Convex Optimization and Congestion Control

Fabian Wirth

University of Würzburg

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Outline

• Part I:	Convexity and Convex Functions	Lectures 1, 2, 3
• Part II:	Convex Optimization	Lectures 4 and 5
• Part III:	Numerical Methods	Lectures 6 and 7
• Part IV:	Congestion Control	Lecture 8
• Part V:	Utility Based Congestion Control	Lecture 9
• Part VI:	Miscellaneous Problems in Networks	Lecture 10 (we shall see)

Part I: Convexity and Convex Functions

- I.1: Convex Sets
- I.2: Operations on Convex Sets and Construction of Convex Sets
- I.3: Separation
- I.4: Faces , Extreme Points and Recession Cones
- I.5: Duality
- I.6: Convex Functions
- I.7: Subgradients
- I.8: Optimality

Outline

Separation

Separation makes the heart grow fonder

A fundamental property of convex sets is that they can be separated from points not lying in the set.

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A fundamental property of convex sets is that they can be separated from points not lying in the set.

Furthermore nonintersecting convex sets can be separated by a hyperplane.

Separation

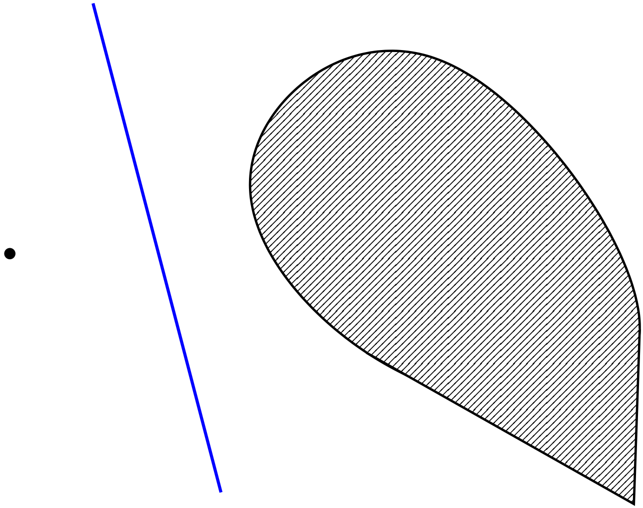


Figure: Separating Hyperplane

Separation

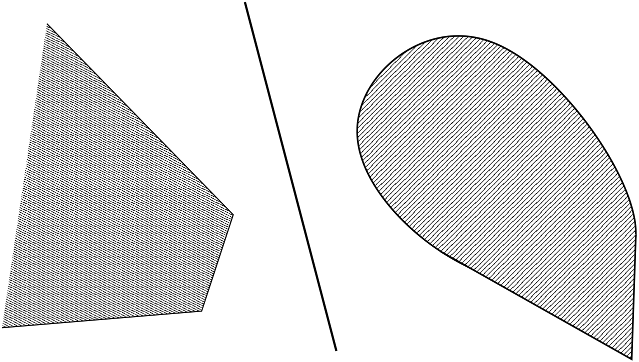
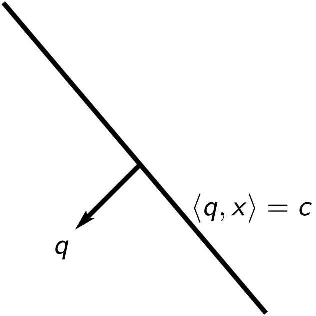


Figure: Separating Hyperplane

Hyperplanes

Definition

A hyperplane H in $\mathbb{R}^n$ is an $n-1$ dimensional affine linear subspace.



