Convex Optimization and Congestion Control

Fabian Wirth

University of Würzburg

23.07. – 03.08.2012

Outline

• Part I: Convexity and Convex Functions	Lectures 1, 2, 3
• Part II: Convex Optimization	Lectures 4 and 5
• Part III: Numerical Methods	Lectures 6 and 7
• Part IV: Congestion Control	Lecture 8
• Part V: Utility Based Congestion Control	Lecture 9
• Part VI: Miscellaneous Problems in Networks	Lecture 10 (we shall see)

Part I: Convexity and Convex Functions

- I.1: Convex Sets
- I.2: Operations on Convex Sets and Construction of Convex Sets
- I.3: Separation
- I.4: Faces , Extreme Points and Recession Cones
- I.5: Duality
- I.6: Convex Functions
- I.7: Subgradients
- I.8: Optimality

Outline

The TGI Problem

Problem

There are n PhD students interested in taking part in a course on convex optimization (and also a lecturer). In the spirit of the course the question is

where should the course take place, optimally speaking?

The TGI Problem

Problem

There are n PhD students interested in taking part in a course on convex optimization (and also a lecturer). In the spirit of the course the question is

where should the course take place, optimally speaking?

What does optimal mean?

A. Environment friendly:
Minimize the total distance traveled by all participants.
B. Fair (?): Minimize the maximum distance any one person has to travel.
As we shall see, both problems are convex optimization problems. We will determine later, whether we should have met here.

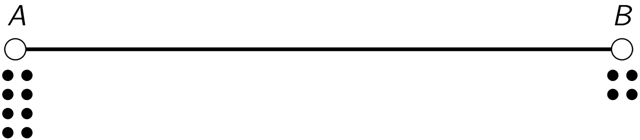
The Social Problem

Watching the Big Game

There once were 12 friends all looking forward to see the final of the world cup together in a nice pub.
8 of them live in A and the other 4 in B.
Technicalities like "where do they serve the best pint?" aside:
Where should the pub be located in order to minimize the total distance traveled by all of them?

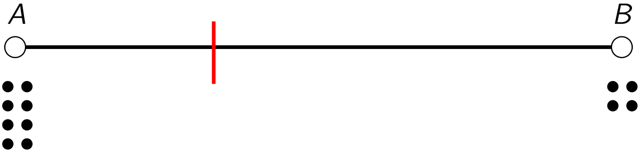
The Social Problem

Where should they meet to minimize total traveled distance?



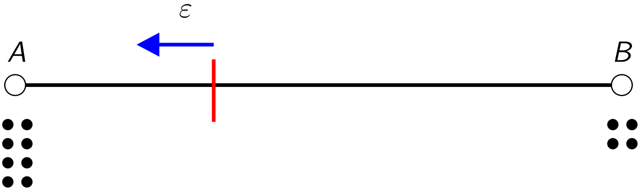
The Social Problem

Where should they meet to minimize total traveled distance?



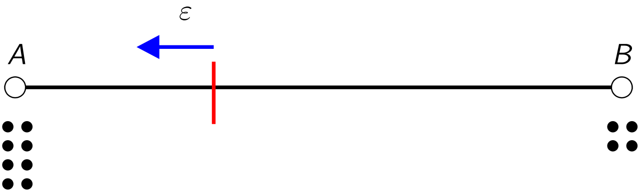
The Social Problem

Where should they meet to minimize total traveled distance?



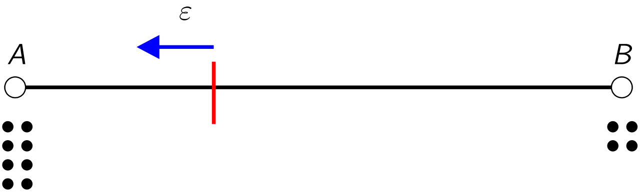
The Social Problem

Where should they meet to minimize total traveled distance?



A shift of the pub to the left gives a gain of -8ε decrease in total travel and an increase of 4ε.

Where should they meet to minimize total traveled distance?



A shift of the pub to the left gives a gain of -8ε decrease in total travel and an increase of 4ε.

The optimal pub is located in A.

Question

Given m points $x_1, \ldots, x_m$ in $\mathbb{R}^n$, is there a point x^* minimizing the sum of the distances

$$\|x_1 - x^*\| + \ldots + \|x_m - x^*\|? \tag{1}$$

And is yes, is it unique and what is it?

