Sample path large deviations of Poisson shot noise with heavy tail semi-exponential distributions

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Abstract

It is shown that the sample paths of Poisson shot noise with heavy-tailed semi-exponential distributions satisfy a large deviation principle with a rate function that is insensitive to the shot shape. This demonstrates that, on the scale of large deviations, paths to rare events do not depend on the shot shape.

Keywords: Heavy tail distributions; sample path large deviations; Poisson shot noise.

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1 Introduction

Shot noise processes have an extensive range of applications from Physics [10], through Electrical Engineering [9] and Queueing Theory [8, 4]. They have also been used, for example, in Risk Theory to model the delay in claim settlement [7, 5]. It has recently been shown that Poisson Shot Noise (PSN) with i.i.d. heavy tailed semi-exponential shot values satisfies a scalar Large Deviation Principle (LDP) with a rate function that is insensitive to the shot shape [11]. In this note we extend this result proving that a sample path LDP holds for this process and, again, the resulting rate function is insensitive to the shot shape. The insensitivity manifests itself through the LDP having the same rate function as for a compound Poisson process with similarly distributed increments. Thus, on the scale of large deviations, the paths to rare events do not depend on the shot shape. The main result of this note can be viewed as the heavy-tailed counterpart of the sample path LDP for PSN under light tail conditions [4].

Our proof is inspired by Gantert's work on the centered partial sums of i.i.d. heavy tailed semi-exponential distributions [6]. The main novel difficulties stem from a lack of independent increments in PSN. These are overcome using the regenerative properties of the Poisson process in conjunction with delicate estimates to create a process with independent increments that is exponentially equivalent to that under study.

PSN is the following process:

$$S(t) = \sum_{n=1}^{N(t)} H(t - T_n, Z_n), \quad t > 0$$

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where $\{N(t)\}_{t>0}$ is a homogeneous Poisson process on $(0,\infty)$ with intensity $\lambda > 0$, $\{T_n\}_{n\geq 1}$ are the points of the Poisson process and $\{Z_n\}_{n\geq 1}$ form a sequence of i.i.d. random variables taking values in a measurable space $(E, \mathcal{E})$. The *shot shape* $H : \mathbb{R} \times E \to [0,\infty)$ is assumed measurable and is such that $H(t,z) = 0$ for $t \leq 0$, and, for any $z \in E$, $H(t,z)$ is non-decreasing with respect to t. Throughout this paper we assume that the sequences $\{T_n\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 1}$ are independent, and denote by $H(\infty, z)$ the *shot value*, i.e. the limit of $H(t,z)$, as $t \to \infty$.

2 Sample path large deviations

We shall prove that the sample paths of PSN satisfy the LDP in $D[0,1]$, the space of càdlàg functions defined on $[0,1]$, equipped with the L_1 topology induced by the norm $||f|| = \int_0^1 |f(t)|dt$. The idea is first to prove an LDP for the finite-dimensional distributions of the process and then lift this LDP to a principle for the process in $D[0,1]$ equipped with the topology of point-wise convergence using the Dawson-Gärtner theorem (see Theorem 4.6.1 in [3]). Finally, one strengthens to the L_1 topology by demonstrating exponential tightness and establishing the upper and lower LDP bounds.

We begin by introducing basic definitions and recalling the scalar LDP for PSN with heavy tail semi-exponential distributions as proved in [11]. We say that a family of random variables $\{V_\alpha\}_{\alpha>0}$ taking values in a topological space (M,τ) obeys an LDP with rate function I and speed $v : [0,\infty) \mapsto [0,\infty)$ if $I : M \mapsto [0,\infty]$ is a lower semi-continuous function, v is a measurable function such that $v(\alpha) \to \infty$ as $\alpha \to \infty$, and the following inequalities hold:

$$\limsup_{\alpha\to\infty} \frac{1}{v(\alpha)} \log P(V_\alpha \in C) \leq - \inf_{x\in C} I(x), \quad \text{for all } C \text{ closed}$$

and

$$\liminf_{\alpha\to\infty} \frac{1}{v(\alpha)} \log P(V_\alpha \in O) \geq - \inf_{x\in O} I(x), \quad \text{for all } O \text{ open.}$$

Lower semi-continuity of I means that its level sets, $\{x \in M : I(x) \leq c\}$ for $c \geq 0$, are closed. If the level sets are compact, the rate function I is said to be good. The reader is referred to [3] for an introduction to large deviations theory.

We write $f(x) \sim g(x)$ if f and g are two non-negative functions such that $f(x)/g(x) \to 1$ as $x \to \infty$ and for a non-negative r.v., X, we define $\overline{F}(x) = P(X > x)$, $x \geq 0$. Let $r \in (0,1)$ be a constant. We say that $\overline{F}$ or X is heavy tail semi-exponential if $\overline{F}(x) \sim a(x)\exp\{-x^r L(x)\}$, where a and L are non-negative slowly varying functions, i.e. $\lim_{x\to\infty} L(tx)/L(x) = 1$ for all $t > 0$ and the same holds for a. As is well-known a semi-exponential r.v. X has finite moments of all orders, but $\mathrm{E}[e^{\theta X}] = \infty$ for all $\theta > 0$. See, for example, [2] for an introduction to semi-exponential distributions.