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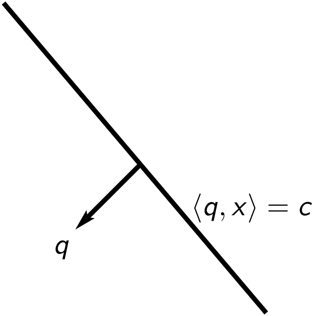
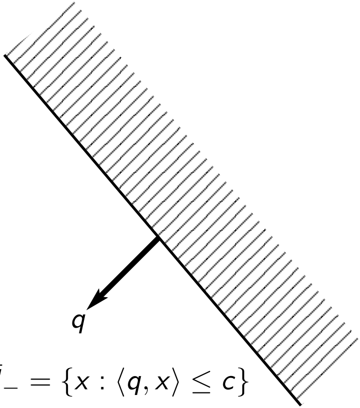


Hyperplanes have a representation by an orthogonal vector q and a constant value c of the scalar product of n with the elements of the hyperplane.

Half spaces



$$\overline{H}_{-} = \{x : \langle q, x \rangle \leq c\}$$

Figure: Half space

Notation

Given $q \in \mathbb{R}^n, q \neq 0,\ c \in \mathbb{R}$ we denote the hyperplane by

$$H(q,c) = \{x : \langle q, x \rangle = c\}$$

the half spaces by

$$\begin{aligned} H_- &= \{x : \langle q, x \rangle < c\} \\ H_+ &= \{x : \langle q, x \rangle > c\} \\ \overline{H}_- &= \{x : \langle q, x \rangle \leq c\} \\ \overline{H}_+ &= \{x : \langle q, x \rangle \geq c\} \end{aligned}$$

Closures of Convex Sets

Lemma

Let $C \subset \mathbb{R}^n$ be convex, then

(i) the closure $\mathrm{cl}\, C$ is convex.

(ii) the interior int C is convex. (possibly empty)

(iii) the relative interior $\mathrm{ri}\, C$ is convex.

The projection onto a convex set

Theorem

Let $K \subset \mathbb{R}^n$ be convex and $x \in \mathbb{R}^n$ be an arbitrary point.
Then

(i) **(unique distance minimization)**
there is a unique $y_0 \in \mathrm{cl}\, K$ (the closure of K) such that

$$\|x - y_0\| \leq \|x - y\| \qquad \text{for all } y \in \mathrm{cl}\, K\,.$$

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(ii) **(projection property)**
y_0 is characterized by

$$\langle x - y_0, y - y_0 \rangle \leq 0 \qquad \text{for all } y \in \mathrm{cl}\, K\,. \tag{1}$$

Geometric interpretation of the projection property

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y_0 is characterized by

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So $q = x - y_0$ and $c = \langle x - y_0, y_0 \rangle$ define a hyperplane H going through y_0 and so that K is to the other side of the hyperplane.

Distance to Convex Sets

We define the distance of a point to convex set by

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The separation "distance" between two convex sets C, K is then defined by

$$\mathrm{dist}(C, K) := \inf\{\|x - y\| \mid x \in C, y \in K\}\,.$$

Note that this is not a metric on the space of convex sets. The "distance" is already 0, if the closures of C and K intersect.

Projection onto a convex set

Definition

Let $K \subset \mathbb{R}^n$ be closed and convex. The projection onto K is defined by $\pi_K : \mathbb{R}^n \to K$

$$\pi_K(x) := y \in K\,, \text{ where } \|x - y\| = \operatorname{dist}(x, K)\,.$$

By the previous theorem this projection is uniquely defined.

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Proposition

Let $K \subset \mathbb{R}^n$ be closed and convex. Then π_K, the projection onto K, is globally Lipschitz continuous with constant $L = 1$.

Note: It is sometimes said that π_K is reducing.

Separation: First Pass

Corollary

Let K be closed and convex and $x \notin K$. Then there exists $q \in \mathbb{R}^n$, $c_1 > c_2 \in \mathbb{R}$ such that

$$\langle q, x \rangle = c_1$$

and

$$\langle q, y \rangle \leq c_2 \qquad \text{for all } y \in K\,.$$

Convex Sets and Half-Spaces

Proposition

Let $K \subset \mathbb{R}^n$ be closed and convex. Then K is equal to the intersection of the closed half spaces containing K.

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Application of this fact.