The Fermat-Torricelli problem

Question

Given m points $x_1, \dots, x_m$ in $\mathbb{R}^n$, is there a point x^* minimizing the sum of the distances

$$\|x_1 - x^*\| + \ldots + \|x_m - x^*\|? \tag{1}$$

And is yes, is it unique and what is it?

Existence is clear: The sum in (1) is nonnegative, continuous in x^* and goes to ∞ as $\|x^*\| \to \infty$. By continuity a minimum has to exist.

The Fermat-Torricelli problem

Question

Given m points $x_1, \dots, x_m$ in $\mathbb{R}^n$, is there a point x^* minimizing the sum of the distances

$$\|x_1 - x^*\| + \ldots + \|x_m - x^*\|? \tag{1}$$

And is yes, is it unique and what is it?

Existence is clear: The sum in (1) is nonnegative, continuous in x^* and goes to ∞ as $\|x^*\| \to \infty$. By continuity a minimum has to exist.

How can we determine a/the minimal point?

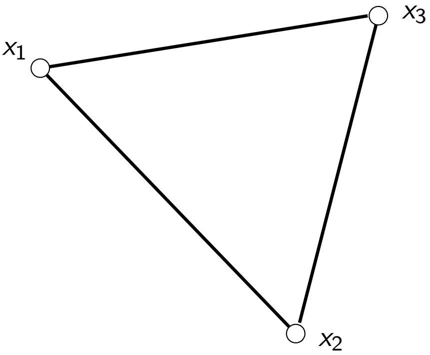
The Fermat-Torricelli problem

Even simpler:

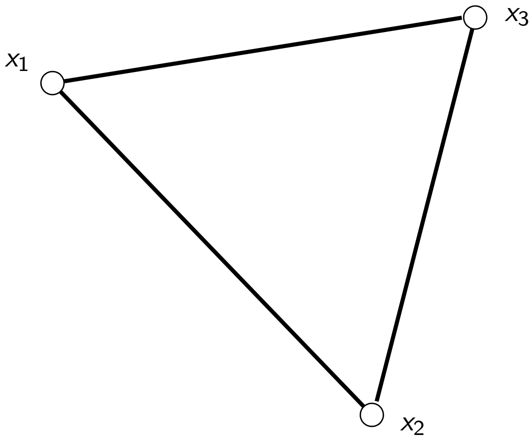
The original Fermat-Torricelli problem

Given x_1, x_2, x_3 in $\mathbb{R}^2$ what is the point in $\mathbb{R}^2$ minimizing

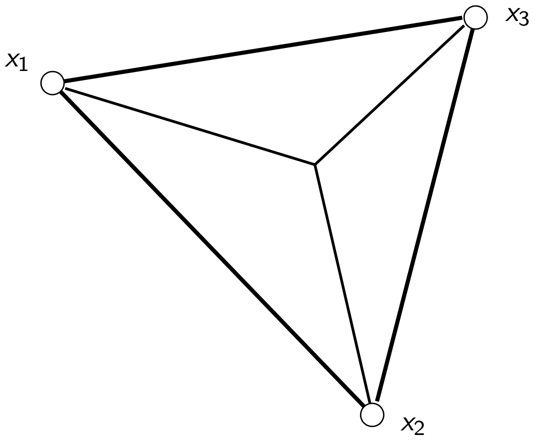
$$\|x_1 - x^*\| + \|x_2 - x^*\| + \|x_3 - x^*\|? \tag{2}$$



The Fermat-Torricelli problem



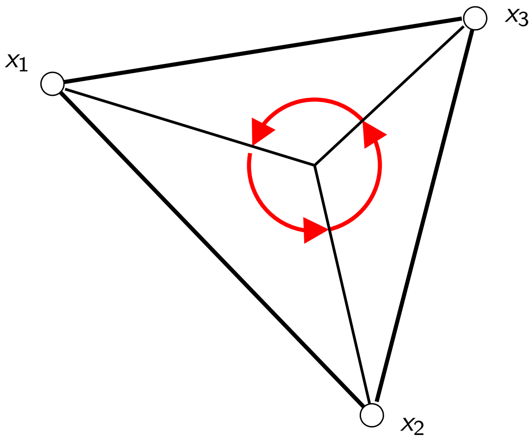
The Fermat-Torricelli problem



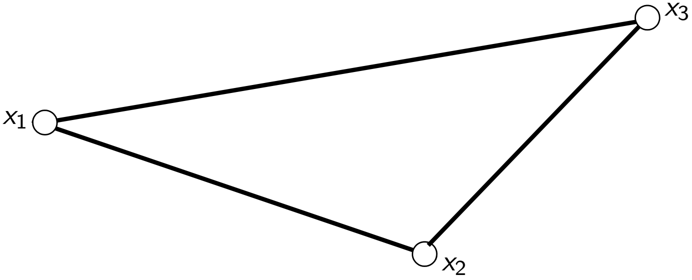
The Fermat-Torricelli problem

If ...

