linear systems and is both appealing and plausible. Moreover, if it were true, it would have significant implications for the stability theory of positive switched linear systems. In this paper, we shall extend some earlier work and show that the above conjecture is true for some specific classes of positive systems. However, one of the the major contributions of the paper is to construct a counterexample which proves that, in general, the conjecture is false. However, this in turn gives rise to a number of open questions for future research, some of which we discuss towards the end of the paper.

The layout of the paper is as follows. In Section 2 we present the mathematical background and notation necessary to state the main results of the paper. Then in Section 3, we present necessary and sufficient conditions for the uniform asymptotic stability of switched second order positive linear systems. In Section 4 we show by means of an abstract construction that the results derived in the preceding sections do not generalise to higher dimensional systems. In Section 5, we demonstrate that these results also fail to generalise for the more restrictive case of matrices with constant diagonals and we make some observations on the computation of the joint Lyapunov exponent for positive switched systems in Section 6. Finally, our conclusions are presented in Section 7.

II. Mathematical Preliminaries

In this section we present a number of preliminary results that shall be needed later and introduce the main notations used throughout the paper.

(i) Notation

Throughout, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^n$ stands for the vector space of all n-tuples of real numbers and $\mathbb{R}^{n\times n}$ is the space of $n \times n$ matrices with real entries. For x in $\mathbb{R}^n$, x_i denotes the i^{th} component of x, and the notation $x \succ 0$ ($x \succeq 0$) means that $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \succeq 0\}$ denotes the non-negative orthant in $\mathbb{R}^n$. Similarly, for a matrix A in $\mathbb{R}^{n\times n}$, a_{ij} or $A(i,j)$ denotes the element in the (i,j) position of A, and $A \succ 0$ ($A \succeq 0$) means that $a_{ij} > 0 (a_{ij} \geq 0)$ for $1 \leq i,j \leq n$. $A \succ B$ ($A \succeq B$) means that $A - B \succ 0$ ($A - B \succeq 0$). We write A^T for the transpose of A and $exp(A)$ for the usual matrix exponential of $A \in \mathbb{R}^{n\times n}$.

For P in $\mathbb{R}^{n\times n}$ the notation $P > 0$ ($P \geq 0$) means that the matrix P is positive (semi-)definite, and $PSD(n)$ denotes the cone of positive semi-definite matrices in $\mathbb{R}^{n\times n}$. The spectral radius of a matrix A is the maximum modulus of the eigenvalues of A and is denoted by $\rho(A)$. Also we shall denote the maximal real part of any eigenvalue of A by $\mu(A)$. If $\mu(A) < 0$ (all the eigenvalues of A are in the open left half plane) A is said to be *Hurwitz* or *Hurwitz-stable*.

Given a set of points, $\{x_1, \ldots, x_m\}$ in a finite-dimensional linear space V, we shall use the notations $CO(x_1, \ldots, x_m)$ and $Cone(x_1, \ldots, x_m)$ to denote the convex hull and the cone generated by $x_1, \ldots, x_m$ respectively. Formally:

$$CO(x_1, \ldots, x_m) = \{\sum_{i=1}^{m} \alpha_i x_i : \alpha_i \geq 0, 1 \leq i \leq m, \text{ and } \sum_{i=1}^{m} \alpha_i = 1\};$$

$$Cone(x_1, \ldots, x_m) = \{\sum_{i=1}^{m} \alpha_i x_i : \alpha_i \geq 0, 1 \leq i \leq m\}.$$

A closed *convex cone* in $\mathbb{R}^n$ is a set $\Omega \subseteq \mathbb{R}^n$ such that, for any $x, y \in \Omega$ and any $\alpha, \beta \geq 0$, $\alpha x + \beta y \in \Omega$. A convex cone is said to be: *solid* if the interior of Ω is non-empty; *pointed* if $\Omega \cap (-\Omega) = \{0\}$; *polyhedral* if $\Omega = Cone(x_1, \ldots, x_m)$ for some finite set $\{x_1, \ldots, x_m\}$ of vectors in $\mathbb{R}^n$. We shall call a closed convex cone that is both solid and pointed, a *proper convex cone*.