Proposition

Let $M \subset \mathbb{R}^n$ then

$$\operatorname{conv} B_\varepsilon(M) = B_\varepsilon(\operatorname{conv} M) .$$

Separation of Convex Sets

Theorem

Let C_1, C_2 be convex such that

$$C_1 \cap C_2 = \emptyset \,.$$

Then there exists a hyperplane H given by $q \neq 0$, $c \in \mathbb{R}$ such that

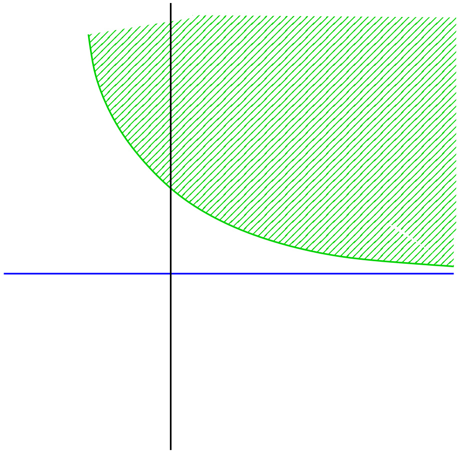
$$C_1 \subset \overline{H}_+ = \{x \in \mathbb{R}^n \mid \langle x, q \rangle \geq c\}$$

and

$$C_2 \subset \overline{H}_- = \{x \in \mathbb{R}^n \mid \langle x, q \rangle \leq c\} \,.$$

Note: Both inequalities are not strict !!

Separation by strict inequalities is not always possible



C_1 is the blue line, C_2 the green hashed area.

Why not a stronger statement?

Why it is not always possible to do the following?

Separate two convex sets by a hyperplane H given by $q \neq 0$, $c \in \mathbb{R}$ such that

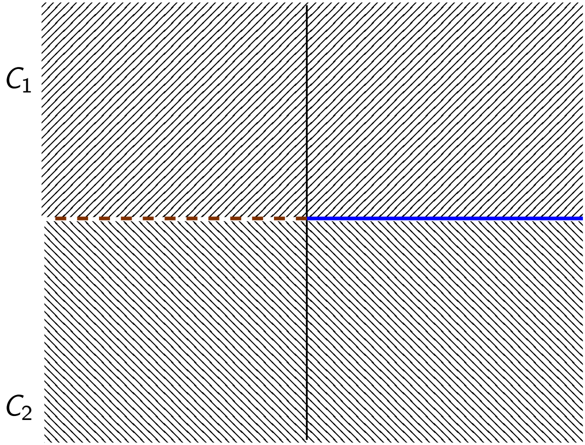
$$\overline{H}_+ = \{x \in \mathbb{R}^n \mid \langle x, q \rangle \geq c\}$$

contains one of the sets C_1, C_2 and

$$H_- = \{x \in \mathbb{R}^n \mid \langle x, q \rangle < c\}$$

contains the other one.

That's Why!



The blue line belongs to C_1, the dashed brown line to C_2.
Separation in the sense just suggested is not possible, even though $C_1 \cap C_2 = \emptyset$.

Strong Separation

Definition

Two convex sets can be *strongly separated*, if there exists a hyperplane H and an $\varepsilon > 0$ such that

$$C_1 \subset \overline{H}_{+,\varepsilon} = \{x \in \mathbb{R}^n \mid \langle x, q \rangle \geq c + \varepsilon\}$$

and

$$C_2 \subset \overline{H}_{-,\varepsilon} = \{x \in \mathbb{R}^n \mid \langle x, q \rangle \leq c - \varepsilon\} .$$

Strong Separation

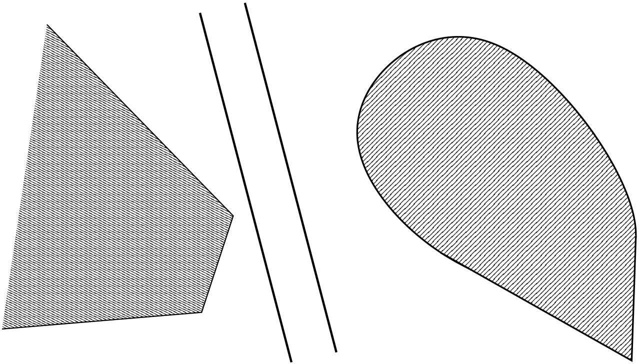


Figure: Strong separation means two hyperplanes.

