Convex Optimization and Congestion Control

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Outline

• Part I:	Convexity and Convex Functions	Lectures 1, 2, 3
• Part II:	Convex Optimization	Lectures 4 and 5
• Part III:	Numerical Methods	Lectures 6 and 7
• Part IV:	Congestion Control	Lecture 8
• Part V:	Utility Based Congestion Control	Lecture 9
• Part VI:	Miscellaneous Problems in Networks	Lecture 10 (we shall see)

Part IV: Congestion Control

- IV.1: Basics of TCP
- IV.2: Dynamics of Deterministic AIMD
- IV.3: Utility Based Congestion Control

Outline

1. Modeling of TCP flows
2. Utility Based Congestion Control

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2. Utility Based Congestion Control

User and Resources

Setup

Consider a situtation in which users $i = 1, \ldots, n$ use a network or resources $\mathcal{R}$.

User and Used Resources

We identify user i with his subset of used resources, i.e.

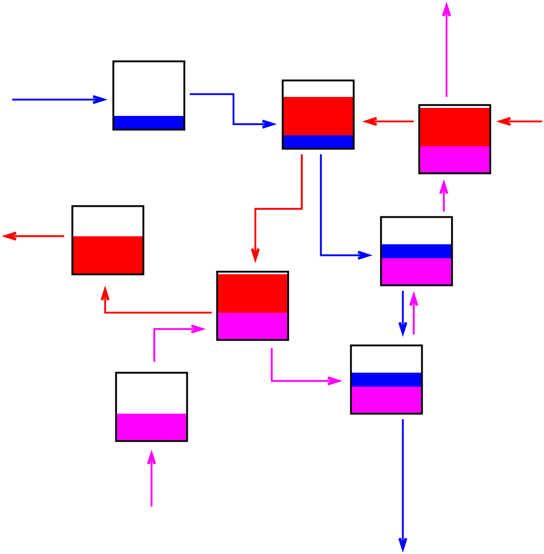
$$i \subset \mathcal{R}\,.$$

Capacity Constraint

Each resource is limited, so we need

$$\sum_{i:r\in i} x_i \leq C_r\,.$$

User and Resources



Fairness

Definition

An allocation vector $x = (x_1, \ldots, x_n)$ is called *min-max fair*, if

(i) it satisfies the constraint conditions,

(ii) if y is another vector satisfying the constraints and for some i we have $y_i > x_i$, then there is an index j such that

$$y_j < x_j \leq x_i \, .$$

Bottlenecks

Definition

Given an allocation vector $x = (x_1, \dots, x_n)$ a resource r is called a *bottleneck* for user i, if

(i) full use of the resource, i.e.

$$\sum_{j, r \in j} x_j = C_r \,,$$

(ii) $x_i \geq x_j$ for all users j with $r \in j$.

Characterization of Min-Max Fairness

Lemma

An allocation vector $x = (x_1, \ldots, x_n)$ is min-max fair if and only if every user has a bottleneck resource.

Corollary

In a max-min fair allocation vector $x = (x_1, \ldots, x_n)$ there are at least two entries that are equal to the minimal entry.

Utility Functions

Utility

We assume from now on that the utility obtained by user i of having x_i bandwidth/rate is given by the utility function U_i.

Assumptions

(i) U_i are strictly concave and increasing, $i = 1, \ldots .n$.

(ii) $U_i(x_i) \to -\infty$ as $x_i \to 0$.

(iii) $U_i'(x_i) \to 0$ as $x_i \to \infty$.

Note: Condition (iii) still allows for bounded and unbounded utility functions, but functions like $s \mapsto s + log(s)$ are ruled out.

Optimization Problem

Definition

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{n} U_i(x_i) \\
\text{subject to} \quad & \sum_{i:r\in i} x_i \leq C_r \\
& x_i \geq 0\,, \quad i = \{1, \ldots, n\}\,.
\end{aligned}
$$

Barrier Functions

Definition

For every $r \in \mathcal{R}$ let $f_r : [0, \infty) \to [0, \infty)$ be a continuous, non-decreasing function. We assume that

$$\int_0^y f_r(s) ds \to \infty \,, \qquad \text{for } y \to \infty \,. \tag{1}$$

Use these integrals as a general form of barrier function.

$$V(x) = \sum_{i=1}^{n} U_i(x_i) - \sum_{r \in \mathcal{R}} \int_0^{\sum_{i:r \in i} x_i} f_r(s) ds \tag{2}$$

Lemma

Assume that for each i the utility function U_i is continuously differentiable, nondecreasing and strictly concave. The function V is strictly concave.