(ii) Positive LTI systems and Metzler matrices

The LTI system

$$\Sigma_A : \dot{x}(t) = Ax(t), \qquad A \in \mathbb{R}^{n\times n}, x(0) = x_0$$

is said to be positive if $x_0 \succeq 0$ implies that $x(t) \succeq 0$ for all $t \geq 0$. See [3] for a description of the basic theory and several applications of positive linear systems. The system Σ_A is positive if and only if the off-diagonal entries of the matrix A are non-negative. Matrices of this form are known as Metzler matrices. The next result concerning positive combinations of Metzler Hurwitz matrices was pointed out in [10].

Lemma 2.1: Let A_1, A_2 be Metzler and Hurwitz. Then $A_1 + \gamma A_2$ is Hurwitz for all $\gamma > 0$ if and only if $A_1 + \gamma A_2$ is non-singular for all $\gamma > 0$.

(iii) Common Quadratic Lyapunov Functions and Stability

It is well known that the existence of a common quadratic Lyapunov function (CQLF) for the family of stable LTI systems $\Sigma_{A_i} : \dot{x} = A_i x \quad i \in \{1, \ldots, k\}$ is sufficient to guarantee that the associated switched system $\Sigma_S : \dot{x} = A(t)x \quad A(t) \in \{A_1, \ldots, A_k\}$ is uniformly asymptotically stable under arbitrary switching. Throughout the paper, when we speak of the stability (uniform asymptotic stability) of a switched linear system, we mean stability (uniform asymptotic stability) under arbitrary switching.

Note that any initial state $x_0 \in \mathbb{R}^n$ can be written as $x_0 = u - v$ where $u, v \succeq 0$. Hence, for linear systems, uniform asymptotic stability with respect to initial conditions in the positive orthant is equivalent to uniform asymptotic stability with respect to arbitrary initial conditions in $\mathbb{R}^n$. In particular, if a positive switched linear system fails to be uniformly asymptotically stable (UAS) for initial conditions in the whole of $\mathbb{R}^n$, then it is also not UAS for initial conditions in the positive orthant.

Formally checking for the existence of a CQLF amounts to looking for a single positive definite matrix $P = P^T > 0$ in $\mathbb{R}^{n \times n}$ satisfying the k Lyapunov inequalities $A_i^T P + P A_i < 0 \quad i \in \{1, \ldots, k\}$. If such a P exists, then $V(x) = x^T P x$ defines a CQLF for the LTI systems Σ_{A_i}. While the existence of such a function is sufficient for the uniform asymptotic stability of the associated switched system, it is in general not necessary for stability [8], and CQLF existence can be a conservative condition for stability. However, recent work has established a number of system classes for which this is not necessarily the case [11], [12]. The results in these papers relate the existence of an unbounded solution to a switched linear system to the Hurwitz-stability of the convex hull of a set of matrices and are based on the following theorem.

Theorem 2.1: [13], [14] Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Hurwitz matrices. A sufficient condition for the existence of an unstable switching signal for the system $\dot{x} = A(t)x,\ A(t) \in \{A_1, A_2\}$, is that $A_1 + \gamma A_2$ has an eigenvalue with a positive real part for some positive γ.

Any trajectory of a positive system originating in the positive orthant will remain there as time evolves. Consequently, to demonstrate the stability of such systems, one need not search for a CQLF, but rather the existence of a *copositive* Lyapunov function. Formally, $V(x) = x^T P x$ is a copositive CQLF if the symmetric matrix $P \in \mathbb{R}^{n\times n}$ is such that $x^T P x > 0$ for $x \in \mathbb{R}^n_+$, $x \neq 0$, and $x^T(A_i^T P + PA_i)x^T < 0 \quad i \in \{1, \dots, k\},\ \forall x \succeq 0, x \neq 0$.

III. Second Order Positive Linear Systems

In this section, we shall show that the conjecture in [9] is true for second order positive switched linear systems. First, we recall the result of [11] which described necessary and sufficient conditions for the existence of a CQLF for a pair of general second order LTI systems.

Theorem 3.1: Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz. Then a necessary and sufficient condition for Σ_{A_1}, Σ_{A_2} to have a CQLF is that the matrix products A_1A_2 and $A_1A_2^{-1}$ have no negative eigenvalues.

It is only necessary to check one of the products in the above theorem if the individual systems $\Sigma_{A_1}, \Sigma_{A_2}$ are positive systems.

Lemma 3.1: Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz and Metzler. Then the product A_1A_2 has no negative eigenvalue.