Strong Separation

Theorem

Two convex sets $C_1, C_2 \subset \mathbb{R}^n$ can be strongly separated if and only if

$$\operatorname{dist}(C_1, C_2) := \inf\{\|x - y\| \mid x \in C_1, y \in C_2\} > 0\,.$$

Remark

If $C_1, C_2 \subset \mathbb{R}^n$ are both closed and at least one of the sets is bounded, then

$$\operatorname{dist}(C_1, C_2) > 0 \iff C_1 \cap C_2 = \emptyset\,.$$

Support

Definition

Let C be a convex set and x be a boundary point of C. A hyperplane H is said to be supporting in x (with respect to the convex set C), if H is represented by $q \in \mathbb{R}^n, q \neq 0$, $c \in \mathbb{R}$ and

(i) $x \in H$, i.e. $\langle x, q \rangle = c$,

(ii) $C \subset \overline{H}_+$, or $C \subset \overline{H}_-$.

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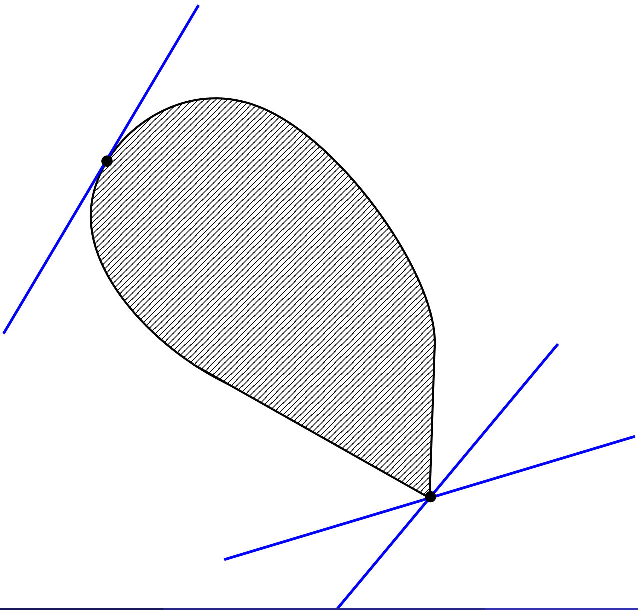
If C is convex, then for every boundary point $x \in \partial C$, there exists a supporting hyperplane.

Proof: Apply the separation theorem to x and $\mathrm{ri}\, C$. Or if $x \in \mathrm{ri}\, C$, take any hyperplane containing $\mathrm{aff}\, C$. If $x \in \partial C$ with respect to $\mathrm{aff}\, C$, then take a supporting hyperplane in $\mathrm{aff}\, C$ and extend it to a hyperplane in $\mathbb{R}^n$ by adding the orthogonal complement of $\mathrm{aff}\, C$.

Support



Outline

Faces

Definition

Let C be convex. A convex set $C' \subset C$ is a *face* of C, if for every segment $[x, y] \subset C$ such that

$$\mathrm{ri}\,[x, y] \cap C' \neq \emptyset$$

we have

$$[x, y] \subset C' .$$

A face C' is called *exposed* if there exists a hyperplane $H(q, c)$ such that $C' \subset H$ and $(C \setminus C') \subset H_+(q, c)$.
(Note: we are considering the open half-space.)

Note: C and $\emptyset$ are always faces of C.

Extreme Points

Definition

Let $C \subset \mathbb{R}^n$ be convex. A point $x \in C$ is an extreme point of C, if from the condition

$$x = \lambda y + (1-\lambda) z\,, \quad y, z \in C, \lambda \in (0,1) \tag{2}$$

it follows that $x = y = z$.

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it follows that $x = y = z$.

Extreme points are just the zero-dimensional faces.