Lemma

$V(x) \to -\infty$ for $x \to 0$ and for $\|x\| \to \infty$.

Convex Optimization Problem

$$\text{maximize} \quad V(x)$$

$$x_i \geq 0\,, \quad i = \{1, \ldots, n\}\,.$$

By the previous lemmas a unique optimum exists.

Characterization of the Optimum

$$U_i'(x_i^*) - \sum_{r:r\in i} f_r\left(\sum_{j:r\in j} x_j^*\right) = 0\,, \quad i = 1, \ldots, n\,.$$

Prices

Barrier Functions as Prices

Price for router r

$$p_r = f_r\Big(\sum_{j:r\in j} x_j\Big)\,. \tag{3}$$

Total price per user

$$q_i = \sum_{r\in i} p_r\,.$$

Optimality Condition

$$U_i'(x_i^*) - q_i(x^*) = 0\,.$$

Rate Adjustement

Differential equation

$$\dot{x}_i = k_i(x_i)\left(U_i'(x_i) - q_i(x(t))\right) ,$$

where $k_i : [0, \infty) \to (0, \infty)$ is a continuous positive function that regulates the speed of the differential equation.

Rate Adjustement

Differential equation

$$\dot{x}_i = k_i(x_i)\left(U_i'(x_i) - q_i(x(t))\right) ,$$

where $k_i : [0, \infty) \rightarrow (0, \infty)$ is a continuous positive function that regulates the speed of the differential equation.

Theorem

Assuming that there is an optimal point x^* for the optimization problem the solution of the differential equation will converge to x^* for any initial condition $x > 0$.

The Routing Matrix

Routing

Define the matrix $R \in \mathbb{R}^{|\mathcal{R}| \times n}$ by

$$R_{ri} = \begin{cases} 1 & r \in i \\ 0 & \text{else.} \end{cases}$$

We then have the relations

$$y = Rx$$

where y_r is the total used resource at router r.
Also

$$q = R^T p$$

where p is the vector of prices at the routers and q is the vector of total prices for the users.

The Dual Algorithm

The Optimization Problem

The original optimization problem was

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} U_i(x_i) & \\ \text{subject to} \quad & \sum_{i:r\in i} x_i \leq C_r & r \in \mathcal{R} \\ & x_i \geq 0 & i = 1, \ldots, n\,. \end{aligned}$$

Note: As we are assuming $U_i(x_i) \to -\infty$ for $x_i \to 0$, we can ignore the condition $x_i \geq 0, i = 1, \ldots, n$.

The KKT Conditions

The corresponding KKT conditions are

$$
\begin{aligned}
\sum_{i:r\in i} x_i^* - C_r &\leq 0 \quad r \in \mathcal{R} \\
\lambda_r^* &\geq 0 \quad r \in \mathcal{R} \\
\lambda_r^* \left(\sum_{i:r\in i} x_i^* - C_r \right) &= 0 \quad r \in \mathcal{R} \\
U_i'(x_i^*) - \sum_{r\in i} \lambda_r^* &= 0 \quad i = 1, \dots, n
\end{aligned}
$$

We interpret the Lagrange multipliers as prices p and recall $y = Rx$, $q = R^\top p$.

Reformulation

KTT Conditions II

In this reformulation the KKT conditions become

$$\begin{aligned} y_r - C_r \quad & \leq 0 \quad r \in \mathcal{R} \\ p_r \quad & \geq 0 \quad r \in \mathcal{R} \\ p_r\left(y_r - C_r\right) \quad & = 0 \quad r \in \mathcal{R} \\ U_i'(x_i^*) - q_i \quad & = 0 \quad i = 1, \ldots, n \end{aligned}$$

And the last condition may be rewritten as

$$x_i^* = \left(U_i'\right)^{-1}\left(q_i\right)$$

The Dual Algorithm

We now consider an algorithm in which routers adjust prices and sources adjust their rate according to the price information received.

The Dynamic System

$$
\begin{aligned}
x_i &= \left(U_i'\right)^{-1}(q_i) \\
q &= R^\top p \\
y &= Rx \\
\dot{p}_r &= h_r(p_r) \begin{cases} (y_r - C_r) & \text{if} \quad p_r > 0 \\ \max\{0, (y_r - C_r)\} & \text{if} \quad p_r = 0 \end{cases}
\end{aligned}
$$

with a positive non-decreasing continuous function h_r.

