Convex Optimization and Congestion Control

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23.07. – 03.08.2012

Outline

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Part II: Convex Optimization

- II.1: Optimization Problems
- II.2: The Dual Problem
- II.3: Duality Gap and Strong Duality
- II.4: The Karush-Kuhn-Tucker Conditions

Outline

Optimization

General Problem

Given $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$, $C \subset U$ the general minimization problem is

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in C\,, \end{aligned}$$

Nomenclature

- f objective function or cost function
- C constraint set
- $x \in \mathbb{R}^n$ is called feasible, if $x \in C$.
- If $C = \emptyset$, then the problem is infeasible, otherwise the problem is feasible.

The Value of the problem

General Problem

Given $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$, $C \subset U$ the general minimization problem is

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in C\,, \end{aligned}$$

The value of the minimization problem

The optimal value of the problem is defined by

$$p^* := \inf\{f(x) \mid x \in C\}\,. \tag{1}$$

A point x^* is optimal, if $x^* \in C$ and $f(x^*) = p^*$.

Note: $p^* = \infty$ or $p^* = -\infty$ is possible.

Analytic Description of the Constraints

Analytic description

$$
\begin{array}{rll}
\text{minimize} & f(x) & \\
\text{subject to} & g_j(x) \leq 0 & \qquad j = 1, \ldots, m \\
 & h_j(x) = 0 & \qquad j = 1, \ldots, p\,.
\end{array}
$$

Domain of the problem

$$
\mathcal{D} := \operatorname{dom} f \;\cap\; \bigcap_{j=1}^{m} \operatorname{dom} g_j \;\cap\; \bigcap_{j=1}^{p} \operatorname{dom} h_j\,.
$$

g_j define the inequality constraints
h_j define the equality constraints.

Convex Optimization Problems

Definition

The optimization problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in C\,, \end{aligned}$$

is called convex if

(i) the cost function f is convex;

(ii) the feasibility set is convex.

Note: In order to be able to compute we usually need an analytic description of the constraint set.

Convex Problem in Standard form

Definition

Given $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ convex as well as $A \in \mathbb{R}^{p \times n}, b \in \mathbb{R}^p$. The standard form of the convex optimization problem is

$$\begin{aligned} \text{minimize} \quad & f(x) & \\ \text{subject to} \quad & g_j(x) \leq 0 & j = 1, \ldots, m \\ & Ax = b\,. & \end{aligned}$$

Linear Optimization Problems

General Linear Program

Given $c \in \mathbb{R}^n$, $d \in \mathbb{R}$ describing the cost and $G \in \mathbb{R}^{m \times n}$, $h \in \mathbb{R}^m$, $A \in \mathbb{R}^{p \times n}$, $b \in \mathbb{R}^p$ the corresponding linear optimization problem is

$$\begin{aligned} \text{minimize} \quad & c^\top x + d \\ \text{subject to} \quad & Gx \leq h \\ & Ax = b\,. \end{aligned}$$

Note: The inequalities $Gx \leq h$ are to be understood componentwise, that is, row by row.

Optimal Points of Linear Optimization Problems

Theorem

Consider a general linear program. If an optimal point x^* exists, then the optimum is attained on some faces of the feasible set

$$\{x \in \mathbb{R}^n \mid Gx \leq h\,, \quad Ax = b\}\,.$$

If the feasible set is a bounded polytope, then the optimal value of the problem is attained in an extreme point of the feasible set.

Optimal Points of Linear Optimization Problems

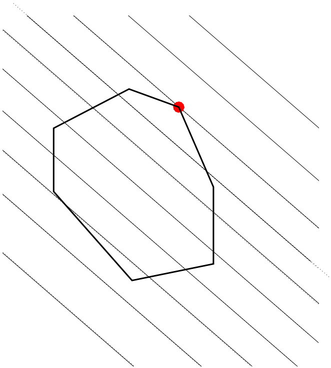

Figure: Linear program over a bounded convex polytope.

Standard Forms of Linear Programs

There are two standard forms of linear programs:

Standard Form with Equality Constraints

$$\begin{aligned} \text{minimize} \quad & c^\top x + d \\ \text{subject to} \quad & Ax = b \\ & x \geq 0\,. \end{aligned}$$

Standard Form without Equality Constraints

$$\begin{aligned} \text{minimize} \quad & c^\top x + d \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

These forms can be always achieved by introducing slack variables and positive variables.

Example: Minimum Cost Network Flow Problem

Network

A network is a directed graph $G(V, E)$ with vertex set $V = \{1, \dots, n\}$ and edge set $E \subset V \times V$.

To each edge $(i, j) \in E$ we associate

(i) edge capacity $u_{ij} > 0$, the maximal flow along the edge

(ii) edge cost c_{ij}, the cost for one unit of flow along that edge.

To each node $i \in V$ we associate

(i) external supply b_i, the flow entering the network at node i.

- $b_i > 0$: source node
- $b_i = 0$: transient node
- $b_i < 0$: destination node

Example: Minimum Cost Network Flow Problem

x_{ij} - amount of flow sent along edge (i, j)

The Optimization Problem

$$\begin{aligned} \text{minimize} \quad & \sum_{(i,j)\in E} c_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = b_i \,, \quad i = 1, \ldots, n \\ & 0 \le x_{ij} \le u_{ij} \,, \qquad\qquad\qquad i, j = 1, \ldots, n \end{aligned}$$

Most likely we want to assume network flow conservation, i.e. $\sum_{i=1}^{n} b_i = 0$.