Extreme Points

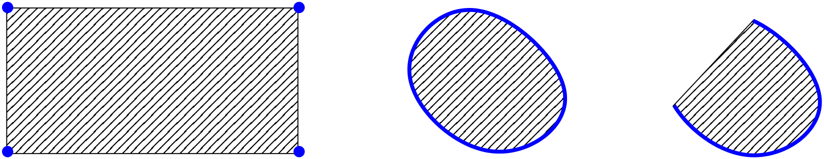

Figure: extreme points

Not every extreme point is exposed.

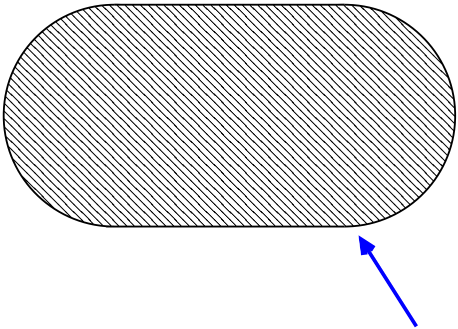

Figure: Extreme points that are not exposed.

On other hand, of course, every exposed point is an extreme point.

Extreme Points and Convex Hulls

Theorem

Every bounded convex set is the convex hull of its extreme points.

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Proposition

Bounded convex polytopes are precisely the bounded convex sets that have a finite number of extreme points.

Extreme Points of Compact Sets

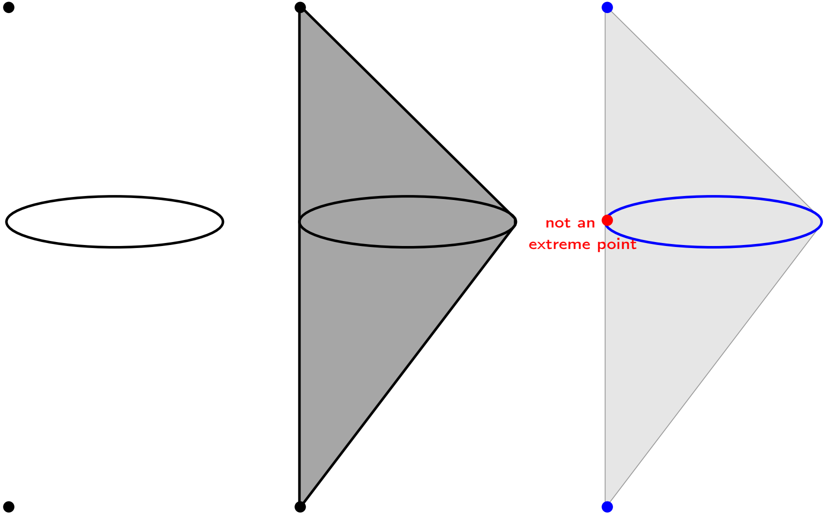

Figure: $ext(C)$ needs not to be compact for C compact

Recession Cones

Definition

Let C be a nonempty convex set. We say C recedes in the direction $y \in \mathbb{R}^n$ if

$$x + \lambda y \in C\,, \qquad \text{for all } x \in C, \lambda \geq 0\,.$$

The recession cone 0^+C of C is the set

$$0^+C := \{y \in \mathbb{R}^n \mid C \text{ recedes in direction } y\}\,.$$

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Lemma

Let C be a nonempty convex set. The recession cone 0^+C is a convex cone. It satisfies

$$0^+C = \{y \in \mathbb{R}^n \mid C + y \subset C\}\,.$$

Proposition

(i) If C is a closed convex set containing the origin, then

$$0^+C = \bigcap_{\varepsilon>0} \varepsilon C = \{y \mid \varepsilon^{-1} y \in C\,, \quad \text{for all } \varepsilon > 0\}\,.$$