The Dual Algorithm

Rank Assumption

Assume R has full row rank, so that $p_1 \neq p_2$ implies $q_1 = R^\top p_1 \neq R^\top p_2 = q_2$.

Theorem

If R has full row rank, the dual algorithm is globally asymptotically stable in the global optimum of the optimization problem with respect to initial conditions $p > 0$.
A Lyapunov function is given by

$$W(p) = \sum_{r \in \mathcal{R}} (C_r - y_r^*)\, p_r + \sum_{i=1}^{n} \int_{q_i^*}^{q_i} \left(x_i^* - (U_i')^{-1}(s) \right) ds \,.$$

Here x^*, p^* denotes the unique optimal value of the optimization problem. $y^* := Rx^*$ and $q^* := R^\top p^*$.

The Primal-Dual Algorithm

We now consider the combination of the two approaches. I.e. we combine

Simultaneous Update

$$
\begin{aligned}
\dot{x}_i &= k_i(x_i)\left(U_i'(x_i) - q_i(x(t))\right) \\
\dot{p}_r &= h_r(p_r)\begin{cases} (y_r - C_r) & \text{if} \quad p_r > 0 \\ \max\{0, (y_r - C_r)\} & \text{if} \quad p_r = 0 \end{cases}
\end{aligned}
$$

Theorem

The primal-dual algorithm is globally asymptotically stable in (x^*, p^*). A Lyapunov function is given by

$$W(x) = \sum_{i=1}^{n} \int_{x_i^*}^{x_i} \frac{1}{k_i(s)} (s - x_i^*)\, ds + \sum_{r \in \mathcal{R}} \int_{p_r^*}^{p_r} \frac{1}{h_r(s)} (s - p_r^*)\, ds\,.$$

One Bit Marking

A

ssume that instead of transferring price information each router sets a "price bit" with the probability of its current price.
This assumes of course that prices are in the interval $[0,1]$.
The probability that packet is marked along the route i is then

$$q_i = 1 - \prod_{r:r \in i} (1 - p_r)\,.$$

This information could be sent back by setting one bit in acknowledgements.

The Dynamic Equations

$$\dot{x}_i = k_i(x_i)\left((1 - q_i)U_i'(x_i) - q_i\right)\,.$$

One Bit Marking

Theorem

The one bit marking controllers are globally asymptotically stable.
A Lyapunov function is given by the convex function

$$W(x) = -\sum_{i=1}^{n} \int_0^{x_i} \log(1 + U_r'(s))ds - \sum_{r \in \mathcal{R}} \int_0^{y_r} \log(1 - f_r(s))ds\,.$$

TCP and Resource Allocation

A Differential Model for TCP

Let $w_i(t)$ be the window size of user i at time t.
T_i be the round trip time (RTT)
$q_r(t)$ the fraction of packets lost
$x_i(t) = w_i(t)/T_i$ the transmission rate
β_i the decrease factor

$$\dot{w}_i = \frac{x_i(t - T_i)(1 - q_i(t))}{w_i} - \beta x_i(t - T_i) q_i(t) w_i$$

Small RTT's

If the round trip time T_i is small, then using the approximation $x_i(t) = x_i(t - T_i)$ we obtain the approximate system

$$
\begin{aligned}
\dot{x}_i &= \frac{1 - q_i(t)}{T_i^2} - \beta x_i^2 q_i(t) \\
&= \left(\beta x_i^2 + \frac{1}{T_i^2}\right) \left(\frac{1}{\beta T_i^2 x_i^2 + 1} - q_i(t)\right)
\end{aligned}
$$

$x_i(t) \approx x_i(t - T_i)$

Small RTT's

If the round trip time T_i is small, then using the approximation $x_i(t) = x_i(t - T_i)$ we obtain the approximate system

$$\begin{aligned}\dot{x}_i &= \frac{1 - q_i(t)}{T_i^2} - \beta x_i^2 q_i(t) \\ &= \left(\beta x_i^2 + \frac{1}{T_i^2}\right)\left(\frac{1}{\beta T_i^2 x_i^2 + 1} - q_i(t)\right)\end{aligned}$$

This corresponds to the utility function given by

$$U_i' = \frac{1}{\beta T_i^2 x_i^2 + 1}$$

so

$$U_i(x_i) = \frac{\arctan(x_i T_i \sqrt{\beta})}{\sqrt{\beta} T_i}$$

Thank you !