Theorem 2.1 *[Proposition 2.1 [11]] Let a and L be positive slowly varying functions, $r \in (0,1)$ a positive constant, and define $\beta = \lambda\mathrm{E}[H(\infty, Z)]$. If*

$$\liminf_{t\to\infty} \frac{1}{t^r L(t)} \log P(H(t,Z) \geq bt) \geq -b^r \quad \textit{for all } b > 0$$

and

$$P(H(\infty, Z) \geq t) \leq a(t)\exp(-t^r L(t)) \quad \textit{for all } t \textit{ sufficiently large,}$$

then $\{S(t)/t\}_{t>0}$ *obeys an LDP in* $\mathbb{R}$ *with speed* $t^r L(t)$ *and good, non-convex rate function*

$$I^{(\beta)}(x) = \begin{cases} (x-\beta)^r & \text{if } x \geq \beta \\ \infty & \text{if } x < \beta. \end{cases} \tag{1}$$

Using this scalar LDP we first prove that the LDP holds for finite-dimensional distributions. In doing so, we encounter the primary difficulty when compared to partial sums processes: the increments of PSN are not independent. This is overcome by the construction of an exponentially equivalent process with independent increments.

Theorem 2.2 *Under the assumptions of Theorem 2.1, for any integer* $k \geq 1$ *and real numbers* $0 = t_0 < t_1 < t_2 < \ldots < t_k \leq 1$*, the family of random vectors* $\{S(\alpha t_1)/\alpha, \ldots, S(\alpha t_k)/\alpha\}_{\alpha>0}$ *satisfies an LDP in* $\mathbb{R}^k$ *with speed* $\alpha^r L(\alpha)$ *and good rate function*

$$I^{(\beta)}_{t_1,\ldots,t_k}(x_1,\ldots,x_k) = \sum_{i=1}^{k} (t_i - t_{i-1})^r I^{(\beta)}\left(\frac{x_i - x_{i-1}}{t_i - t_{i-1}}\right),$$

where $x_0 = 0$ *and the function* $I^{(\beta)}$ *is defined in* (1).

Proof: We divide the proof in 5 steps.
Step 1: an approximation with independent increments. Let $0 = t_0 < t_1 < t_2 < \ldots < t_k \leq 1$ be fixed. For $i = 1, \ldots, k$, let $\{N^{(i)}(t)\}_{t>0}$ be i.i.d. copies of the Poisson process $\{N(t)\}_{t>0}$ and $\{Z_n^{(i)}\}_{n\geq 1}$ be i.i.d. copies of the process $\{Z_n\}_{n\geq 1}$ that are independent of the Poisson processes $\{N^{(i)}(t)\}$. For $0 < s \leq t \leq 1$ and $i = 1, \ldots, k$, we define

$$S^{(i)}(s,t) = \sum_{n=1}^{N^{(i)}(s)} H(t - T_n^{(i)}, Z_n^{(i)}).$$

By the regenerative property of the Poisson process and the i.i.d. property of the sequence $\{Z_n\}_{n\geq 1}$, the following equality in distribution holds:

$$\begin{aligned}(S(t_1), S(t_2), \ldots, S(t_k)) \stackrel{d}{=} (&S^{(1)}(t_1,t_1), S^{(1)}(t_1,t_2) + S^{(2)}(t_2-t_1, t_2-t_1), \ldots, \\ &S^{(1)}(t_1,t_k) + S^{(2)}(t_2-t_1, t_k-t_1) + \ldots + S^{(k)}(t_k - t_{k-1}, t_k - t_{k-1})).\end{aligned}$$

For $\alpha > 0$, set

$$\Sigma^{(1)}(\alpha, t_1, \ldots, t_k) = \alpha^{-1}\left(S(\alpha t_1), S(\alpha t_2) - S(\alpha t_1), \ldots, S(\alpha t_k) - S(\alpha t_{k-1})\right),$$

$$\begin{aligned}\Sigma^{(2)}(\alpha, t_1, \ldots, t_k) = \alpha^{-1}\Bigg(&S^{(1)}(\alpha t_1, \alpha t_1), S^{(2)}(\alpha(t_2-t_1), \alpha(t_2-t_1)) + \left[S^{(1)}(\alpha t_1, \alpha t_2) - S^{(1)}(\alpha t_1, \alpha t_1)\right], \\ &\ldots, S^{(k)}(\alpha(t_k - t_{k-1}), \alpha(t_k - t_{k-1})) + \\ &\sum_{i=1}^{k-1}\left[S^{(i)}(\alpha(t_i - t_{i-1}), \alpha(t_k - t_{i-1})) - S^{(i)}(\alpha(t_i - t_{i-1}), \alpha(t_{k-1} - t_{i-1}))\right]\Bigg)\end{aligned}$$

and

$$\Sigma^{(3)}(\alpha, t_1, \ldots, t_k) = \alpha^{-1}\left(S^{(1)}(\alpha t_1, \alpha t_1), S^{(2)}(\alpha(t_2-t_1), \alpha(t_2-t_1)), \ldots, S^{(k)}(\alpha(t_k - t_{k-1}), \alpha(t_k - t_{k-1}))\right).$$

Step 2: exponential equivalence. Next we shall show that the families of random vectors $\{\Sigma^{(2)}(\alpha, t_1, \ldots, t_k)\}_{\alpha>0}$ and $\{\Sigma^{(3)}(\alpha, t_1, \ldots, t_k)\}_{\alpha>0}$ are exponentially equivalent at the speed $\alpha^r L(\alpha)$. This claim follows if we can prove that for fixed $s < t$ the following holds