Quadratic Optimization Problems

Quadratic Program

In addition to the previous data let $P \in \mathcal{H}_n$ be positive semidefinite.

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} x^\top P x + c^\top x + d \\ \text{subject to} \quad & Gx \leq h \\ & Ax = b\,. \end{aligned}$$

Quadratically Constrained Quadratic Program

Here the constraints are also given by some positive semidefinite matrices Q_i.

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} x^\top P x + c^\top x + d \\ \text{subject to} \quad & \frac{1}{2} x^\top Q_i x + q_i^\top x + r_i \leq 0\,, \quad i = 1, \ldots, m\,, \\ & Ax = b\,, \end{aligned}$$

Semidefinite Optimization Problems

Semidefinite Program

Let $x = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^\top \in \mathbb{R}^n$, $G, F_1, \dots, F_n \in \mathcal{H}_n$,
$A \in \mathbb{R}^{p \times n}, b \in \mathbb{R}^p$.
The corresponding semidefinite program is

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & x_1 F_1 + x_2 F_2 + \dots + x_n F_n + G \leq 0\,, \\ & Ax = b\,. \end{aligned}$$

Note: inequality has to be understood in the sense of positive semidefinite matrices.

$P \geq Q :\Leftrightarrow P - Q$ is positive semidefinite.

Standard Form Semidefinite Program

Semidefinite Program

Let the optimization variable $X \in \mathcal{H}_n$ and consider the optimization problem

$$\begin{array}{rl} \text{minimize} & \operatorname{trace} CX \\ \text{subject to} & \operatorname{trace} A_i X = b_i\,, \quad i = 1, \ldots, m\,, \\ & X \geq 0\,, \end{array}$$

Note that $\operatorname{trace} CX = \sum_{i,j=1}^{n} c_{ij} x_{ij}$ can represent any linear map from $\mathcal{H}_n$ to $\mathbb{R}$.

Optimal Points of Semidefinite Programs

Theorem

Consider a general semidefinite program. If an optimal point x^* exists, then the optimum is attained on some faces of the feasible set

$$\{x \in \mathbb{R}^n \mid x_1 F_1 + x_2 F_2 + \ldots + x_n F_n + G \leq 0\,, Ax = b\}\,.$$

If the feasible set is bounded, then the optimal value of the problem is attained in an extreme point of the feasible set.

Outline

The Lagrangian

$$\begin{aligned}
\text{minimize} \quad & f(x) & & \qquad (2)\\
\text{subject to} \quad & g_j(x) \leq 0 & i = 1, \dots, m & \qquad (3)\\
& h_j(x) = 0 & i = 1, \dots, p\,. & \qquad (4)
\end{aligned}$$

Lagrangian approach

Idea: Make the constraints part of the objective function.

$$L(x, \lambda, \nu) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x) + \sum_{j=1}^{p} \nu_j h_j(x)\,, \qquad (5)$$

with domain of definition

$$\mathrm{dom}\, L = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p\,.$$

Lagrange Dual Function

$$L(x, \lambda, \nu) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x) + \sum_{j=1}^{p} \nu_j h_j(x)\,, \qquad (6)$$

with domain of definition

$$\mathrm{dom}\, L = \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p\,.$$

Lagrange multipliers: λ associated with the inequality constraints
ν associated with the equality constraints

Lagrange dual function

The *Lagrange dual function* is defined as

$$g_L(\lambda, \nu) := \inf_{x \in \mathcal{D}} L(x, \lambda, \nu)\,.$$

Lagrange Dual Function

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Concavity

The dual function is concave as an infimum of linear functions.

Lemma: Weak Duality

For all $\lambda \geq 0$, $\nu \in \mathbb{R}^p$ we have

$$g_L(\lambda, \nu) \leq p^*\,.$$

The Dual Optimization Problem

Consider the primal problem

$$\begin{aligned} \text{minimize} \quad & f(x) & \\ \text{subject to} \quad & g_j(x) \leq 0 & \qquad j = 1, \ldots, m \\ & Ax = b\,. & \end{aligned}$$

Definition

The Lagrange dual problem is

$$\begin{aligned} \text{maximize} \quad & g_L(\lambda, \nu) \\ \text{subject to} \quad & \lambda \geq 0\,. \end{aligned}$$

The value of this optimization problem is denoted by

$$d^* := \sup\{g_L(\lambda, \nu) \mid \lambda \in \mathbb{R}^m, \lambda \geq 0, \nu \in \mathbb{R}^p\}\,.$$

Outline

Duality Gap

Recall that by our previous lemma

$$g_L(\lambda, \nu) \leq p^*, \qquad \text{for all } \lambda \geq 0, \nu \in \mathbb{R}^p.$$

Definition

Consider the primal problem and the associated dual problem. The **duality gap** of the problem is given by the difference

$$p^* - d^* \geq 0.$$

If the duality gap is equal to zero, then we say that *strong duality* holds.

Strong Duality

Even for convex problems strong duality need not hold, but it very often does.

Consider the convex optimization problem

$$
\begin{array}{rll}
\text{minimize} & f(x) & \\
\text{subject to} & g_j(x) \leq 0 & j = 1, \ldots, m \\
& Ax = b\,, &
\end{array}
$$

Theorem: Slater's constraint qualification

Consider the convex optimization problem. If there exists an $x \in \operatorname{ri} \mathcal{D}$ such that

$$
Ax = b \quad \text{and} \quad g_j(x) < 0\,, \quad j = 1, \ldots, m
$$

then strong duality holds.