Convex Optimization and Congestion Control

Fabian Wirth

University of Würzburg

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Outline

• Part I:	Convexity and Convex Functions	Lectures 1, 2, 3
• Part II:	Convex Optimization	Lectures 4 and 5
• Part III:	Numerical Methods	Lectures 6 and 7
• Part IV:	Congestion Control	Lecture 8
• Part V:	Utility Based Congestion Control	Lecture 9
• Part VI:	Miscellaneous Problems in Networks	Lecture 10 (we shall see)

Part I: Convexity and Convex Functions

- I.1: Convex Sets
- I.2: Operations on Convex Sets and Construction of Convex Sets
- I.3: Separation
- I.4: Faces , Extreme Points and Recession Cones
- I.5: Duality
- I.6: Convex Functions
- I.7: Subgradients
- I.8: Optimality

Outline

Minima and Maxima

Definition

For a function $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^n$, the point x^* is a local minimum of f, if there exists an open neighborhood U of x^* such that

$$f(x^*) \leq f(x)\,, \quad \forall x \in U \cap D\,. \tag{1}$$

The local minimum is called strict, if the inequality in (1) is strict for all $x^* \neq x \in U \cap D$. The minimum is global, if $U = D = \mathbb{R}^n$.

Definition

x^* is a local (global) maximum of f, if it is a local (global) minimum of $-f$.

The Minima of Convex Functions

Proposition

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be convex. Then

(i) every local minimum is a global minimum;

(ii) the set of global minima of f is convex;

(iii) if f is strictly convex, there exists at most one minimum, which is then necessarily global.

Minima of Convex Functions

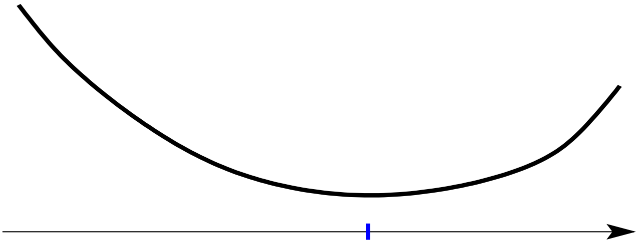

Figure: The set of minima of a convex function is convex.

Minima of Convex Functions

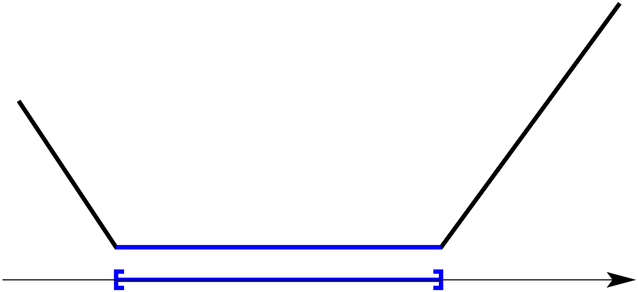

Figure: The set of minima of a convex function is convex.

Minima of Convex Functions

Figure: The set of minima of a convex function is convex but possibly unbounded.

Sublevel Sets of Convex Functions

Lemma

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be convex. Then the c-sublevel set

$$f^{-1}(-\infty, c) := \{x \in \mathrm{dom}\, f \mid f(x) < c\}$$

is convex.

Maxima of Convex Functions

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be convex and let $C \subset \operatorname{dom} f$ be convex.

(i) If there exists an $x^* \in \operatorname{ri} C$ such that

$$f(x^*) = \sup\{f(x) \mid x \in C\} \tag{2}$$

then f is constant on C.

Maxima of Convex Functions

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(ii) Let $W \subset C$ be the set of points x^* satisfying (2). Then W is a union of faces of C.

Maxima of Convex Functions

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then f is constant on C.

(ii) Let $W \subset C$ be the set of points x^* satisfying (2). Then W is a union of faces of C.