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^r L(\alpha)} \log P\left(\sum_{n=1}^{N(\alpha s)} [H(\alpha t - T_n, Z_n) - H(\alpha s - T_n, Z_n)] > \alpha\delta\right) = -\infty, \quad \forall\, \delta > 0.$$

By the Chernoff bound we have that, for any $\theta > 0$,

$$\begin{aligned}
&P\left(\sum_{n=1}^{N(\alpha s)} [H(\alpha t - T_n, Z_n) - H(\alpha s - T_n, Z_n)] > \alpha\delta\right) \\
&\leq \mathrm{e}^{-\theta\alpha\delta}\mathrm{E}\left[\mathrm{e}^{\theta\sum_{n=1}^{N(\alpha s)}[H(\alpha t - T_n, Z_n) - H(\alpha s - T_n, Z_n)]}\right] \\
&= \mathrm{e}^{-\theta\alpha\delta}\exp\left(\lambda\int_0^{\alpha s} \mathrm{E}\left[\mathrm{e}^{\theta(H(\alpha t - u, Z_1) - H(\alpha s - u, Z_1))} - 1\right]\,\mathrm{d}u\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{\alpha^r L(\alpha)} \log P\left(\sum_{n=1}^{N(\alpha s)} [H(\alpha t - T_n, Z_n) - H(\alpha s - T_n, Z_n)] > \alpha\delta\right) \\
&\leq -\frac{\theta\delta}{(L(\alpha)/\alpha^{1-r})} + \frac{\lambda}{\alpha^r L(\alpha)}\int_0^{\alpha s} \mathrm{E}\left[\mathrm{e}^{\theta(H(\alpha t - u, Z_1) - H(\alpha s - u, Z_1))} - 1\right]\,\mathrm{d}u.
\end{aligned}$$

Let $y > \beta$ be arbitrarily fixed. In Step 5 we shall show that if we take $\theta = d\alpha^{r-1}L(\alpha)$, with $0 < d < (y-\beta)^{r-1}$, then

$$\lim_{\alpha\to\infty} \frac{\lambda}{\alpha^r L(\alpha)}\int_0^{\alpha s} \mathrm{E}\left[\mathrm{e}^{\theta(H(\alpha t - u, Z_1) - H(\alpha s - u, Z_1))} - 1\right]\,\mathrm{d}u = 0, \tag{2}$$

so that

$$\limsup_{\alpha\to\infty} \frac{1}{\alpha^r L(\alpha)} \log P\left(\sum_{n=1}^{N(\alpha s)} [H(\alpha t - T_n, Z_n) - H(\alpha s - T_n, Z_n)] > \alpha\delta\right) \leq -d\delta$$

and the claim follows by letting first d tend to $(y-\beta)^{r-1}$ and then y tend to β.

Step 3: large deviations for the family $\{\Sigma^{(3)}(\alpha, t_1, \ldots, t_k)\}_{\alpha>0}$. By Theorem 2.1 and the definition of slowly varying function we have that, for any fixed $t > 0$, the stochastic process $\{S(\alpha t)/(\alpha t)\}_{\alpha>0}$ obeys an LDP on $\mathbb{R}$ with speed $\alpha^r L(\alpha)$ and good rate function $t^r I^{(\beta)}(x)$. Using the contraction principle (Theorem 4.2.1 in [3]) we have that $\{S(\alpha t)/\alpha\}_{\alpha>0}$ obeys an LDP on $\mathbb{R}$ with speed $\alpha^r L(\alpha)$ and good rate function $t^r I^{(\beta)}(x/t)$. Due to the independence of the processes $\{S^{(i)}(t,t)\}_{t>0}$ $(i = 1, \ldots, k)$, Exercise 4.2.7 in [3] yields that $\{\Sigma^{(3)}(\alpha, t_1, \ldots, t_k)\}_{\alpha>0}$ obeys an LDP on $\mathbb{R}^k$ with speed $\alpha^r L(\alpha)$ and good rate function

$$\widetilde{I}^{(\beta)}_{t_1,\ldots,t_k}(x_1, \ldots, x_k) = \sum_{i=1}^{k} (t_i - t_{i-1})^r I^{(\beta)}\left(\frac{x_i}{t_i - t_{i-1}}\right). \tag{3}$$

Step 4: conclusion of the proof. By construction $\Sigma^{(1)}(\alpha,t_1,\ldots,t_k)\overset{d}{=}\Sigma^{(2)}(\alpha,t_1,\ldots,t_k)$, for all $\alpha>0$. Combining Steps 2 and 3 with Theorem 4.2.13 in [3], we deduce that the family $\{\Sigma^{(1)}(\alpha,t_1,\ldots,t_k)\}_{\alpha>0}$ obeys an LDP on $\mathbb{R}^k$ with speed $\alpha^r L(\alpha)$ and good rate function $\widetilde{I}^{(\beta)}_{t_1,\ldots,t_k}$ defined in equation (3). The claim follows by an application of the contraction principle with the function $(x_1,\ldots,x_k)\mapsto(x_1,x_1+x_2,\ldots,x_1+\cdots+x_k)$.