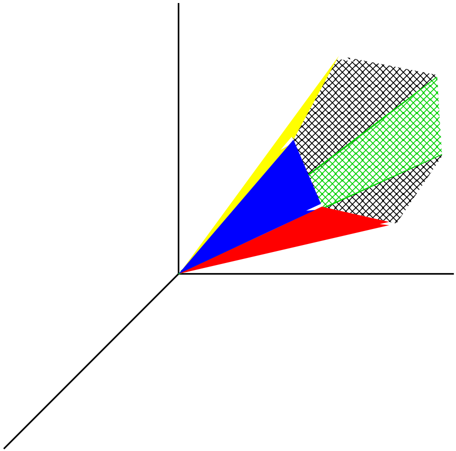
(ii) If $C_i, i \in I$ is a family of convex sets, such that its intersection is nonempty, then

$$0^+\left(\bigcap_{i\in I} C_i\right) = \bigcap_{i\in I} 0^+ C_i\,.$$

(iii) A nonempty closed and convex set is bounded if and only if $0^+C = \{0\}$.

Extreme Rays

For convex cones extreme points are not very interesting: Only 0 can be an extreme point.



Extreme Rays, extreme directions

For a convex cone an extreme ray is a one-dimensional face, which is not a straight line.

More generally:

Definition

Let $C \subset H$ be a nonempty convex set. We say that y is an *extreme direction* of C, if there exists an $x \in C$ such that

$$x + \{\lambda y \mid \lambda \geq 0\}$$

is a face of C.

Convex Hulls of Points and Directions

Definition

Given a set of points $M_1 \subset \mathbb{R}^n$ and a set of directions $M_2 \subset \mathbb{R}^n$, the convex hull of (M_1, M_2) is defined as the smallest convex set containing

$$M_1 + \{\lambda x \mid x \in M_2, \lambda \geq 0\}.$$

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Theorem

Let $C \subset \mathbb{R}^n$ be a closed convex set containing no lines. Let S be the set of extreme points and extreme directions of C. Then

$$C = \operatorname{conv} S .$$

Outline

Convex Functions

We now consider functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$.

Definition

The *domain* of such a function is defined as the set, where the function has finite values, i.e.

$$\operatorname{dom} f := \{x \in \mathbb{R}^n \mid f(x) \in \mathbb{R}\}.$$

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With this notation we may as well consider the finite function

$$f : \operatorname{dom} f \subset \mathbb{R}^n \to \mathbb{R} .$$

Convex Functions

Definition

A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is called *convex*, if

(i) $\operatorname{dom} f$ is convex,

(ii) for all $x, y \in \operatorname{dom} f$, $\lambda \in [0,1]$ we have

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y) \, .$$

The function f is called *strictly convex* if in addition the inequality is strict, whenever $x \neq y$, $\lambda \in (0,1)$.

Convex Functions

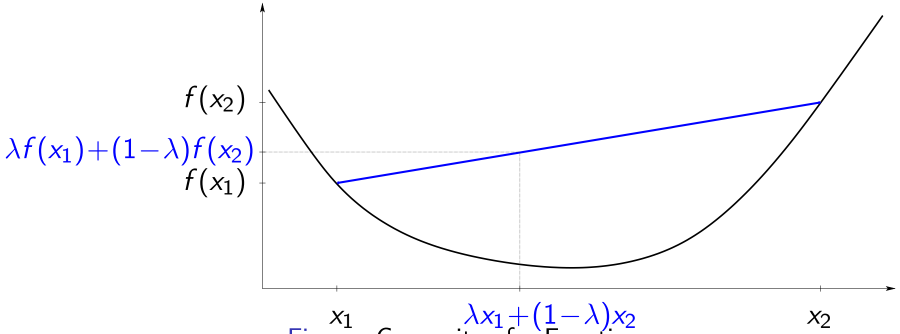

Figure: Convexity of a Function

Convex - Concave

Definition

f is called concave if $-f$ is convex.
f is called strictly concave, if $-f$ is strictly convex.

Thus all results we obtain on convex functions also give information about concave functions.

The Epigraph

Definition

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$. The *epigraph* of f defined by

$$\operatorname{epi} f := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \operatorname{dom} f\,, \quad y \geq f(x)\}\,.$$

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Lemma

A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex if and only if $\mathrm{epi}\, f$ is convex.