The solution is in a point inside the triangle such that the between any two lines from the corner meeting in this point is 120° ($2\pi/3$).

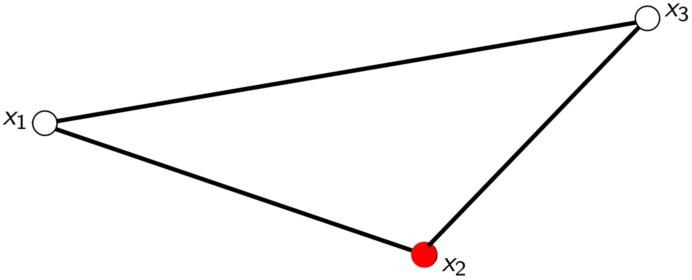


The solution changes with the shape of the triangle.



The "very obtuse" case

If an angle in the triangle is larger than 120° (or $2\pi/3$), then the optimal point is in the corresponding corner of the triangle.



Outline

Convex Sets

We consider convex subsets of $\mathbb{R}^n$.
Much of the geometry discussed here extends very well to Hilbert spaces.
Some remarks on this are in the notes.

Line Segments

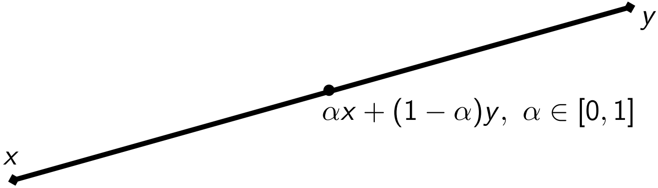
Given two points $x, y \in \mathbb{R}^n$ the line segment $[x, y]$ is defined by

$$[x, y] := \{\alpha x + (1 - \alpha) y \mid \alpha \in [0, 1]\}$$

Line Segments

Given two points $x, y \in \mathbb{R}^n$ the line segment $[x, y]$ is defined by

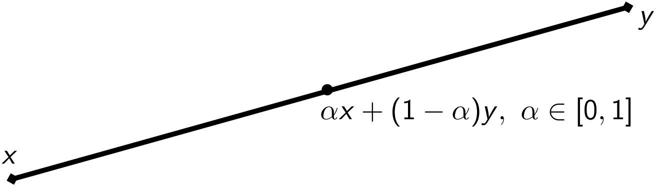
$$[x, y] := \{\alpha x + (1 - \alpha)y \mid \alpha \in [0, 1]\}$$



Line Segments

Given two points $x, y \in \mathbb{R}^n$ the line segment $[x, y]$ is defined by

$$[x, y] := \{\alpha x + (1 - \alpha) y \mid \alpha \in [0, 1]\}$$



For $\alpha \in [0, 1]$ the vector

$$\alpha x + (1 - \alpha) y = y + \alpha (x - y)$$

is called a convex combination of x and y.

Convex Sets

Definition

A subset K of $\mathbb{R}^n$ is called convex, if for all $x, y \in K$, $\alpha \in [0,1]$ the convex combination

$$\alpha x + (1-\alpha) y \in K\,.$$

Convex Sets

Definition

A subset K of $\mathbb{R}^n$ is called convex, if for all $x, y \in K$, $\alpha \in [0,1]$ the convex combination

$$\alpha x + (1-\alpha) y \in K\,.$$

Equivalently:

K is convex, if for all $x, y \in K$ the line segment

$$[x, y] \subset K\,.$$

Convex Sets

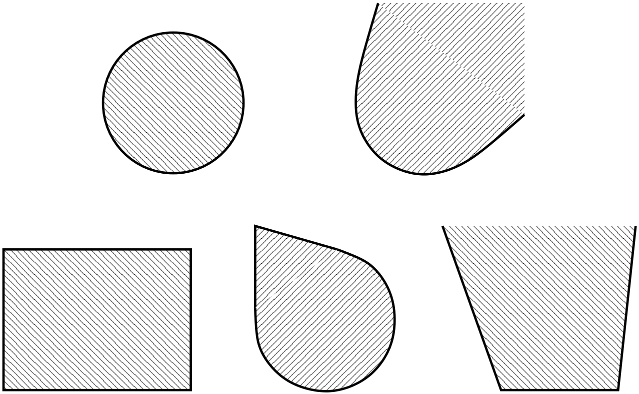


Figure: Convex sets

Nonconvex Sets

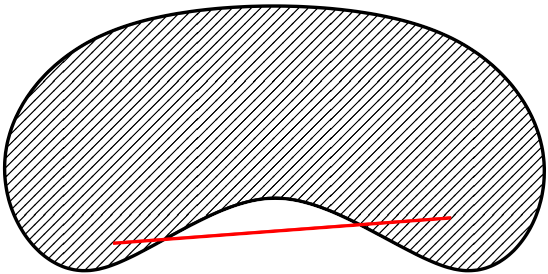


Figure: Definitely not a convex set.

Examples of Convex Sets

Important examples of convex sets are the following

- affine sets
- hyperplanes
- half-spaces
- convex polytopes
- Euclidean balls and ellipsoids
- convex cones

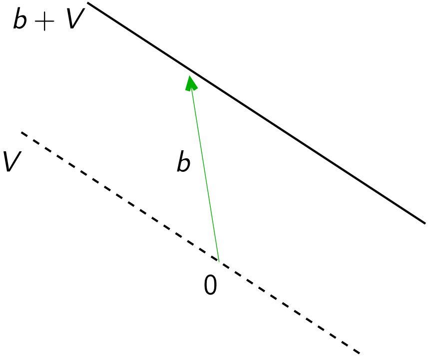
Affine Sets

Shifting Linear Subspaces

An affine set is a linear subspace of $\mathbb{R}^n$ shifted by a constant vector. I.e.

$$\{b + v \mid v \in V\}$$

for some $b \in \mathbb{R}^n$ and some linear subspace V.



Affine Sets

Image Representation

Any affine set can be viewed as the image of a linear map $F \in \mathbb{R}^{n \times n}$ shifted by a vector b, so that we arrive at a description

$$M = \{Fx + b \mid x \in \mathbb{R}^n\} . \quad (3)$$

Affine Sets

Image Representation

Any affine set can be viewed as the image of a linear map $F \in \mathbb{R}^{n \times n}$ shifted by a vector b, so that we arrive at a description

$$M = \{Fx + b \mid x \in \mathbb{R}^n\}. \tag{3}$$

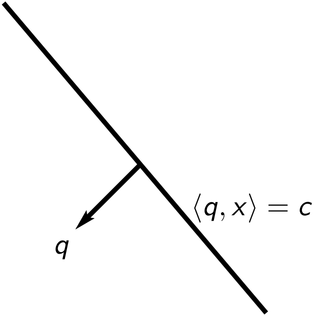
Lemma

Provided the intersection is nonempty, the intersection of affine sets is an affine set.

Hyperplanes

Definition

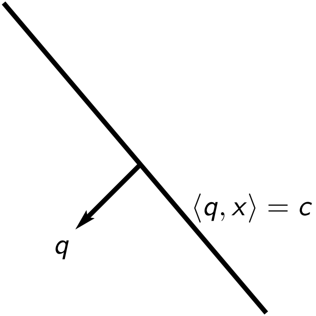
A hyperplane H in $\mathbb{R}^n$ is an $n-1$ dimensional affine linear subspace.



Hyperplanes

Definition

A hyperplane H in $\mathbb{R}^n$ is an $n-1$ dimensional affine linear subspace.



Hyperplanes have a representation by an orthogonal vector q and a constant value c of the scalar product of n with the elements of the hyperplane.

Half spaces

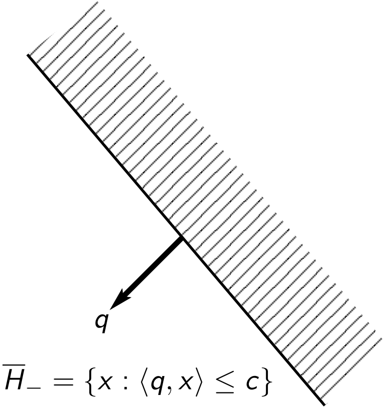
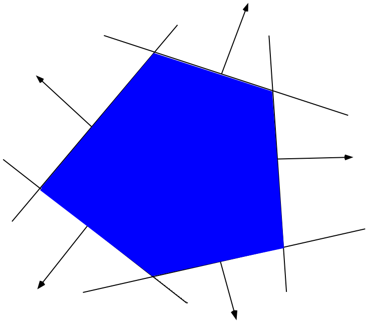