Step 5: proof of equation (2). All that remains is the establishment of the assertion in equation (2). Let $k\geq 1$ be an integer such that $r<k/(k+1)$, which exists as $r\in(0,1)$. By the inequality

$$\mathrm{e}^x-1\leq x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^{k+1}}{(k+1)!}\,\mathrm{e}^x,\quad\forall x\geq 0$$

we have, $\forall\ \alpha>0$, $t\geq s\geq 0$ and $0\leq u\leq\alpha s$,

$$\begin{aligned}&\frac{1}{\alpha^r L(\alpha)}\mathrm{E}[\mathrm{e}^{d\alpha^{r-1}L(\alpha)(H(\alpha t-u,Z_1)-H(\alpha s-u,Z_1))}-1]\\&\leq\frac{d}{\alpha}\mathrm{E}[H(\alpha t-u,Z_1)-H(\alpha s-u,Z_1)]\\&\quad+\frac{1}{2}\,d^2\alpha^{r-2}L(\alpha)\mathrm{E}[H^2(\alpha t,Z_1)]+\ldots+\frac{1}{k!}\,d^k\alpha^{(k-1)r-k}L^{k-1}(\alpha)\mathrm{E}[H^k(\alpha t,Z_1)]\\&\quad+\frac{1}{(k+1)!}\,d^{k+1}\alpha^{kr-(k+1)}L^k(\alpha)\mathrm{E}[H^{k+1}(\alpha t,Z_1)\mathrm{e}^{d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}].\end{aligned}$$

So

$$\begin{aligned}&\frac{1}{\alpha^r L(\alpha)}\int_0^{\alpha s}\mathrm{E}[\mathrm{e}^{d\alpha^{r-1}L(\alpha)(H(\alpha t-u,Z_1)-H(\alpha s-u,Z_1))}-1]\,\mathrm{d}u\\&\leq\frac{d}{\alpha}\int_0^{\alpha s}\mathrm{E}[H(\alpha t-u,Z_1)-H(\alpha s-u,Z_1)]\,\mathrm{d}u\\&\quad+\frac{s}{2}\,d^2\alpha^{r-1}L(\alpha)\mathrm{E}[H^2(\alpha t,Z_1)]+\ldots+\frac{s}{k!}\,d^k\alpha^{(k-1)(r-1)}L^{k-1}(\alpha)\mathrm{E}[H^k(\alpha t,Z_1)]\\&\quad+\frac{s}{(k+1)!}\,d^{k+1}\alpha^{k(r-1)}L^k(\alpha)\mathrm{E}[H^{k+1}(\alpha t,Z_1)\mathrm{e}^{d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}].\end{aligned}$$

By the assumption on the distribution of $H(\infty,Z_1)$ we have $\mathrm{E}[H^n(\infty,Z_1)]<\infty$ for any $n\geq 1$, so that all the terms in the third line of the above inequality go to zero as $\alpha\to\infty$. By a change of variable we deduce

$$\frac{d}{\alpha}\int_0^{\alpha s}\mathrm{E}[H(\alpha t-u,Z_1)-H(\alpha s-u,Z_1)]\,\mathrm{d}u=ds\int_0^1\mathrm{E}[H(\alpha(t-zs),Z_1)-H(\alpha s(1-z),Z_1)]\,\mathrm{d}z$$

and this latter term goes to zero as $\alpha\to\infty$ by the dominated convergence theorem. Therefore, we only need to prove

$$\lim_{\alpha\to\infty}\alpha^{k(r-1)}L^k(\alpha)\mathrm{E}[H^{k+1}(\alpha t,Z_1)\mathrm{e}^{d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}]=0.$$

Note that for an arbitrary fixed $T>0$

$$\begin{aligned}&\mathrm{E}[H^{k+1}(\alpha t,Z_1)\mathrm{e}^{d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}]\\&=\mathrm{E}[H^{k+1}(\alpha t,Z_1)\mathrm{e}^{d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}\mathbf{1}\{H(\alpha t,Z_1)<T\}]\\&\qquad+\mathrm{E}[H^{k+1}(\alpha t,Z_1)\mathrm{e}^{d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}\mathbf{1}\{H(\alpha t,Z_1)\geq T\}]\\&\leq T^{k+1}\mathrm{e}^{d\alpha^{r-1}L(\alpha)T}+\mathrm{E}[H^{k+1}(\alpha t,Z_1)\mathrm{e}^{d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}\mathbf{1}\{H(\alpha t,Z_1)\geq T\}].\end{aligned}$$

Since $\lim_{\alpha\to\infty} \alpha^{r-1}L(\alpha) = 0$, the claim follows if we prove

$$\lim_{\alpha\to\infty} \alpha^{k(r-1)}L^k(\alpha)\mathrm{E}[H^{k+1}(\alpha t, Z_1)\mathrm{e}^{d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}\mathbf{1}\{H(\alpha t, Z_1) \geq T\}] = 0. \tag{4}$$

By the choice of $k \geq 1$ we have that $k(r-1) + r(1+\varepsilon)^{-1} < 0$, for all $\varepsilon > 0$. An application of Hölder's inequality with conjugate exponents $(1+\varepsilon)/\varepsilon$ and $1+\varepsilon$ yields

$$\begin{aligned}
&\mathrm{E}[H^{k+1}(\alpha t, Z_1)\mathrm{e}^{d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}\mathbf{1}\{H(\alpha t, Z_1) \geq T\}] \\
&\leq (\mathrm{E}[H^{(k+1)(1+\varepsilon)/\varepsilon}(\alpha t, Z_1)\mathbf{1}\{H(\alpha t, Z_1) \geq T\}])^{\varepsilon/(1+\varepsilon)} \\
&\qquad \times (\mathrm{E}[\mathrm{e}^{(1+\varepsilon)d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}\mathbf{1}\{H(\alpha t, Z_1) \geq T\}])^{1/(1+\varepsilon)}.
\end{aligned}$$

