Sample path large deviations for order statistics

Ken R. Duffy* Claudio Macci† Giovanni Luca Torrisi‡

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Abstract

We consider the sample paths of the order statistics of i.i.d. random variables with common distribution function F. If F is strictly increasing but possibly having discontinuities, we prove that the sample paths of the order statistics satisfy the large deviation principle in the Skorohod M_1 topology. Sanov's Theorem is deduced in the Skorohod M_1' topology as a corollary to this result. A number of illustrative examples are presented, including applications to the sample paths of trimmed means and Hill Plots.

1 Introduction

Let $\{X_n : n \geq 1\}$ be a sequence of i.i.d. real-valued random variables with distribution function $F(x) = P(X_1 \leq x)$ that is assumed to be strictly increasing, but possibly having discontinuities. Define $a := \inf\{x : F(x) > 0\} \in [-\infty, \infty)$ and $b = \inf\{x : F(x) = 1\} \in (-\infty, \infty]$. For each $n \geq 1$, let $X_{1,n}, \ldots, X_{n,n}$ denote the ascending order statistics of $X_1, \ldots, X_n$, so that $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ and define $X_{0,n} := a$ and $X_{n+1,n} := b$. For each $n \geq 1$ define the sample path of the order statistics by

$$X_n(t) := X_{[(n+1)t],n} \text{ for all } t \in [0,1], \tag{1}$$

where $[x]$ is the greatest integer that is less than x.

The purpose of the present article is to prove the functional Large Deviation Principle (LDP) for order statistics in the sense of Varadhan [34]. We consider $X_n(\cdot)$ as a random element of the space of non-decreasing càdlàg functions (right continuous functions with left hand limits) ϕ such that $\phi(0) \geq a$ and $\phi(1) = b$.

*Hamilton Institute, National University of Ireland Maynooth, Ireland. E-mail: ken.duffy@nuim.ie

†Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, I-00133 Rome, Italy. e-mail: macci@mat.uniroma2.it

‡Istituto per le Applicazioni del Calcolo "Mauro Picone", Consiglio Nazionale delle Ricerche (CNR), Via dei Taurini 19, 00185, Rome, Italy. e-mail: torrisi@iac.rm.cnr.it

We equip this space with the Skorohod M_1 topology [31] everywhere, apart from for Sanov's Theorem where we use the Skorohod M_1' topology [25][37]. We prove that the random paths $\{X_n(\cdot)\}$ satisfy the Large Deviation Principle (LDP). That is, for all Borel sets B

$$-\inf_{\phi\in B^\circ} J^F(\phi) \leq \liminf_{n\to\infty} \frac{1}{n}\log P(X_n(\cdot)\in B) \leq \limsup_{n\to\infty} \frac{1}{n}\log P(X_n(\cdot)\in B) \leq -\inf_{\phi\in\bar{B}} J^F(\phi) \tag{2}$$

where B° denotes the interior of B and $\bar{B}$ denotes its closure. The rate function J^F takes values in $[0,\infty]$, is lower semi-continuous and has compact level sets (i.e. it is a good rate function, see e.g. [7]).

The sequence of order statistics sample paths $\{X_n(\cdot)\}$ defined in equation (1) is closely related to the sequence of empirical distribution functions $\{F_n\}$ defined by

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_n\leq x\}},$$

for all $n \geq 1$. Indeed, the right continuous generalized inverse of $X_n(\cdot)$ is approximately $F_n(\cdot)$ (in a sense that is made precise in the proof of Corollary 5). This relationship suggests that one way to prove that order statistics sample paths satisfy the LDP is to begin with Sanov's Theorem [28], the LDP for empirical measures in the space of probability measures, and to deduce the LDP for $\{X_n(\cdot)\}$ from it. In a recent article, this is the approach taken by Boistard [4] in order to prove large deviation results for L-statistics such as the trimmed mean and Gini's mean difference. For distribution functions with lighter than exponential tails she strengthens the topology in Sanov's Theorem from the topology of weak convergence to the topology generated by the L_2-Wasserstein metric. This enables her to deduce the LDP for L-statistics of not-necessarily bounded random variables by use of the contraction principle. The case of L-statistics for exponentially distributed random variables falls outside the conditions of her general approach, but is treated using alternate arguments.