Proof: As A_1, A_2 are both Hurwitz, the determinant of A_1A_2 must be positive. Also, the diagonal entries of A_1A_2 must both be positive. Hence the trace of A_1A_2 is positive. It now follows easily that the product A_1A_2 cannot have any negative eigenvalues.

Combining Theorem 3.1 and Lemma 3.1 yields the following result.

Theorem 3.2: Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz and Metzler. Then the following statements are equivalent:

(a) Σ_{A_1} and Σ_{A_2} have a CQLF;

(b) Σ_{A_1} and Σ_{A_2} have a common copositive quadratic Lyapunov function;

(c) The switched system $\dot{x} = A(t)x$, $A(t) \in \{A_1, A_2\}$ is uniformly asymptotically stable;

(d) The matrix product $A_1 A_2^{-1}$ has no negative eigenvalues.

Proof : (a) $\Leftrightarrow$ (d): From Lemma 3.1 it follows that the matrix product $A_1 A_2$ cannot have a negative eigenvalue. Hence, the equivalence of (a) and (d) follows from Theorem 3.1.

(b) $\Leftrightarrow$ (d): If $A_1 A_2^{-1}$ has no negative eigenvalues, then Σ_{A_1} and Σ_{A_2} have a CQLF. Thus, they certainly have a copositive common quadratic Lyapunov function. Conversely, suppose that $A_1 A_2^{-1}$ has a negative eigenvalue. It follows that $A_1 + \gamma_0 A_2$ has a real, non-negative eigenvalue for some $\gamma_0 > 0$. Since, $A_1 + \gamma_0 A_2 = N - \alpha_0 I$, where $N \succeq 0$, it follows that the eigenvector corresponding to this eigenvalue is the Perron eigenvector of N and consequently lies in the positive orthant [2]. It follows that a copositive Lyapunov function cannot exist.

(c) $\Leftrightarrow$ (d): Suppose that $A_1 A_2^{-1}$ has a negative eigenvalue; namely, $A_1 + \gamma A_2$ is non-Hurwitz for some $\gamma > 0$. It now follows from Theorem 2.1 that there exists some switching signal for which the switched system $\Sigma_S : \dot{x} = A(t)x \quad A(t) \in \{A_1, A_2\}$ is not uniformly asymptotically stable. This proves that (c) implies (d). Conversely, if $A_1 A_2^{-1}$ has no negative eigenvalues, then Σ_{A_1}, Σ_{A_2} have a CQLF and the associated switched systems is uniformly asymptotically stable. This completes the proof.

The equivalence of (c) and (d) in the previous theorem naturally gives rise to the following question. Given a finite set $\{A_1, \ldots, A_k\}$ of Metzler, Hurwitz matrices in $\mathbb{R}^{2\times 2}$, does the Hurwitz stability of $CO(A_1, ..., A_k)$ imply the uniform asymptotic stability of the associated switched system? This is indeed the case and follows from the following theorem, which can be thought of as an edge theorem for positive systems. This theorem extends a result presented recently in [15] by removing the restrictive assumption that the diagonal entries of all the system matrices are equal to -1.

Theorem 3.3: Let $A_1, \ldots, A_k$ be Hurwitz, Metzler matrices in $\mathbb{R}^{2\times 2}$. Then the positive switched linear system,

$$\dot{x} = A(t)x \quad A(t) \in \{A_1, \ldots, A_k\}, \tag{1}$$

is uniformly asymptotically stable if and only if each of the switched linear systems,

$$\dot{x} = A(t)x \quad A(t) \in \{A_i, A_j\}, \tag{2}$$

for $1 \leq i < j \leq k$ is uniformly asymptotically stable.

Outline of Proof

(a) First, we show that $\mathbb{R}^2_+$, can be partitioned into a finite collection of wedges, Sec_j, $1 \leq j \leq m$ such that, for $1 \leq j \leq m$ there exists a quadratic form $x^T P_j x$, which is non-increasing along each trajectory of (1) within Sec_j. Formally, for $x \in Sec_j$ and $1 \leq i \leq k$, $x^T(A_i^T P_j + P_j A_i)x \leq 0$.

(b) Using level sets of the quadratic forms in (a), we show that the system (1) has uniformly bounded trajectories.

