Correction to “D-Stability and Delay-Independent Stability of Homogeneous Cooperative Systems”

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Abstract—We correct some errors in the statements and proofs presented in Section V of the above mentioned manuscript.

In the recent paper [1], the results of Section V are incorrect as stated. In this brief note, we first present an example to show that the statement of Theorem 5.1 in [1] is incorrect and then state a corrected version of this result under an additional technical assumption on the vector field $f$.

Example 1
Consider the 2-dimensional cooperative system given by

$$\dot{x} = f(x_1, x_2) = \left( -\frac{x_1}{1+x_1} + x_2 - \frac{x_2}{1+x_2} \right).$$

Using a similar argument as employed in Example 3.19 of [2], it can be shown that the origin is a globally asymptotically stable equilibrium of this system. However, it is clear that choosing $a = (1, 2)$, there can be no vector $v \geq a$ with $f(v) \ll 0$.

The proof of Theorem 5.1 in [1] implicitly assumes that for any $a \in \mathbb{R}_+^n$, both of the sets

$$\Omega_1 := \{ x \geq a : f_1(x) < 0 \}$$

$$\Omega_2 := \{ x \geq a : f_2(x) < 0 \}$$

are non-empty. In general, this will not be the case as is clear from the above example.

In order for the conclusion of Theorem 5.1 in [1] to hold, it is necessary to impose some additional assumption on the vector field $f$.

Assumption A:

(i) For any $x_1 \in \mathbb{R}_+$, there exists some $K_{x_1}$ and $\varepsilon_{x_1} > 0$ such that

$$\frac{\partial f_2}{\partial x_2}(x_1, t) \leq -\varepsilon_{x_1} \text{ for all } t \geq K_{x_1}.$$

(ii) For any $x_2 \in \mathbb{R}_+$, there exists some $K_{x_2}$ and $\varepsilon_{x_2} > 0$ such that

$$\frac{\partial f_1}{\partial x_1}(t, x_2) \leq -\varepsilon_{x_2} \text{ for all } t \geq K_{x_2}.$$

In the following result, which is a correction of Theorem 5.1 in [1], $f$ is required to satisfy the same conditions as in Section V of [1] as well as Assumption A above.

**Theorem 5.1** Assume that the system $\dot{x}(t) = f(x(t))$ has a GAS equilibrium at the origin. Then given any $a \in \mathbb{R}_+^n$, there exists $v \geq a$ with $f(v) \ll 0$.

**Proof:** If there exist $u \geq a$ and $w \geq a$ with $f_1(u) < 0$ and $f_2(w) < 0$ then the sets $\Omega_1$, $\Omega_2$ in (1) are both non-empty and the argument presented in Theorem 5.1 of [1] implies the existence of a $v \geq a$ with $f(v) \ll 0$. We now prove that, provided Assumption A is satisfied, these sets are both non-empty.

We shall show that there exists $u \geq a$ with $f_1(u) < 0$. (The proof that there is a $w \geq a$ with $f_2(w) < 0$ is identical.) By Assumption A, there exists some constants $K$, $\epsilon > 0$ such that

$$\frac{\partial f_1}{\partial x_1}(t, a_2) < -\epsilon$$

for all $t \geq K$. For $t \geq K$, we have

$$f_1(t, a_2) = f_1(K, a_2) + \int_K^t \frac{\partial f_1}{\partial x_1}(s, a_2) ds \leq f_1(K, a_2) - \epsilon(t - K).$$

This immediately implies that for $t > K + \frac{f_1(K, a_2)}{\epsilon}$, we must have $f_1(t, a_2) < 0$. This completes the proof.

Provided the vector field $f$ satisfies the additional Assumption A and $g$ is non-decreasing, the other results of Section V of [1] are correct as stated.

**References**