The work in this paper differs from Boistard [4] in two significant ways: (1) the method of proof and (2) the topology in which the result holds. We take a completely different approach to prove the LDP for $\{X_n(\cdot)\}$. We begin by proving that the sample paths of the order statistics for i.i.d. uniformly distributed random variables satisfy the LDP. This is achieved by using an alternate characterization of the distribution of the sample paths of the order statistics of uniformly distributed random variables in terms of self-normalized sums of i.i.d. exponentially distributed random variables. By recalling a version of Mogul'skii's Theorem [19] due to Puhalskii [22] and then applying Puhalskii's extension of the contraction principle [22][24] with a function that embodies this representation, we obtain the LDP for the sample paths of the order statistics of the i.i.d. uniformly distributed random variables. An additional application of the contraction principle recovers the result for more general distributions than the uniform. When the underlying distribution function is strictly increasing (although possibly discontinuous), this approach leads to the functional LDP holding in the Skorohod

M_1 topology. From this result we deduce the LDP for trimmed means for any strictly increasing distribution function.

We comment that the topology of uniform convergence would be too strong for these results as even in the limit it is possible to have discontinuous sample paths. This is embodied by the resultant rate functions being finite at discontinuous paths.

An expression for the rate function, J^F in equation (2), is given in equation (6). If $F(x) = \int_a^x f(y)\,dy$ where $f(y) > 0$ almost everywhere, this reduces to the formula in equation (9). This functional large deviation principle enables not only the calculation of the exponential decay in the probability of seeing unlikely sample paths of order statistics, but also the identification of the most likely paths of the order statistics given a rare event occurred.

We illustrate the merits of this LDP by deducing the sample path LDP for the trimmed means of any strictly increasing distribution function, even those with infinite mean. We also establish the large deviation principle for Hill Plots, which enables estimates on the likelihood that Hill's [13] widely-used methodology misclassifies a non-Pareto law as being a Pareto law.

This article is organized as follows. In Section 2 we introduce the basic set-up and notation. The functional LDP for order statistics is presented in Section 3. Applications of the results are presented in Section 4.

2 Notation and terminology

We equip the real line $\mathbb{R}$ and its subsets with the Euclidean metric $\rho_1(x, y) = |x - y|$, but we equip the extensions of the real line ($\mathbb{R} \cup \{+\infty\}$, $\mathbb{R} \cup \{-\infty\}$ and $\mathbb{R} \cup \{-\infty, +\infty\}$) with an alternate metric, $\rho_2(x, y) = |\arctan(x) - \arctan(y)|$, to ensure that they are Polish spaces [9]. The metrics ρ_1 and ρ_2 are topologically equivalent when restricted to $[0, 1]$. The use of ρ_1 or ρ_2 is solely a technicality with the usage being dependent on if we are working with real or extended-real valued functions.

Let $D[0, 1]$ denote the space of real (or extended-real) valued càdlàg functions on the closed interval $[0, 1]$ equipped with the Skorohod M_1 topology [31][37] induced by the metric

$$d_{M_1}(\phi_1, \phi_2) = \inf_{(u_j, r_j) \in \Pi(\phi_j), j=1,2} \max\{\|u_1 - u_2\|_\infty, \|r_1 - r_2\|_\infty\}$$

where $\|u\|_\infty = \sup_{s\in[0,1]} |u(s)|$ and $\Pi(\phi)$ is the set of all parametric representations (u, r) of ϕ. A parametric representation (u, r) is a continuous nondecreasing function of the interval $[0, 1]$ onto the completed graph Γ_ϕ of ϕ, where the function u gives the spatial component, while the function r gives the time component. In this context completed graph of ϕ means

$$\Gamma_\phi = \{(u, t) \in \mathbb{R} \times (0, \infty) : u \in [\min\{\phi(t-), \phi(t)\}, \max\{\phi(t-), \phi(t)\}]\} \cup \{(\phi(0), 0)\},$$

where $\phi(t-)$ denotes the left limit of ϕ at t and we define an order on Γ_ϕ by saying that $(u_1, t_1) \leq (u_2, t_2)$ if either (i) $t_1 < t_2$ or (ii) $t_1 = t_2$ and $|\phi(t_1-) - u_1| \leq |\phi(t_2-) - u_2|$.