(c) Finally, we show that for sufficiently small $\epsilon > 0$ the same conclusion will hold if we replace each system matrix A_i with $A_i + \epsilon I$. This then establishes the uniform asymptotic stability of the system (1).

Proof : It is immediate that if the system (1) is uniformly asymptotically stable (for arbitrary switching), then each of the systems (2) is also.

Now suppose that for each i, j with $1 \leq i < j \leq k$, the system (2) is uniformly asymptotically stable. We can assume without loss of generality that for all $a > 0$ and $1 \leq i < j \leq k$, the matrix $A_i - aA_j$ is not zero. Let $\mathbb{R}^2_+$ be the nonnegative orthant in $\mathbb{R}^2$. For any vector $x \in \mathbb{R}^2$,

$$Cone(A_1x, \ldots, A_kx) = \cup_{1 \leq i < j \leq k} Cone(A_ix, A_jx).$$

Moreover, as the switched system (2) is uniformly asymptotically stable for $1 \leq i < j \leq k$ and the system matrices are Metzler, $Cone(A_ix, A_jx) \cap \mathbb{R}^2_+ = \{0\}$ for all $1 \leq i < j \leq k$ and nonzero $x \in \mathbb{R}^2_+$. Therefore $Cone(A_1x, \ldots, A_kx) \cap \mathbb{R}^2_+ = \{0\}$.

For a nonzero vector $x \in \mathbb{R}^2$ define $arg(x)$, the argument of x in the usual way, viewing x as a complex number. Let $(l(x), u(x)), 1 \leq l(x), u(x) \leq k$ be a pair of integers such that $arg(A_{l(x)}x) \leq arg(A_ix) \leq arg(A_{u(x)}x)$. Then clearly

$$Cone(A_1x, \ldots, A_kx) = Cone(A_{l(x)}, A_{u(x)}).$$

For $1 \leq i, j \leq k$ define

$$D_{(i,j)} \quad = \quad \{y \in \mathbb{R}^2_+, y \neq 0 : Cone(A_1 y, \ldots, A_k y) = Cone(A_i y, A_j y)\}$$

Here (i, j) is a pair of integers, not necessarily ordered and possibly equal and $arg(A_i y) \leq arg(A_j y)$. It now follows that $\mathbb{R}^2_+ - \{0\} = \cup_{1 \leq i,j \leq k} D_{(i,j)}$. Note that $D_{(i,j)} \cup \{0\}$ is a closed cone, not necessarily convex and that if $x \in D_{(i,j)}$ and $arg(A_i x) < arg(A_m x) < arg(A_j x)$ for $m \neq i, j$ then x belongs to the interior of $D_{(i,j)}$.

Consider the set $Symp = \{\hat{a} =: (a, 1-a)^T : 0 \leq a \leq 1\}$, and define $d_{(i,j)} = Symp \cap D_{(i,j)}$. We shall write $\hat{a} < \hat{b}$ if and only if $a < b$. The sets $d_{(i,j)}$ are closed and their (finite) union is equal to $Symp$. Moreover, the only way for $x \in Symp$ not to lie in the interior of some $d_{(i,j)}$ is if there exists $b > 0, 1 \leq l \neq m \leq k$ such that $A_l x = b A_m x$. As we assumed that for all $a > 0, 1 \leq i < j \leq k$ the matrix $A_i - a A_j$ is not zero, it follows that there exists a finite subset $Sing =: \{0 \leq \hat{a}_1 < \ldots < \hat{a}_q \leq 1\}$ such that all vectors in $Symp - Sing$ belong to the interior of some $d_{(i,j)}$.

It now follows that $Symp$ can be partitioned into a finite family of closed intervals, each of them contained in some $d_{(i,j)}$. This in turn defines a partition of $\mathbb{R}^2_+ - \{0\}$ into finitely many closed cones/wedges Sec_j, $1 \leq j \leq m$, each of which is contained in some $D_{(L(j),U(j))}$. We shall label the rays which define this partition $r_1, \ldots, r_{m+1}$ where r_1 is the y-axis, r_{m+1} is the x-axis and the rays are enumerated in the clockwise direction.