Figure: Half space

Convex Polytopes

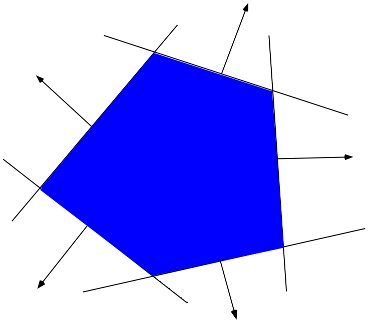
Given k orthogonal conditions by $(q_i, c_i) \in \mathbb{R}^n \times \mathbb{R}, i = 1, \dots, k,\ k \geq 2$ the corresponding polytopic set is

$$P := \{x \in \mathbb{R}^n \mid \langle q_i, x\rangle \leq c_i, i = 1, \dots, k\}.$$



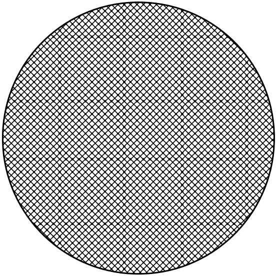
Given k orthogonal conditions by $(q_i, c_i) \in \mathbb{R}^n \times \mathbb{R}, i = 1, \ldots, k,\ k \geq 2$ the corresponding polytopic set is

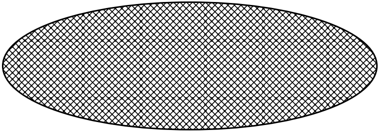
$$P := \{x \in \mathbb{R}^n \mid \langle q_i, x \rangle \leq c_i, i = 1, \ldots, k\}.$$



Note: Polytopic sets are not necessarily bounded.

Euclidean Balls and Ellipsoids





$B_\epsilon(x^*) :=$
$\{x \in \mathbb{R}^n \mid \|x - x^*\|_2 \leq c\}$
Closed Euclidean ball
of radius c centered at x^*

$\{x \in \mathbb{R}^n \mid (x - x^*)^\top P(x - x^*) \leq c\}$
Ellipsoid defined by positive definite
matrix P centered at x^*

Convex Cones

Definition

A cone C in $\mathbb{R}^n$ is a set with the property

$$x \in C \;\Rightarrow r\, x \in C \quad \text{for all } r > 0.$$

A cone is called pointed, if it does not contain a whole line.

A convex cone is a cone that is a convex set.

Proposition

A cone C is convex if and only if

$$C = C + C\,.$$

Convex Cones

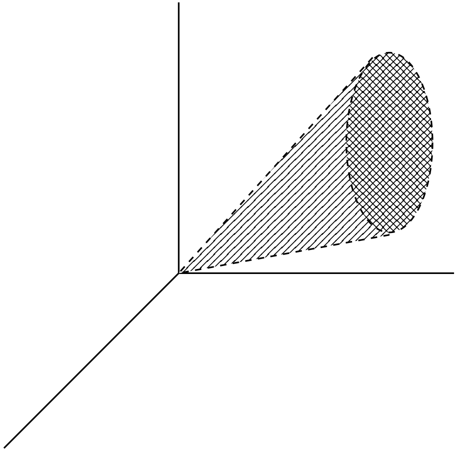


Figure: Ice cream cone

Lorentz cone

Definition

The Lorentz cone in $\mathbb{R}^n$ is defined by

$$L^n := \left\{ x \in \mathbb{R}^n \mid x_n \geq \sqrt{x_1^2 + \ldots + x_{n-1}^2} \right\},$$

Convex Cones

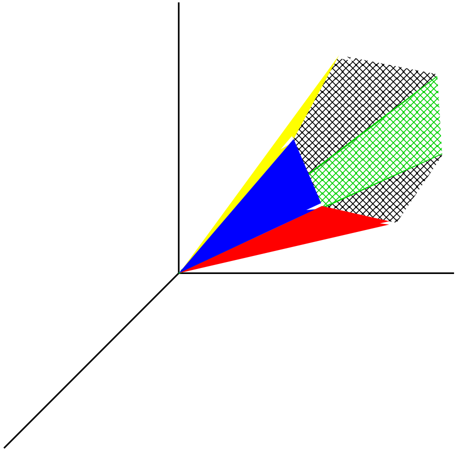


Figure: Polytopic cone

Operations on Convex Sets

The following operations result in convex sets, **if** the original sets are convex

- Intersection
- scalar multiplication
- Minkowski sum
- linear/affine transformations