In Corollary 5 we shall also consider $D[0,1]$ equipped with the weaker M_1' topology [25][37] which is defined in the same way as M_1 except that we change Γ_ϕ to

$$\Gamma_\phi' = \{(u,t) \in \mathbb{R} \times (0,\infty) : u \in [\min\{\phi(t-),\phi(t)\}, \max\{\phi(t-),\phi(t)\}]\},$$

where $\phi(0-) = 0$.

For each $-\infty \le a < b \le \infty$, let $\mathcal{V}_{a,b}^+ \subset D[0,1]$ denote the closed set of non-decreasing functions ϕ such that $\phi(t) \ge a$, for all $t \in [0,1]$, and $\phi(1) = b$. We will equip each $\mathcal{V}_{a,b}^+$ with the Skorohod M_1 topology, apart from for Sanov's Theorem where we will use the M_1' variant. For both topologies, the space is metrizable as a separable metric space.

For each function $\phi \in D[0,1]$ we use the following notation for its Lebesgue decomposition with respect to Lebesgue measure:

$$\phi(t) = \phi^{(a)}(t) + \phi^{(s)}(t) = \int_0^t \dot{\phi}^{(a)}(s)ds + \phi^{(s)}(t),$$

where $\phi^{(a)}$ is its absolutely continuous component with $\phi^{(a)}(0) := 0$ and $\phi^{(s)}$ is its singular component.

The quantile function, $F^{-1} : [0,1] \mapsto [a,b]$ defined by

$$F^{-1}(u) := \inf\{x : F(x) > u\} \text{ if } u \in [0,1) \text{ and } F^{-1}(1) := b, \tag{3}$$

is the right continuous generalized inverse of F.

3 Functional LDP for order statistics

Theorem 3 is the cornerstone result. It proves the functional LDP in the Skorohod M_1 topology for the sample paths of order statistics where the distribution function F is strictly increasing, but possibly discontinuous. In order to do so, we shall appeal to the following version of Mogul'skii's Theorem[1].

Theorem 1 (Puhalskii [22]) *If $\{Y_i\}$ is i.i.d. with $P(Y_i > 0) = 1$ and $E(\exp(\theta Y_1)) < \infty$ for some $\theta > 0$, then the sample paths*

$$S_n(t) := \frac{1}{n} \sum_{i=1}^{[(n+1)t]} Y_i \tag{4}$$

[1]The version of Mogul'skii's Theorem reported in Theorem 5.1.2 [7] is insufficient for our needs, as it does not encompass the case of exponentially distributed random variables. See also [17][20][26][21].

satisfy the LDP in $D[0,1]$*, equipped with the Skorohod* M_1 *topology, with a rate function that is finite only for functions,* ϕ*, that are non-decreasing and of finite variation. For such a* ϕ*, the rate function is*

$$I(\phi) = \int_0^1 I_l(\dot{\phi}^{(a)}(t))dt + \phi^{(s)}(1), \tag{5}$$

where $I_l(x)$ *is the rate function for the partial sums* $\{n^{-1}\sum_{i=1}^n Y_i\}$*.*

This is a deduction from Lemma 3.2 of [22] (see also [23]). Under the conditions of Theorem 1, it proves that the LDP holds for $\{S_n\}$ in the space of divergent càdlàg functions on the interval $[0,\infty)$ equipped with the topology of weak convergence. We first restrict the argument to $[0,1]$ by the contraction principle, which gives the rate function in equation (5). To obtain the final result we note equivalence between the topology of weak convergence and the Skorohod M_1 topology for monotone functions (see, for example, Corollary 12.5.1 [37]).

We shall also make extensive use of Puhalskii's extension of the contraction principle [22]. In particular, we have the following, which can be found as Corollary 3.1.15 [24].