Convex Functions

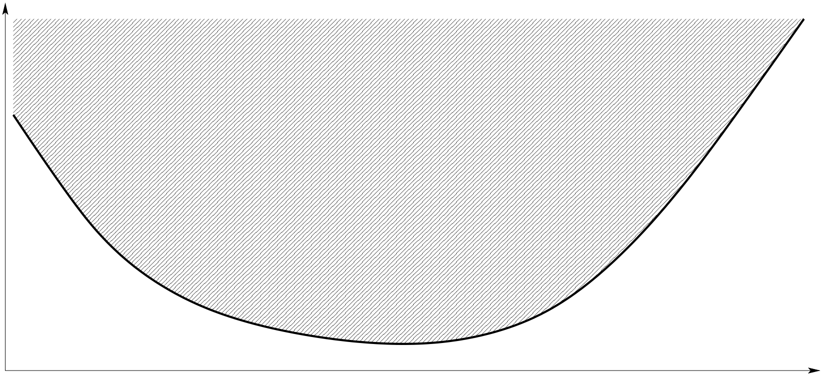

Figure: The epigraph of a convex function is convex.

Convexity and Differentiability

Theorem

Consider $f : \mathrm{dom}\, f \subset \mathbb{R}^n \to \mathbb{R}$ and assume that $\mathrm{dom}\, f = \mathrm{ri}\,\mathrm{dom}\, f$ is convex.

(i) If f is continuously differentiable on $\mathrm{dom}\, f$, then f is convex if and only if

$$f(x) + \langle y - x, \nabla f(x) \rangle \leq f(y)\,, \qquad \text{for all } x, y \in \mathrm{dom}\, f\,. \tag{3}$$

If the inequality is strict whenever $x \neq y$, then f is strictly convex.

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If the inequality is strict whenever $x \neq y$, then f is strictly convex.

(ii) Assume f is twice continuously differentiable. Then f is convex if and only if the Hessian of f satisfies

$$Hf(x) \geq 0\,, \qquad \text{for all } x \in \mathrm{dom}\, f\,. \tag{4}$$

If the Hessian is positive definite everywhere on $\mathrm{dom}\, f$, then f is strictly convex.

Rearranging the condition for differentiable functions we obtain equivalently:

$$\langle y, \nabla f(x) \rangle - f(y) \leq \langle x, \nabla f(x) \rangle - f(x) \,.$$

The Epigraph and Differentiability

Rearranging the condition for differentiable functions we obtain equivalently:

$$\langle y, \nabla f(x)\rangle - f(y) \leq \langle x, \nabla f(x)\rangle - f(x)\,.$$

Thinking in terms of the epigraph of f we can write this as

$$\left\langle \begin{pmatrix} y \\ f(y) \end{pmatrix}, \begin{pmatrix} \nabla f(x) \\ -1 \end{pmatrix} \right\rangle \leq \left\langle \begin{pmatrix} x \\ f(x) \end{pmatrix}, \begin{pmatrix} \nabla f(x) \\ -1 \end{pmatrix} \right\rangle \tag{5}$$

And we see that in fact

$$n := \begin{pmatrix} \nabla f(x) \\ -1 \end{pmatrix}, \quad c := \left\langle \begin{pmatrix} x \\ f(x) \end{pmatrix}, \begin{pmatrix} \nabla f(x) \\ -1 \end{pmatrix} \right\rangle$$

defines a supporting hyperplane of the epigraph in the point $(x, f(x))$

The Epigraph and Differentiability

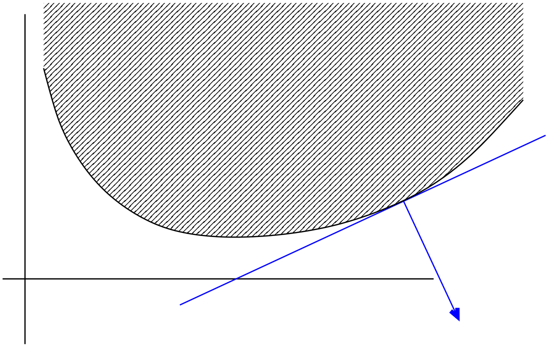

Figure: Supporting hyperplane of the epigraph.