Now, by assumption, the switched system $\dot{x} = A(t)x \quad A(t) \in \{A_{L(j)}, A_{U(j)}\}$ is uniformly asymptotically stable for all $1 \leq j \leq m$. Thus, it follows from Theorem 3.2 that, for $1 \leq j \leq m$, there exist quadratic forms $x^T P_j x$, $P_j = P_j^T > 0$, such that $A^T_{L(j)} P_j + P_j A_{L(j)} < 0$, $A^T_{U(j)} P_j + P_j A_{U(j)} < 0$. As $Sec_j \subset D_{(L(j),U(j))}$, it follows that $x^T P_j A_i x \leq 0$ for all $x \in Sec_j$ and all i with $1 \leq i \leq k$.

Now, choose a point $T_1 = (0, y)^T$, $y > 0$ and consider the level curve of $x^T P_1 x$ which passes through T_1. This curve intersects the second ray r_2 at some point T_2 and the level curve of $x^T P_2 x$ going through T_2 intersects the third ray r_3 at some point T_3. We can continue this process until we reach some point T_{m+1} on the x-axis. This gives us a domain bounded by the y-axis, the chain of ellipsoidal arcs defined above, and the x-axis. This domain is an invariant set for (1), which implies that the trajectories of the system (1) are uniformly bounded. The same conclusion will hold if we replace the system matrices $A_1, \ldots, A_k$ with $\{A_1 + \epsilon I, .., A_k + \epsilon I\}$ for some small enough positive ϵ. This implies that the original system (1) is in fact uniformly asymptotically stable and completes the proof of the theorem.

IV. Higher Dimensional Systems

Motivated by results such as those described in the previous section, a number of authors have recently formulated the following conjecture.

Conjecture 1: Let $A_1, \ldots, A_k$ be a finite family of Hurwitz, Metzler matrices in $\mathbb{R}^{n\times n}$. Then the following statements are equivalent:

(i) All matrices in the convex hull, $CO(A_1, \ldots, A_k)$, are Hurwitz;

(ii) The switched linear system, $\dot{x} = A(t)x \quad A(t) \in \{A_1, \ldots, A_k\}$, is uniformly asymptotically stable.

In the remainder of this section, we shall present a counterexample to Conjecture 1, based on arguments first developed by Gurvits in [16](which extended the results in [17], [18]).

Lemma 4.1: Let $A_1, \ldots, A_k$ be a finite family of matrices in $\mathbb{R}^{n\times n}$. Assume that there exists a proper polyhedral convex cone Ω in $\mathbb{R}^n$ such that $exp(A_i t)(\Omega) \subseteq \Omega$ for all $t \geq 0$ and $1 \leq i \leq k$.

Then there is some integer $N \geq n$ and a family of Metzler matrices $A_1^M, \ldots A_k^M$ in $\mathbb{R}^{N\times N}$ such that:

(i) All matrices in $CO(A_1, \ldots, A_k)$ are Hurwitz if and only if $CO(A_1^M, \ldots, A_k^M)$ consists entirely of Hurwitz matrices;

(ii) The switched linear system $\dot{x} = A(t)x$, $A(t) \in \{A_1, \ldots, A_k\}$ is uniformly asymptotically stable if and only if the positive switched linear system $\dot{x} = A(t)x$, $A(t) \in \{A_1^M, \ldots, A_k^M\}$ is uniformly asymptotically stable.

Proof:

As Ω is polyhedral, solid and pointed, we can assume without loss of generality that there exist vectors $z_1, \ldots, z_N$ in $\mathbb{R}^n$, with $N \geq n$, such that $\Omega = Cone(z_1, \ldots, z_N)$. Also, (see Theorem 8 in [19]) for $1 \leq i \leq k$, $exp(A_i t)(\Omega) \subseteq \Omega$ for all $t \geq 0$ if and only if there is some $\tau > 0$ such that $(I + \tau A_i)(\Omega) \subseteq \Omega$.

Define a linear operator $\Phi : R^N \rightarrow R^n$ by $\Phi(e_i) = z_i$ for $1 \leq i \leq N$ where $e_1, \ldots, e_N$ is the standard basis of $\mathbb{R}^N$. We shall now show how to construct Metzler matrices $A_i^M \in \mathbb{R}^{N \times N}$ satisfying the requirements of the lemma.