Theorem 2 (Puhalskii [24]) *Assume* $\{X_n\}$ *satisfies the LDP in a Hausdorff topological space* E *with rate function* I_E*. If* $f : E \mapsto E'$*, where* E' *is a Tychonoff space, is continuous at all* x *such that* $I_E(x) < \infty$*, then* $\{f(X_n)\}$ *satisfies the LDP in* E' *with rate function* $I_{E'}(y) = \inf\{I_E(x) : f(x) = y\}$*.*

All of the spaces we consider are metric spaces, so the topological conditions of this theorem are met.

Armed with Theorems 1 and 2, we now prove our cornerstone result.

Theorem 3 (LDP for order statistics) *The sample paths* $\{X_n(\cdot)\}$ *satisfy the LDP in* $\mathcal{V}_{a,b}^+$ *equipped with the Skorohod* M_1 *topology with the rate function*

$$J^F(\chi) = \inf_{\phi\in\mathcal{V}_{0,1}^+}\left\{-\int_0^1 \log(\dot{\phi}^{(a)}(t))dt : F^{-1}(\phi(t)) = \chi(t) \text{ for all } t \in [0,1]\right\}. \tag{6}$$

Note that $J^F(\chi) = 0$ *if* $\chi(t) = F^{-1}(t)$*.*

Proof: Begin by considering $\{U_n : n \geq 1\}$, a sequence of i.i.d. random variables that are uniformly distributed on $[0,1]$. For each $n \geq 1$, let $U_{1,n}, \ldots, U_{n,n}$ be the order statistics of $U_1, \ldots, U_n$, with $U_{0,n} := 0$ and $U_{n+1,n} := 1$. For each $n \geq 1$ define the sample path of the order statistics by $U_n(t) := U_{[(n+1)t],n}$ for all $t \in [0,1]$.

The distribution of the sample path $U_n(\cdot)$ is equal to a distribution that can be constructed from a sequence of i.i.d. exponentially distributed random variables. Let $\{Y_n\}$ be i.i.d.

exponentially distributed random variables with mean 1. Define the self-normalized random functions $\{N_n\}$ by

$$N_n(t) := \left(\sum_{i=1}^{[(n+1)t]} Y_i\right) / \left(\sum_{j=1}^{n+1} Y_j\right), \text{ if } \sum_{j=1}^{n+1} Y_j > 0. \tag{7}$$

As a consequence of Proposition 8.2.1 in Shorack and Wellner [30], $N_n(\cdot)$ is equal in distribution to $U_n(\cdot)$. As an application of Theorem 1, the sample paths

$$S_n(t) := \frac{1}{n} \sum_{i=1}^{[(n+1)t]} Y_i$$

satisfy the LDP in $D[0,1]$ with rate function that is finite only for functions ϕ that are non-decreasing and of finite variation, $I(\phi) = \int_0^1 I_l(\dot{\phi}^{(a)}(t))dt + \phi^{(s)}(1)$, where $I_l(x) = x - \log(x) - 1$ is the rate function for the partial sums of i.i.d. exponentially distributed random variables $\{n^{-1}\sum_{i=1}^n Y_i\}$.

Define $g : \{\phi \in D[0,1] : \phi(1) \neq 0\} \mapsto D[0,1]$ by $g(\phi)(t) = \phi(t)/\phi(1)$. Note that if $S_n(1) \neq 0$, then $g(S_n) = N_n$, where $S_n(\cdot)$ is defined in equation (4) and $N_n(\cdot)$ is defined in equation (7). The map g is continuous at all ϕ such that $\phi(1) > 0$ as if $\phi_n \to \phi$ in $D[0,1]$ equipped with the Skorohod M_1 topology, then $\phi_n(1) \to \phi(1)$. As $I_l(0) = -\log(0) - 1 = \infty$, $I(\phi) < \infty$ only if $\phi(1) > 0$. Puhalskii's extension of the contraction principle, Theorem 2, only requires g to be continuous at all limit points where the rate function is finite in order for the usual contraction principle result to hold (e.g. Theorem 4.2.1 [7]). Thus as g is continuous at all ϕ such that $I(\phi) < \infty$, we deduce that $\{N_n(\cdot)\}$ satisfies the LDP in $\mathcal{V}_{0,1}^+$ with the following rate function:

$$\begin{aligned} J^U(\psi) &= \inf\{I(\phi) : g(\phi) = \psi\} = \inf\{I(\phi) : \phi(t)/\phi(1) = \psi(t) \text{ for all } t \in [0,1]\} \\ &= \inf_{\phi(1)>0} I(\phi(1)\psi) = \inf_{z>0} I(z\psi). \end{aligned}$$