Note that

$$(\mathrm{E}[H^{(k+1)(1+\varepsilon)/\varepsilon}(\alpha t, Z_1)\mathbf{1}\{H(\alpha t, Z_1) \geq T\}])^{\varepsilon/(1+\varepsilon)} \leq (\mathrm{E}[H^{(k+1)(1+\varepsilon)/\varepsilon}(\infty, Z_1)])^{\varepsilon/(1+\varepsilon)} \in (0,\infty).$$

Thus (4) follows if we show

$$\lim_{\alpha\to\infty} \alpha^{k(r-1)}L^k(\alpha)(\mathrm{E}[\mathrm{e}^{(1+\varepsilon)d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}\mathbf{1}\{H(\alpha t, Z_1) \geq T\}])^{1/(1+\varepsilon)} = 0.$$

This in turn follows if

$$\limsup_{\alpha\to\infty} \frac{1}{\alpha^r L(\alpha)}\mathrm{E}[\mathrm{e}^{(1+\varepsilon)d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}\mathbf{1}\{H(\alpha t, Z_1) \geq T\}] < \infty. \tag{5}$$

Indeed (5) gives, for all α large enough and a positive constant $K_1 > 0$,

$$(\mathrm{E}[\mathrm{e}^{(1+\varepsilon)d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}\mathbf{1}\{H(\alpha t, Z_1) \geq T\}])^{1/(1+\varepsilon)} \leq K_1\alpha^{r/(1+\varepsilon)}L(\alpha)^{1/(1+\varepsilon)}.$$

Then

$$\begin{aligned}
&\limsup_{\alpha\to\infty} \alpha^{k(r-1)}L^k(\alpha)(\mathrm{E}[\mathrm{e}^{(1+\varepsilon)d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}\mathbf{1}\{H(\alpha t, Z_1) \geq T\}])^{(1+\varepsilon)^{-1}} \\
&\leq K_1 \lim_{\alpha\to\infty} \alpha^{k(r-1)+r(1+\varepsilon)^{-1}}L^{k+(1+\varepsilon)^{-1}}(\alpha) = 0,
\end{aligned}$$

where the latter equality follows because $k(r-1) + r(1+\varepsilon)^{-1} < 0$ and L is slowly varying. In the remainder of the proof we establish the veracity of equation (5). Note that, if X is a non-negative r.v., $z > 0$ and $0 < U < \infty$, we have

$$\mathrm{E}[\mathrm{e}^{zX}\mathbf{1}\{X \geq U\}] \leq \int_U^\infty z\mathrm{e}^{zs}P(X > s)\,\mathrm{d}s + \mathrm{e}^{zU}P(X \geq U).$$

Then, for all α big enough,

$$\begin{aligned}
&\frac{\mathrm{E}[\mathrm{e}^{(1+\varepsilon)d\alpha^{r-1}L(\alpha)H(\alpha t,Z_1)}\mathbf{1}\{H(\alpha t, Z_1) \geq T\}]}{\alpha^r L(\alpha)} \\
&\leq \frac{(1+\varepsilon)d}{\alpha}\int_T^\infty \mathrm{e}^{(1+\varepsilon)d\alpha^{r-1}L(\alpha)s}\, P(H(\infty, Z_1) > s)\,\mathrm{d}s + \frac{\mathrm{e}^{(1+\varepsilon)d\alpha^{r-1}L(\alpha)T}}{\alpha^r L(\alpha)}.
\end{aligned}$$

Therefore, for (5) it suffices to check

$$\limsup_{\alpha\to\infty} \int_T^\infty \frac{\mathrm{e}^{K_2\alpha^{r-1}L(\alpha)s}}{\alpha} P(H(\infty, Z_1) > s)\,\mathrm{d}s < \infty, \quad \text{for some } T > 0. \tag{6}$$

where we set $K_2 = (1+\varepsilon)d$. Note that the sequence

$$\limsup_{\alpha\to\infty} \int_T^M \frac{\mathrm{e}^{K_2\alpha^{r-1}L(\alpha)s}}{\alpha} P(H(\infty, Z_1) > s)\,\mathrm{d}s, \quad M \geq 1$$

is non-decreasing with supremum

$$\limsup_{\alpha\to\infty} \int_T^\infty \frac{\mathrm{e}^{K_2\alpha^{r-1}L(\alpha)s}}{\alpha} P(H(\infty, Z_1) > s)\,\mathrm{d}s.$$

Now, let $M \geq 1$ and $y > \beta$ be arbitrarily fixed and note that for all $\alpha > M/(y-\beta)$

$$\begin{aligned}\int_T^M \frac{\mathrm{e}^{K_2\alpha^{r-1}L(\alpha)s}}{\alpha} P(H(\infty, Z_1) > s)\,\mathrm{d}s &\leq \int_T^{\alpha(y-\beta)} \frac{\mathrm{e}^{K_2\alpha^{r-1}L(\alpha)s}}{\alpha} P(H(\infty, Z_1) > s)\,\mathrm{d}s \\ &\leq \int_T^\infty \frac{\mathrm{e}^{K_2\alpha^{r-1}L(\alpha)s}}{\alpha} P(H(\infty, Z_1) > s)\,\mathrm{d}s.\end{aligned}$$