First, we note the following readily verifiable facts:

(i) For any trajectory,$x(t) = \sum_{1 \leq i \leq N} \alpha_i(t) z_i$, $\alpha_i(t) \geq 0$, in Ω, $\lim_{t \rightarrow \infty} x(t) = 0$ if and only if $\lim_{t \rightarrow \infty} \alpha_i(t) = 0$ for $1 \leq i \leq N$;

(ii) For each $i \in \{1, \ldots, k\}$ and $q \in \{1, \ldots, N\}$, we can write (non-uniquely)

$$A_i(z_q) = \sum_{p=1}^{N} a_{pq} z_p \qquad \text{where } a_{pq} \geq 0 \text{ if } p \neq q.$$

In this way, we can associate a Metzler matrix, $A_i^M = (a_{pq} : 1 \leq p, q \leq N)$ in $\mathbb{R}^{N \times N}$ with each of the system matrices A_i in $\mathbb{R}^{n \times n}$.

(iii) By construction, $\Phi A_i^M = A_i \Phi$ and $\Phi(exp(A_i^M t)) = (exp(A_i t))\Phi$ for all $t \geq 0$. Hence, A_i^M is Hurwitz if and only if A_i is Hurwitz for $1 \leq i \leq k$.

From points (i) and (iii) above we can conclude that all matrices in the convex hull $CO(A_1, ..., A_k)$ are Hurwitz if and only if all matrices in the convex hull $CO(A_1^M, ..., A_k^M)$ are Hurwitz. Moreover, the switched linear system $\dot{x} = A(t)x,$, $A(t) \in \{A_1, \ldots, A_k\}$ is uniformly asymptotically stable if and only if the positive switched linear system $\dot{x} = A(t)x$, $A(t) \in \{A_1^M, \ldots, A_k^M\}$ is uniformly asymptotically stable. This proves the lemma.

It follows from Lemma 4.1 that if Conjecture 1 was true, then the same statement would also hold for switched linear systems having an invariant proper polyhedral convex cone.

Given a matrix $A \in \mathbb{R}^{n \times n}$, define the linear operator $\hat{A}$, on the space of $n \times n$ real symmetric matrices, by $\hat{A}(X) = A^T X + XA$. It is a straightforward exercise to verify that if $x_1(t)$ and $x_2(t)$ are solutions of the system $\dot{x} = A^T x$ with initial conditions $x_1(0) = x_1$, $x_2(0) = x_2$, then $x_1(t)x_2(t)^T + x_2(t)x_1(t)^T$ is a solution of the linear system $\dot{X} = \hat{A}(X)$, with initial conditions $x_1 x_2^T + x_2 x_1^T$. The following result follows easily by combining this observation with standard facts about the existence and uniqueness of solutions to linear systems.

Lemma 4.2: Consider a family, $\{A_1, \ldots, A_k\}$, of matrices in $\mathbb{R}^{n \times n}$. Then:

(i) $CO(A_1, \ldots, A_k)$ consists entirely of Hurwitz stable matrices if and only if all of the operators in $CO(\hat{A}_1, \ldots, \hat{A}_k)$ are Hurwitz stable;

(ii) The cone, $PSD(n)$, of positive semi-definite matrices in $\mathbb{R}^{n\times n}$ is an invariant cone for the switched system $\dot{X} = \hat{A}(t,X) \quad \hat{A}(t,X) \in \{\hat{A}_1(X), \ldots, \hat{A}_k(X)\}$;

(iii) The system $\dot{X} = \hat{A}(t,X)$, $\hat{A}(t,X) \in \{\hat{A}_1(X), \ldots, \hat{A}_k(X)\}$ is uniformly asymptotically stable if and only if the system $\dot{x} = A(t)x$, $A(t) \in \{A_1, \ldots, A_k\}$ is uniformly asymptotically stable.

The Counterexample

To begin, consider the following two matrices in $\mathbb{R}^{2\times 2}$

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$$

where $a > b \geq 0$. Then, for some $t_1, t_2 > 0$ the spectral radius $\rho((exp(A_1t_1)(exp(A_2t_2)) > 1$. In fact, if we take $a = 2, b = 1$ then this is true with $t_1 = 1, t_2 = 3/2$. By continuity of eigenvalues, if we choose $\epsilon > 0$ sufficiently small, we can ensure that $\rho((exp((A_1 - \epsilon I)t_1))(exp((A_2 - \epsilon I)t_2))) > 1$. Hence, the switched linear system associated with the system matrices $A_1 - \epsilon I$, $A_2 - \epsilon I$ is unstable and moreover, all matrices in the convex hull $CO(\{A_1 - \epsilon I, A_2 - \epsilon I\})$ are Hurwitz.

