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Exercises
Convex Optimization and Congestion Control

1. **Voronoi Sets and Polyhedral Decomposition**

Let $x_0, \ldots, x_k \in \mathbb{R}^n$. The set of points closer to x_0 than to any other of the points x_k is given by

$$V := \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq \|x - x_i\|, i = 1, \ldots, K\}.$$

This is called the Voronoi region around x (w.r.t $x_1, \ldots, x_K$).

(a) Show that V is a convex polytope. Express V in the form $Ax \leq b$, where $A \in \mathbb{R}^{K \times n}, b \in \mathbb{R}^K$ and the inequality has to be understood component-wise.

(b) Given a convex polytope P with nonempty interior, show how to find $x_0, x_1, \ldots, x_k$ such that P is the Voronoi region around x_0.

(c) Extending the definition, we can define

$$V_k = \{x \in \mathbb{R}^n \mid \|x - x_k\| \leq \|x - x_i\|, i \neq k\}.$$

Show that the V_k are a polytopic decomposition of $\mathbb{R}^n$. I.e., the union covers $\mathbb{R}^n$ and $\operatorname{int} V_i \cap \operatorname{int} V_k = \emptyset$ for $i \neq k$.

Can every polytopic decomposition of this type be described by Voronoi regions?

2. **Convexity**

(a) Show that $A \mapsto \lambda_{\max}(A)$ is a convex function on the space of symmetric matrices.

(b) For a symmetric matrix A we will assume that the eigenvalues are listed in their order and with repeated multiplicities, i.e.

$$\sigma(A) = \{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}.$$

With this convention show that

$$\sum_{i=1}^{k} \lambda_i(A)$$

is convex on the space of symmetric matrices. Use the characterization

$$\sum_{i=1}^{k} \lambda_i(A) = \sup\{\operatorname{trace}(V^\top A V) \mid V \in \mathbb{R}^{n \times k}, V^\top V = I\}$$

3. **Products of Convex/Concave Functions**

In general the product of two convex functions is not convex. Give an example showing this. Prove that the following is true nonetheless:

Assume $f, g : [a, b] \subset R \to R$.

(a) If f, g are convex, both nonincreasing (nondecreasing) and positive, then fg is convex.

(b) If f, g are concave and positive and one is nonincreasing, while the other is nondecreasing, then fg is concave.

(c) If f is convex, positive, nondecreasing and g is concave, nonincreasing and positive, then f/g is convex.