Operations on Convex Sets

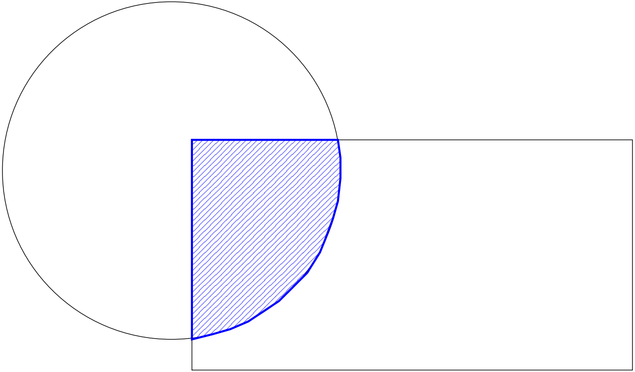


Figure: The intersection of two convex sets is convex.

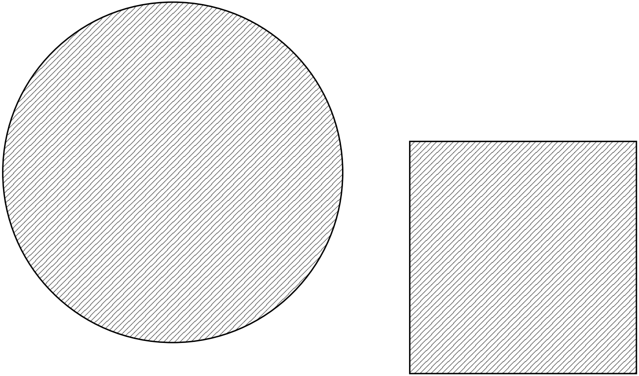


Figure: The union of two convex sets need not be convex.

The Minkowski Sum

Definition

The sum of two sets $C_1, C_2 \subset \mathbb{R}^n$ is defined by

$$C_1 + C_2 := \{x + y \mid x \in C_1, y \in C_2\}$$

The Minkowski Sum

Definition

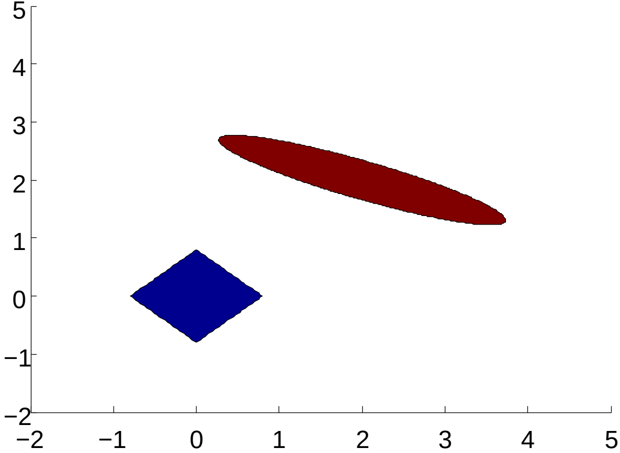
The sum of two sets $C_1, C_2 \subset \mathbb{R}^n$ is defined by

$$C_1 + C_2 := \{x + y \mid x \in C_1, y \in C_2\}$$

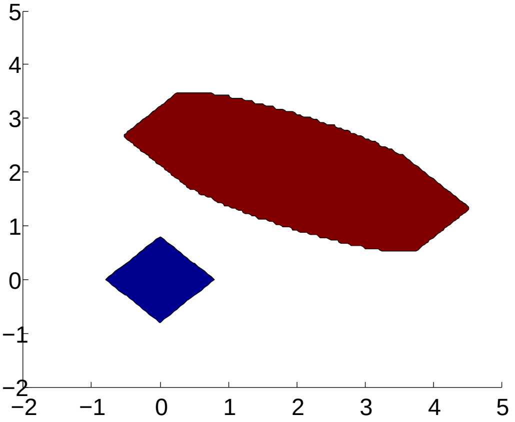
Proposition

If $C_1, C_2 \subset \mathbb{R}^n$ are convex, then $C_1 + C_2$ is convex.

The Minkowski Sum



The Minkowski Sum



Affine transformations

Proposition

Let $A \in \mathbb{R}^{n\times m}, b \in \mathbb{R}^n$.

(i) If $C \subset \mathbb{R}^m$ is convex then the image of C under the map $x \mapsto Ax + b$ is convex, i.e.

$$AC + b := \{Ax + b \mid x \in C\} \quad \text{is convex}\,.$$

(ii) If $D \subset \mathbb{R}^n$ is convex, then the preimage of D under the map $x \mapsto Ax + b$ is convex, i.e.

$$A^{-1}(D - b) := \{x \in \mathbb{R}^m \mid \exists y \in D \text{ such that } Ax + b = y\} \quad \text{is convex}\,.$$

Definition

Let $M \subset \mathbb{R}^n$. The convex hull of M, denoted by $\operatorname{conv} M$ is the smallest convex set containing M.