The above remarks establish the existence of Hurwitz matrices B_1, B_2 in $\mathbb{R}^{2\times 2}$ such that all matrices in $CO(B_1, B_2)$ are Hurwitz and the switched linear system $\dot{x} = A(t)x$, $A(t) \in \{B_1, B_2\}$ is unstable.

Next, consider the Lyapunov operators, $\hat{B}_1$, $\hat{B}_2$ on the symmetric 2×2 real matrices. It follows from Lemma 4.2 that $CO(\hat{B}_1, \hat{B}_2)$ consists entirely of Hurwitz stable operators and that the switched linear system associated with $\hat{B}_1$, $\hat{B}_2$ is unstable, and leaves the proper (not polyhedral) cone $PSD(2)$ invariant. Formally, $exp(\hat{B}_it)(PSD(2)) \subseteq PSD(2)$ for $i = 1, 2$, and all $t \geq 0$.

From examining the power series expansion of $exp(\hat{B}_it)$, it follows that for any $\epsilon > 0$, there exists $\tau > 0$ and two linear operators $\Delta_i, i = 1, 2$ such that $(\tau I + \hat{B}_i + \Delta_i)(PSD(2)) \subseteq PSD(2)$ with $||\Delta_i|| < \epsilon$ for $i = 1, 2$. Combining this fact with standard results on the existence of polyhedral approximations of arbitrary proper cones in finite dimensions (see Theorem 20.4 in [20]), we can conclude that for any $\epsilon > 0$, there exists a proper polyhedral cone $PH_\epsilon \subset PSD(2)$, and two linear operators δ_i, $i = 1, 2$ such that $(\tau I + \hat{B}_i + \Delta_i + \delta_i)(PH_\epsilon) \subseteq PH_\epsilon$

with $||\Delta_i||, ||\delta_i|| < \epsilon$ for $i = 1, 2$.

Recall that $CO(\hat{B}_1, \hat{B}_2)$ consists entirely of Hurwitz-stable operators and that the switched linear system associated with $\hat{B}_1$, $\hat{B}_2$ is unstable. For $\epsilon > 0$, define the linear operators $B_{i,\epsilon} = \hat{A}_i + \Delta_i + \delta_i$ for $i = 1, 2$. By choosing $\epsilon > 0$ sufficiently small, we can ensure that all operators in $CO(B_{1,\epsilon}, B_{2,\epsilon})$ are Hurwitz-stable and that the switched linear system $\dot{x} = A(t)x$, $A(t) \in \{B_{1,\epsilon}, B_{2,\epsilon}\}$ is unstable. Moreover, this switched linear system leaves the proper, polyhedral cone PH_ϵ invariant.

Thus, the statement of Conjecture 1 is not true for switched linear systems with an invariant proper, polyhedral cone and hence, it follows from Lemma 4.1 that Conjecture 1 itself is also false. However, on examining the proof of Lemma 4.1, we see that the dimension of the counterexample is determined by the number of generators of the polyhedral approximation PH_ϵ, and this may be very large.

V. Matrices with constant diagonals

In the recent paper [15], it was shown that for Metzler, Hurwitz matrices $A_1, \dots, A_k$ in $\mathbb{R}^{2\times 2}$ all of whose diagonal entries are equal to -1 ($A_j(i,i) = -1$ for $i = 1, 2$, $j = 1, \dots, k$), the Hurwitz-stability of all matrices in $CO(A_1, \dots, A_k)$ is equivalent to the uniform asymptotic stability of the associated switched linear system. Motivated by this result, Mehmet Akar recently asked if a counterexample to Conjecture 1 exists for the more restrictive system class satisfying: $A_j(i,i) = -1$ for $1 \leq i \leq n$, $1 \leq j \leq k$. We shall now show that such a counterexample does indeed exist. Note that it is enough to provide a counterexample such that each matrix $A_j, 1 \leq j \leq k$ has a constant diagonal in the sense that there are real numbers $c_1, \dots, c_k$ such that $A_j(i,i) = c_j$ for $i = 1, \dots, n$, $j = 1, \dots, k$.