For fixed $z > 0$ and $\psi \in \mathcal{V}_{0,1}^+$, we have that

$$\begin{aligned} I(z\psi) &= \int_0^1 I_l(z\dot{\psi}^{(a)}(t))dt + z\psi^{(s)}(1) \\ &= \int_0^1 \left(z\dot{\psi}^{(a)}(t) - \log(z) - \log(\dot{\psi}^{(a)}(t)) - 1\right) dt + z\psi^{(s)}(1) \\ &= z(1 - \psi^{(s)}(1)) - \log(z) - \int_0^1 \log(\dot{\psi}^{(a)}(t))dt - 1 + z\psi^{(s)}(1) \\ &= z - \log(z) - 1 - \int_0^1 \log(\dot{\psi}^{(a)}(t))dt, \end{aligned}$$

where we have used the fact that $\psi(1) = 1$ to deduce that $\int_0^1 \dot{\psi}^{(a)}(t)dt = 1 - \psi^{(s)}(1)$. However, $\inf_{z>0}(z - \log(z) - 1) = 0$ and is attained at $z = 1$, thus

$$J^U(\psi) = \inf_{z>0} I(z\psi) = -\int_0^1 \log(\dot{\psi}^{(a)}(t))dt. \tag{8}$$

As the order statistics sample path $U_n(\cdot)$ has the same distribution as the self-normalized sample path $N_n(\cdot)$, the sample paths of the order statistics of uniformly distributed random variables $\{U_n(\cdot)\}$ satisfy the LDP in $\mathcal{V}_{0,1}^+$ with the rate function given in equation (8). As $\log(0) = -\infty$, $J^U(\psi) = \infty$ unless the absolutely continuous component of ψ's Lebesgue decomposition is strictly increasing almost everywhere with respect to Lebesgue measure.

Consider a sequence of i.i.d. random variables $\{X_n\}$ with common distribution function $F(\cdot)$. As is well known, e.g. Theorem 14.1 [3], with the quantile function $F^{-1}(u)$ defined in equation (3) and $\{U_n\}$ being i.i.d. random variables distributed uniformly on $[0, 1]$, then $\{F^{-1}(U_n)\}$ is an i.i.d. sequence of random variables with distribution function $F(\cdot)$. With a slight abuse of notation, define the map $F^{-1} : \mathcal{V}_{0,1}^+ \mapsto \mathcal{V}_{a,b}^+$ by $F^{-1}(\phi)(t) = F^{-1}(\phi(t))$ for all $t \in [0, 1]$. As $F^{-1}(u)$ is a non-decreasing function of u, we have that $F^{-1}(U_n(\cdot))$ is exactly the sample path of the order statistics of $F^{-1}(U_1), \ldots, F^{-1}(U_n)$. As we have proved that $\{U_n(\cdot)\}$ satisfies the LDP, to deduce the LDP for the sample paths $\{X_n(\cdot)\}$ of the order statistics of $\{X_n\}$, it suffices to show that the map $F^{-1} : \mathcal{V}_{0,1}^+ \mapsto \mathcal{V}_{a,b}^+$ is sufficiently well behaved that the contraction principle (e.g. Theorem 4.2.1 [7]) can be applied.

As F is assumed to be strictly increasing (although it can have discontinuities), F^{-1} is continuous on $[0, 1]$ (e.g. Lemma 13.6.4 [37]). Note that $F^{-1}(\phi) := F^{-1} \circ \phi$ and Theorem 13.2.3 [37] proves that composition on $D[0, 1] \times D[0, 1]$ is continuous at all (F^{-1}, ϕ) such that F^{-1} is continuous and ϕ is non-decreasing. Thus $F^{-1} : \mathcal{V}_{0,1}^+ \mapsto \mathcal{V}_{a,b}^+$ is continuous and Theorem 3 follows from an application of the contraction principle (e.g. Theorem 4.2.1 [7]).