The Convex Hull

Definition

Let $M \subset \mathbb{R}^n$. The convex hull of M, denoted by conv M is the smallest convex set containing M.

The right question here is: What do you mean by "smallest"?

The convex hull is characterized by the following properties:

Proposition

$$\operatorname{conv} M = \bigcap_{M \subset C, C \text{ convex}} C$$

$$\operatorname{conv} M = \left\{ \sum_{i=1}^{m} \alpha_i x_i \mid m \in \mathbb{N}, x_i \in M, \alpha_i \geq 0, i = 1, \ldots, m, \sum_{1=1}^{m} \alpha_i = 1 \right\}$$

Convex Hull

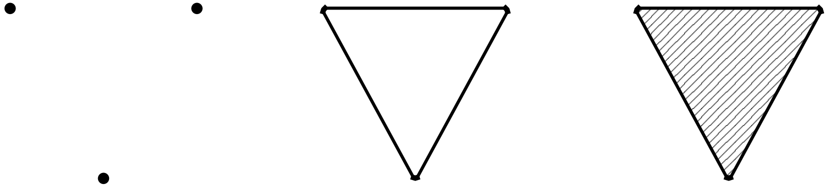


Figure: convex hull

Convex Hull

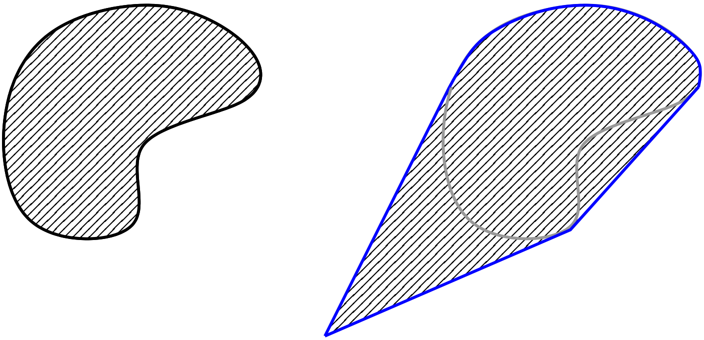


Figure: convex hull

Convex Combinations

Given $x_1, \ldots, x_m$, $\alpha_1, \ldots, \alpha_m \geq 0$ with

$$\sum_{i=1}^{m} \alpha_i = 1$$

the corresponding convex combination is given by

$$\sum_{i=1}^{m} \alpha_i x_i \,.$$

Convex Hull: A Second Look

Proposition

$$\operatorname{conv} M = \left\{ \sum_{i=1}^{m} \alpha_i x_i \mid x_i \in M, \alpha_i \geq 0, \sum \alpha_i = 1 \right\}$$

The convex hull is the set of all possible convex combinations of elements of M.

Carathéodory's Theorem

Theorem

Let $X \subset \mathbb{R}^n$.
For every $x \in \operatorname{conv} X$ there exist $x_0, \ldots, x_n \in X$, such that

$$x = \sum_{k=0}^{n} \alpha_k x_k \,, \text{ where } \quad \alpha_k \geq 0, \; \sum_{k=0}^{n} \alpha_k = 1 \,.$$

Carathéodory's Theorem

Theorem

Let $X \subset \mathbb{R}^n$.
For every $x \in \operatorname{conv} X$ there exist $x_0, \ldots, x_n \in X$, such that

$$x = \sum_{k=0}^{n} \alpha_k x_k \,, \text{ where } \quad \alpha_k \geq 0, \; \sum_{k=0}^{n} \alpha_k = 1 \,.$$

Note: The dimension of the space $\mathbb{R}^n$ appears again in the necessary length of a linear combination.
In n dimensions, for a given point x in the convex hull we need $n+1$ points in X to construct x as a convex combination.

Application: The Convex Hull of Compact Sets is Compact

Theorem

Let $M \subset \mathbb{R}^n$ be compact. Then

$$\operatorname{conv} M \qquad \text{is compact} \ .$$

Application: The Convex Hull of Compact Sets is Compact

Theorem

Let $M \subset \mathbb{R}^n$ be compact. Then

$$\operatorname{conv} M \quad \text{is compact} .$$

Proof: The set

$$M^{n+1} \times \{(\alpha_0, \dots, \alpha_n) \mid \alpha_i \geq 0, \sum_{i=0}^{n} \alpha_i = 1\}$$

is compact. The map

$$(x_0, \dots, x_n, \alpha_0, \dots, \alpha_n) \mapsto \sum_{i=0}^{n} \alpha_i x_i$$

is continuous.