Taking first the limit as $\alpha \to \infty$ and then the limit as $M \to \infty$, we deduce

$$\limsup_{\alpha\to\infty} \int_T^\infty \frac{\mathrm{e}^{K_2\alpha^{r-1}L(\alpha)s}}{\alpha} P(H(\infty, Z_1) > s)\,\mathrm{d}s = \limsup_{\alpha\to\infty} \int_T^{\alpha(y-\beta)} \frac{\mathrm{e}^{K_2\alpha^{r-1}L(\alpha)s}}{\alpha} P(H(\infty, Z_1) > s)\,\mathrm{d}s.$$

We will show that for T large enough,

$$\limsup_{\alpha\to\infty} \int_T^{\alpha(y-\beta)} \frac{\mathrm{e}^{K_2\alpha^{r-1}L(\alpha)s}}{\alpha} P(H(\infty, Z_1) > s)\,\mathrm{d}s = 0, \tag{7}$$

and equation (6) follows.

Proving equation (7) follows exactly as in Proposition 2.2 in [11] from equation (18) to the end of the proof of part (*ii*). To make the current exposition self-contained, we provide these details here. For a fixed $r_0 \in (0, r)$, by Theorem 1.5.4 of [1] we have $L(y)/y^{1-r} \sim \psi_1(y)$ and $a(y)/y^{r_0} \sim \psi_2(y)$, where ψ_1 and ψ_2 are non-increasing functions. So, for any $\varepsilon' > 0$ there exists $y_{\varepsilon'}$ such that for all $y \geq y_{\varepsilon'}$ we have

$$(1-\varepsilon')\psi_1(y) < L(y)/y^{1-r} < (1+\varepsilon')\psi_1(y) \quad \text{and} \quad a(y)/y^{r_0} < (1+\varepsilon')\psi_2(y). \tag{8}$$

By assumption, the tail of $H(\infty, Z_1)$ is bounded above by $a(t)\exp(-t^r L(t))$ for all t large enough, say for all $t \geq \bar{t}$. In the following we take $T > \max\{y_{\varepsilon'}, \bar{t}\}$. By the upper bound on the tail of $H(\infty, Z_1)$, $T > \bar{t}$ and the change of variable $z = s/[\alpha(y-\beta)]$, setting $K_3 = K_2(y-\beta)$ we have

$$\begin{aligned}&\int_T^{\alpha(y-\beta)} \frac{\mathrm{e}^{K_2\alpha^{r-1}L(\alpha)s}}{\alpha} P(H(\infty, Z_1) > s)\,\mathrm{d}s \\ &\leq (y-\beta)\int_{T/[\alpha(y-\beta)]}^1 a(z\alpha(y-\beta))\exp\{K_3\alpha^r L(\alpha)z - (z\alpha)^r(y-\beta)^r L(z\alpha(y-\beta))\}\,\mathrm{d}z. \end{aligned}\tag{9}$$

Since $T > y_{\varepsilon'}$, by (8) and the monotonicity of ψ_1 we have, for all $z \in (T/[\alpha(y-\beta)], 1)$,

$$\frac{L(z\alpha(y-\beta))}{[z\alpha(y-\beta)]^{1-r}} > \left(\frac{1-\varepsilon'}{1+\varepsilon'}\right)\frac{L(\alpha(y-\beta))}{[\alpha(y-\beta)]^{1-r}}$$

and so the right hand side of (9) is less than or equal to

$$(y-\beta)\int_{T/[\alpha(y-\beta)]}^1 a(z\alpha(y-\beta))\,\mathrm{e}^{-K_3\alpha^r L(\alpha)\left[\frac{(y-\beta)^{r-1}}{K_2}\left(\frac{1-\varepsilon'}{1+\varepsilon'}\right)\frac{L(\alpha(y-\beta))}{L(\alpha)}-1\right]z}\,\mathrm{d}z. \tag{10}$$

By the choice of d we can select $\varepsilon > 0$ sufficiently small that $(y-\beta)^{r-1}/K_2 > 1$. Consequently, we can choose ε' sufficiently small that

$$K_4 = \left(\frac{1-\varepsilon'}{1+\varepsilon'}\right)^2 \frac{(y-\beta)^{r-1}}{K_2} - 1 > 0.$$

Since L is slowly varying in correspondence of ε' there exists $t' = t'(y,\beta,\varepsilon')$ such that, for all $\alpha \geq t'$, $L(\alpha(y-\beta))/L(\alpha) > (1-\varepsilon')/(1+\varepsilon')$. Thus, using (10), we have

$$\begin{aligned}
&\int_T^{\alpha(y-\beta)} \frac{\mathrm{e}^{K_2\alpha^{r-1}L(\alpha)s}}{t} P(H(\infty,Z_1) > s)\,\mathrm{d}s \\
&\leq (y-\beta)\left(\sup_{z\in[T/(\alpha(y-\beta)),1]} a(z\alpha(y-\beta))\right)\int_{T/[\alpha(y-\beta)]}^1 \exp\{-K_5\alpha^r L(\alpha)z\}\,\mathrm{d}z \\
&= K_6(\mathrm{e}^{-K_7\alpha^{r-1}L(\alpha)} - \mathrm{e}^{-K_5\alpha^r L(\alpha)})\frac{\sup_{z\in[T/(\alpha(y-\beta)),1]} a(z\alpha(y-\beta))}{\alpha^r L(\alpha)}, \quad \text{for all } \alpha \text{ large enough}
\end{aligned}$$

where $K_5 = K_3K_4$, $K_6 = (y-\beta)/K_5$ and $K_7 = K_5T/(y-\beta)$. Due to the slow variation of L, $K_6(\mathrm{e}^{-K_7\alpha^{r-1}L(\alpha)} - \mathrm{e}^{-K_5\alpha^r L(\alpha)})$ converges to K_6 as $\alpha \to \infty$. So (7) follows if

$$\lim_{\alpha\to\infty} \frac{\sup_{z\in[T/(\alpha(y-\beta)),1]} a(z\alpha(y-\beta))}{\alpha^r L(\alpha)} = 0.$$