Given a Metzler matrix A in $\mathbb{R}^{n\times n}$, let $A(l,l) = \min_{1\leq i\leq n} A(i,i)$, and define the 2×2 blocks:

$$B_{i,j} = \begin{pmatrix} A(i,j) & 0 \\ 0 & A(i,j) \end{pmatrix} \qquad \text{if } i \neq j$$

$$B_{i,i} = \begin{pmatrix} A(l,l) & A(i,i) - A(l,l) \\ A(i,i) - A(l,l) & A(l,l) \end{pmatrix}.$$

Let $Lift(A) \in \mathbb{R}^{2n\times 2n}$ be the block matrix whose (i,j) block is $B_{i,j}$. Next define the linear operator $F \in \mathbb{R}^{n\times 2n}$ by $F(x_1, ..., x_{2n}) = (y_1, ..., y_n)$,, where $y_i = x_{2i-1} + x_{2i}$ for $i = 1, \dots, n$.

It is straightforward to check that for any Metzler $A \in \mathbb{R}^{n\times n}$, $Lift(A) \in \mathbb{R}^{2n\times 2n}$ is Metzler, has a constant diagonal, and $F(Lift(A)) = AF$. The next lemma now follows readily from the previous equation.

Lemma 5.1: Consider a set of Metzler matrices $A_1, ..., A_k$ in $\mathbb{R}^{n\times n}$. Then the following statements hold ;

(i) The convex hull $CO(A_1, ..., A_k)$ is Hurwitz iff the convex hull $CO(Lift(A_1), ..., Lift(A_k))$ is Hurwitz .

(ii) The switched system $\dot{x} = A(t)x$, $A(t) \in \{A_1, ..., A_k\}$ is uniformly asymptotically stable iff the switched system $\dot{x} = A(t)x$, $A(t) \in \{Lift(A_1), ..., Lift(A_k)\}$ is uniformly asymptotically stable .

In the last section, we proved that there exists a positive integer n and a pair of Metzler, Hurwitz matrices A_1, A_2 in $\mathbb{R}^{n\times n}$ which violate Conjecture 1. Now consider one such pair $\{A_1, A_2\}$ and lift it to the pair $\{Lift(A_1), Lift(A_2)\}$. It now follows, using Lemma 5.1, that the pair $Lift(A_1)$, $Lift(A_2)$ provides the required counterexample.

VI. The joint Lyapunov exponent

In this section, we shall make some simple observations concerning the computation of the *joint Lyapunov exponent*, which is a continuous-time analogue of the *joint spectral radius*.

Definition 6.1: Let S be a compact subset of $\mathbb{R}^{n\times n}$. The joint Lyapunov exponent of the associated continuous-time switched linear system Σ_S, $JLE(S)$ is defined as

$$JLE(S) = \inf\{\lambda : \exists \text{ a matrix norm } ||.|| : ||exp(At)|| \leq e^{\lambda t} \text{ for } A \in S, t \geq 0\}$$

Notice that uniform asymptotic stability of the switched linear system Σ_S is equivalent to the inequality $JLE(S) < 0$. A relatively straightforward modification of the proof of Theorem 3.3 yields the following result.

Theorem 6.1: (i) Let $S \subset \mathbb{R}^{2\times 2}$ be a compact set of Metzler matrices. Then

$$JLE(S) = \max_{A,B\in S} JLE(\{A,B\});$$

(ii) $JLE(S) = \max_{M\in CO(S)} \mu(M)$, where $CO(S)$ is the convex hull of S and $\mu(M)$ is the maximal real part of the eigenvalues of M;

(iii) Let $S = \{A_1, ..., A_k\}$ be a finite set of 2×2 Metzler matrices . Then the joint Lyapunov exponent $JLE(S)$ can be computed in $O(k^2)$ arithmetic operations .

VII. Conclusions

In this paper we have presented a counterexample to a recent conjecture presented in [9], and formulated independently by David Angeli, concerning the uniform asymptotic stability of switched positive linear systems. In particular, we have shown that the stability of a positive switched linear system is not in general equivalent to the Hurwitz stability of the convex hull of its system matrices. Furthermore, we have also shown that this conjecture fails for the more restrictive case where the system matrices are required to have constant diagonals. While this conjecture is now known to be false, the lowest dimension for which it fails is still not known. Thus it may be true for other, low-dimensional classes of positive systems. Also, it is not known how large the set of counterexamples is, which means that the conjecture may be true for significant sub-classes of switched positive linear systems.

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