By Carathéodory's theorem conv M is the image of a compact set under a continuous map.

QED

Attention for unbounded sets

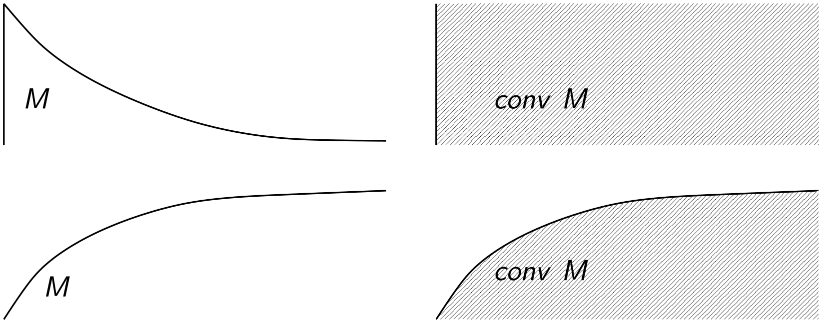


Figure: The convex hull of a closed set need not be closed

Dimension and Relative Interior

Definition

The dimension of a convex set $C \subset \mathbb{R}^n$ is the dimension of the smallest affine space in $\mathbb{R}^n$ containing C.

Recall: The intersection of two affine spaces is an affine space. So there really is a "smallest" affine space containing C.

Dimension and Relative Interior

Definition

The dimension of a convex set $C \subset \mathbb{R}^n$ is the dimension of the smallest affine space in $\mathbb{R}^n$ containing C.

Recall: The intersection of two affine spaces is an affine space. So there really is a "smallest" affine space containing C.

Definition

The relative interior of a convex set $C \subset \mathbb{R}^n$ is the interior of C with respect to the smallest affine space containing C.
The relative interior is denoted by $\mathrm{ri}\, C$.

This means

$$x \in \mathrm{ri}\, C :, \quad \Leftrightarrow \quad \exists \varepsilon > 0 \text{ such that } B_\epsilon(x) \cap \mathrm{aff}\, C \subset C\,.$$

The Affine Hull

Let K be a convex set. The smallest affine set containing K is described by

$$\operatorname{aff} K = \left\{ \sum_{i=1}^{k} \alpha_i x_i \mid x_i \in K, \alpha_i \in \mathbb{R}, \sum_{i=1}^{k} \alpha_i = 1, k = 1, 2, \dots \right\}.$$

Note the difference to relative interior in set topology

In set topology, the relative topology of a subset M of $\mathbb{R}^n$ is defined by the intersection of M with the open sets in $\mathbb{R}^n$.
So given $M \subset \mathbb{R}^n$ the relative interior of M with respect to the relative topology of M is M itself because by definition

$$M = M \cap \mathbb{R}^n$$

is open in M by definition. This is of course very boring.
The key idea for convex sets is that there is a natural set given by C, the affine hull $\mathrm{aff}\, C$. And we always consider the relative interior with respect to the topology in $\mathrm{aff}\, C$.

Relative interior

- The relative interior of a point is the point itself.
- The relative interior of a line segment is the line segment without the end points.

Relative interior

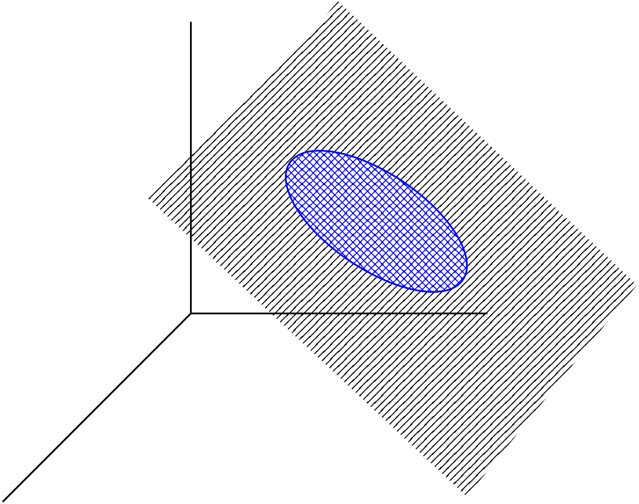


Figure: Relative interior of a 2-dimensional convex set in $\mathbb{R}^3$.