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Exercises
Convex Optimization and Congestion Control

1. **Separating hyperplanes between convex polytopes**

   Consider the two nonempty convex polytopes

   $$P_1 = \{x \in \mathbb{R}^n \mid Ax \leq b\}, \qquad P_2 = \{x \in \mathbb{R}^n \mid Cx \leq d\}.$$

   Assume that $P_1 \cap P_2 = \emptyset$.

   Formulate the problem of finding a separating hyperplane as a linear program or a linear feasibility problem (a linear program with cost function equal to 0). The problem is to find a vector $q \in \mathbb{R}^n$ and a scalar $c \in \mathbb{R}$ such that

   $$\langle q, x \rangle > c, \quad \text{for all } x \in P_1 \quad \text{and} \quad \langle q, x \rangle < c, \quad \text{for all } x \in P_2.$$

   Hint: The vector and scalar must satisfy

   $$\inf_{x \in P_1} q^\top x > c > \sup_{x \in P_2} q^\top x.$$

   Use duality to simplify infimum and supremum in these conditions.

2. **A convex problem without strong duality**

   Consider the optimization problem

   $$\begin{aligned} \text{minimize} \quad & e^{-x} \\ \text{subject to} \quad & x^2/y \leq 0. \end{aligned}$$

   with variables $x, y \in \mathbb{R}$ and domain $\mathcal{D} = \{(x, y) \mid y > 0\}$

   (a) Show that this is a convex optimization problem and determine the optimal value.
   (b) Determine the Lagrange dual problem and find the optimal solution λ^* and the optimal value d^* of the dual problem. What is the duality gap?
   (c) Does Slater's condition hold?

3. **Square Linear Programs**

   Consider the linear optimization problem

   $$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \leq b, \end{aligned}$$

   with A square and invertible. Show that the optimal value is given by

   $$p^* = \begin{cases} c^\top A^{-1} b & A^{-\top} c \leq 0 \\ -\infty & \text{else}. \end{cases}$$