(iii) Let $M \subset \mathrm{dom}\, f$, then

$$\sup\{f(x) \mid x \in M\} = \sup\{f(x) \mid x \in \mathrm{conv}\, M\}\,.$$

Subgradient Characterization

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be convex. Then x^* is a minimum of f, if and only if

$$0 \in \partial f(x^*) \, .$$

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Theorem

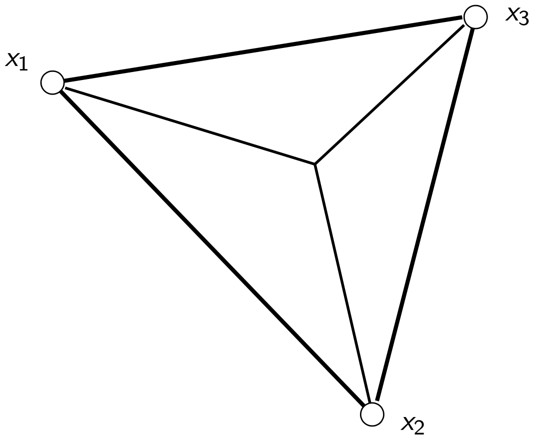
Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be convex and $C \subset \operatorname{dom} f$ be convex. Then x^* is a minimum of f with respect to the set C, if and only if there exists $p \in \partial f(x^*)$ such that

$$\langle y - x^*, p\rangle \geq 0\,, \qquad \text{for all } y \in C\,.$$

Back to Fermat-Torricelli

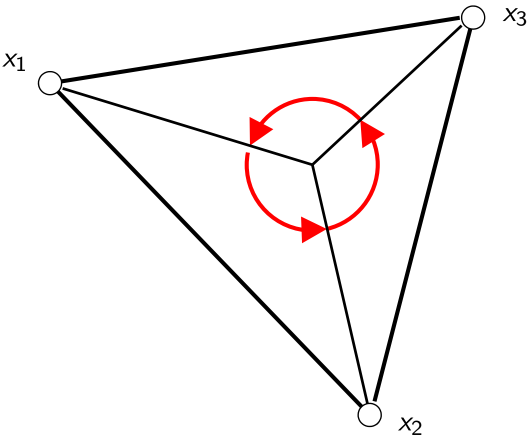
Given x_1, x_2, x_3 in $\mathbb{R}^2$ what is the point in $\mathbb{R}^2$ minimizing

$$\|x_1 - x^*\| + \|x_2 - x^*\| + \|x_3 - x^*\|?$$



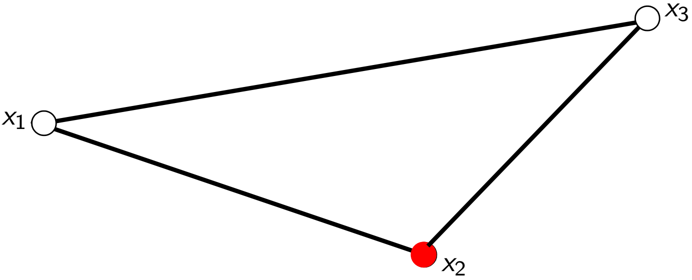
If no angle in the triangle is larger that 120°

The solution is in a point inside the triangle such that the between any two lines from the corner meeting in this point is 120° ($2\pi/3$).



Angles larger than 120°

If an angle in the triangle is larger than 120° (or $2\pi/3$), then the optimal point is in the corresponding corner of the triangle.



The subgradient of a norm

Definition

A norm on $\mathbb{R}^n$ is a function $v : \mathbb{R}^n \to [0, \infty)$ satisfying for all $x, y \in \mathbb{R}^n, \lambda \in \mathbb{R}$:

(i) (positive definiteness): $v(x) \geq 0$, and $v(x) = 0 \Leftrightarrow x = 0$.

(ii) (positive homogeneity): $v(\lambda x) = |\lambda| v(x)$,

(iii) (triangle inequality): $v(x + y) \leq v(x) + v(y)$.

The Dual Norm

Definition

Let v be a norm on $\mathbb{R}^n$ the *dual norm* is defined by

$$v^*(x) := \max\{|\langle l, x\rangle| \mid v(l) \leq 1\}\,.$$

A vector l is called dual to $x \in \mathbb{R}^n$, if $v^*(l) \leq 1$ and

$$\langle x, l\rangle = v(x)$$

The subgradient of a norm

Proposition

Let v be a norm on $\mathbb{R}^n$. Then for all $x \in \mathbb{R}^n$

$$\partial v(x) = \{p \in \mathbb{R}^n \mid v^*(p) \leq 1, \langle p, x \rangle = v(x)\}\,. \tag{3}$$