Since $T > y_{\varepsilon'}$, by (8) and the monotonicity of ψ_2 we have, for all $z \in (T/[\alpha(y-\beta)], 1)$,

$$a(z\alpha(y-\beta)) < (1+\varepsilon')[z\alpha(y-\beta)]^{r_0}\psi_2(z\alpha(y-\beta)) \leq (1+\varepsilon')[\alpha(y-\beta)]^{r_0}\psi_2(T).$$

So

$$\frac{\sup_{z\in[T/(\alpha(y-\beta)),1]} a(z\alpha(y-\beta))}{\alpha^r L(\alpha)} \leq \frac{(1+\varepsilon')(y-\beta)^{r_0}\psi_2(T)}{\alpha^{r-r_0}L(\alpha)}$$

and this latter term goes to zero as $\alpha \to \infty$ due to the slow variation of L and the choice of r_0.
□

Armed with the finite-dimensional LDP in Theorem 2.2 we now complete the programme of proof by establishing the sample path LDP holds in the topology of point-wise convergence and then in the L_1 topology. As a simple transformation of the rate function in question coincides with that for the centered partial sums of i.i.d. heavy-tailed semi-exponential random variables, we can appeal to results in [6] to assert its goodness.

Theorem 2.3 *Under the assumptions of Theorem 2.1, the family $\{S(\alpha\cdot)/\alpha\}_{\alpha>0}$ obeys an LDP on $D[0,1]$ equipped with the topology of point-wise convergence with speed $\alpha^r L(\alpha)$ and good, non-convex rate function*

$$J^{(\beta)}(f) = \begin{cases} \sum (f(t^+) - f(t^-))^r & \text{if } f \in D_\beta[0,1] \\ \infty & \text{otherwise} \end{cases} \tag{11}$$

where the sum is taken over all the points of discontinuity of f and

$$D^{(\beta)}[0,1] = \{f \in D[0,1] : f \text{ is linearly increasing with slope } \beta \text{ between jumps, which are non-negative}\}.$$

Proof: For $k \geq 1$, define the set of indexes

$$\mathcal{I}_k = \{(t_1, \ldots, t_k) : \; t_0 = 0 < t_1 < \ldots < t_k \leq 1\}.$$

By the Dawson-Gärtner Theorem it follows from Theorem 2.2 that $\{S(\alpha\cdot)/\alpha\}_{\alpha>0}$ obeys an LDP on $D[0,1]$ equipped with the topology of point-wise convergence with speed $\alpha^r L(\alpha)$ and good rate function

$$\widetilde{J}^{(\beta)}(f(t_1), \ldots, f(t_k)) = \sup_{k \geq 1, (t_1, \ldots, t_k) \in \mathcal{I}_k} I^{(\beta)}_{t_1,\ldots,t_k}(f(t_1), \ldots, f(t_k)). \tag{12}$$

By the contraction principle with the map $f(t) \mapsto f(t) - \beta t$, $\{S(\alpha\cdot)/\alpha - \beta\cdot\}_{\alpha>0}$ satisfies an LDP in $D[0,1]$ equipped with the topology of point-wise convergence and a rate function $\widetilde{J}^{(0)}$ as defined in equation (12). This rate function coincides with the rate function I_T defined in [6] for the centered partial sums of i.i.d. heavy-tailed semi-exponential distributions. In Lemma 4 of [6] it is established that $\widetilde{J}^{(0)}$ coincides with $J^{(0)}$ (I on page 1358 of [6]) defined in equation (11). Thus the identification of $\widetilde{J}^{(\beta)}$ with the $J^{(\beta)}$ follows from another application of the contraction principle with the map $f(t) \mapsto f(t) + \beta t$. As this rate function mimics that found in [6], its goodness in the L_1 topology is proved in Lemma 8 there. The lack of convexity can be seen noting that if $J^{(\beta)}(f) < \infty$ and $J^{(\beta)}(g) < \infty$, then for any $\gamma \in (0,1)$

$$J^{(\beta)}(\gamma f + (1-\gamma)g) = \gamma^r J^{(\beta)}(f) + (1-\gamma)^r J^{(\beta)}(g) > \gamma J^{(\beta)}(f) + (1-\gamma) J^{(\beta)}(g).$$

In order to strengthen this LDP from the topology of point-wise convergence to the L_1 topology, we prove exponential tightness and use this property to directly prove that the upper and lower large deviation bounds hold in this topology. Exponential tightness alone is not sufficient to establish the LDP in the L_1 topology as, when equipped with the topology of point-wise convergence, $D[0,1]$ is not Hausdorff. For exponential tightness we must establish the existence of compact sets $\{K_L\}_{L>0}$ in the L_1 topology such that

$$\limsup_{\alpha\to\infty} \frac{1}{\alpha^r L(\alpha)} \log P\left(\frac{S(\alpha\cdot)}{\alpha} \in K_L^c\right) \leq -L, \tag{13}$$

where K_L^c denotes the complement of K_L. For $L > 0$, consider the sets

$$K_L = \left\{f \in D[0,1] : \mathrm{var}_{[0,1]}(f) \leq L^{1/r} + \beta\right\},$$

where $\mathrm{var}_{[0,1]}(f)$ is the total variation of f on $[0,1]$. Compactness of K_L is shown in Lemma 5 [6]. Note that

$$P\left(\frac{S(\alpha\cdot)}{\alpha} \in K_L^c\right) \leq P\left(\frac{S(\alpha)}{\alpha} \geq L^{1/r} + \beta\right)$$

and thus equation (13) follows from an application of Theorem 2.1. Using the sets $\{K_L\}_{L>0}$ again, note that, for any closed set C in the L_1 topology, we have that

$$P\left(\frac{S(\alpha\cdot)}{\alpha} \in C\right) \leq P\left(\frac{S(\alpha\cdot)}{\alpha} \in C \cap K_L\right) + P\left(\frac{S(\alpha\cdot)}{\alpha} \in K_L^c\right).$$

As $C \cap K_L$ is closed in the topology of point-wise convergence, we can apply the LDP upper bound in that topology in addition to the identification of $\widetilde{J}^{(\beta)}$ with $J^{(\beta)}$ and the exponential tightness, to obtain the LDP upper bound

$$\limsup_{\alpha\to\infty} \frac{1}{\alpha^r L(\alpha)} \log P\left(\frac{S(\alpha\cdot)}{\alpha} \in C\right) \leq -\inf_{f\in C} J^{(\beta)}(f).$$

To prove the LDP lower bound,

$$\liminf_{\alpha\to\infty} \frac{1}{\alpha^r L(\alpha)} \log P\left(\frac{S(\alpha\cdot)}{\alpha} \in O\right) \geq -\inf_{f\in O} J^{(\beta)}(f),$$

for any open set O in the L_1 topology, it is enough to show that

$$\liminf_{\alpha\to\infty} \frac{1}{\alpha^r L(\alpha)} \log P\left(\frac{S(\alpha\cdot)}{\alpha} \in O_\delta(f)\right) \geq -J^{(\beta)}(f),$$

for all f such that $J^{(\beta)}(f) < \infty$ and

$$O_\delta(f) = \left\{g : \int_0^1 |f(t) - g(t)| \,\mathrm{d}t < \delta\right\}.$$

For $n > 0$ consider the following set

$$O^{(n)}(f) = \left\{g \text{ non-decreasing } : \left|g\left(\frac{k}{n}\right) - f\left(\frac{k}{n}\right)\right| < \frac{1}{n} \text{ for all } k \in \{1, \dots, n\}\right\}.$$

For n sufficiently large, $O^{(n)}(f) \subset O_\delta(f)$. To see this, define the intervals $I_1 = [0, 1/n]$ and $I_k = ((k-1)/n, k/n]$ for $k = 2, \dots, n$. As $g \in O^{(n)}(f)$ and f, g are non-decreasing, on the interval I_k we have

$$|f(t) - g(t)| \leq f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) + \frac{2}{n}$$

and so

$$\int_{I_k} |f(t) - g(t)| \,\mathrm{d}t \leq \frac{1}{n}\left(f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right)\right) + \frac{2}{n^2}.$$

Thus

$$\begin{aligned}
\int_0^1 |f(t) - g(t)| \,\mathrm{d}t &= \sum_{k=1}^n \int_{I_k} |f(t) - g(t)| \,\mathrm{d}t \\
&\leq n\left(\frac{2}{n^2}\right) + \sum_{k=1}^n \frac{1}{n}\left(f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right)\right) \\
&= \frac{2}{n} + \frac{1}{n}(f(1) - f(0)).
\end{aligned}$$

As the right hand side is decreasing in n, we have that $O^{(n)}(f) \subset O_\delta(f)$ for n sufficiently large. As $S(t)$ is non-decreasing almost surely, for n sufficiently large

$$P(S(\alpha\cdot)/\alpha \in O_\delta(f)) \geq P(S(\alpha\cdot)/\alpha \in O^{(n)}(f)).$$

To prove the LDP lower bound and complete the proof, we note that by Theorem 2.2 it follows that

$$\begin{aligned}
&\liminf_{\alpha\to\infty} \frac{1}{\alpha^r L(\alpha)} \log P(S(\alpha\cdot)/\alpha \in O^{(n)}(f)) \\
&= \liminf_{\alpha\to\infty} \frac{1}{\alpha^r L(\alpha)} \log P(|S(\alpha k/n)/\alpha - f(k/n)| < 1/n,\ k \in \{1, \dots, n\}) \\
&\geq -\inf_{(x_1,\dots,x_n)\in\prod_{k=1}^n (f(k/n)-1/n, f(k/n)+1/n)} I^{(\beta)}_{(1/n,2/n,\dots,1)}(x_1, \dots, x_n) \\
&\geq -\sum_{i=1}^n (1/n)^r I^{(\beta)}(n(f(i/n) - f(i-1/n))) \geq -J^\beta(f).
\end{aligned}$$

□

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