■

We now state a corollary of Theorem 3 that follows from the chain rule [35].

Corollary 4 (Distribution functions with positive densities a.e.) *If $F(x) = \int_a^x f(y)\,dy$ and f is continuous and positive almost everywhere, so that $F(x)$ is strictly increasing and continuous, then $J^F(\chi) = \infty$ unless $\chi \in \mathcal{V}_{a,b}^+$ (or equivalently $F \circ \chi \in \mathcal{V}_{0,1}^+$) is strictly increasing in which case*

$$J^F(\chi) = -\int_0^1 \left(\log(f(\chi(t))) + \log(\dot{\chi}^{(a)}(t))\right) dt. \tag{9}$$

In the next subsection we present some illustrative examples based on Theorem 3 and Corollary 4, which, *inter alia*, demonstrate that the rate functions defined in equations (6) and (9) are not convex in general.

3.1 Examples

Example I demonstrates why $J^F(\cdot)$ is finite at paths with discontinuities: they correspond to ranges where no sample has been observed. It also illustrates how the functional LDP enables the deduction of conditional laws of large numbers. We say that the order statistics of random variables with distribution function F can (cannot) emulate the order statistics of random variables with distribution function G if $J^F(G^{-1}) < \infty$ $(= \infty)$. Examples II to V below concern order statistics of given laws that can or cannot emulate the order statistics of other distributions. In particular, Example IV shows that the order statistics of Pareto distributions, even those with finite mean, can emulate those with infinite mean. Example V shows that the order statistics of Pareto distributions can emulate those of any Exponential distribution, but the order statistics of Exponential distributions cannot emulate the order statistics of Pareto distributions with infinite mean. We return to this final point in Section 4.2 Example VII when we consider trimmed means.

Example I: discontinuous paths. If X_1 is uniformly distributed on $[0, 1]$, denoted $F = U$, then $F(x) = \int_0^x dx$. Thus $F^{-1}(u) = u$, so that

$$J^U(\chi) = -\int_0^1 \log(\dot{\chi}^{(a)}(t))\, dt$$

for any $\chi \in \mathcal{V}_{0,1}^+$. As $x \mapsto -\log(x)$ is a strictly convex function, note that J^U is a strictly convex rate function. Define the set $A := \{\phi : \phi(t) \leq 1/3 \text{ for all } t \in [0, 1)\}$. Note that $X_n(\cdot) \in A$ if and only if $X_{n,n} \in [0, 1/3]$, i.e. $X_i \in [0, 1/3]$ for all $i \in \{1, \dots, n\}$, and therefore $P(X_n(\cdot) \in A) = (1/3)^n$. The exponent in the decay of this probability is $-\log(1/3)$. This can also be calculated from the LDP by considering the sample path large deviations for $P(X_n(\cdot) \in A)$ and, in particular, by determining $\inf\{J^U(\chi) : \chi \in A\}$. As $-\log(u)$ is a convex function for $u > 0$ we can use Jensen's inequality to show that the infimum $\inf\{J^U(\chi) : \chi \in A\}$ is attained at $\hat{\chi}(t) = t/3$ for $t \in [0, 1)$ and $\hat{\chi}(1) = 1$. For this path $J^U(\hat{\chi}) = -\log(1/3)$ and therefore $\lim n^{-1} \log P(X_n(\cdot) \in A) = -\log(1/3)$.

The sample path LDP gives more information than the direct calculation. It shows that the most likely path to this event is that $X_{1,n}, \dots, X_{n,n}$ be spread uniformly over $[0, 1/3]$, in the following sense. By, for example, Theorem 3.1 (b) of Lewis, Pfister and Sullivan [16], for any $\varepsilon > 0$

$$\lim_{n\to\infty} P(X_n(\cdot) \notin B_\varepsilon(\hat{\chi}) | X_n(\cdot) \in A) = 0,$$

where $B_\varepsilon(\hat{\chi})$ is the open ball of radius ε around $\hat{\chi} \in \mathcal{V}_{0,1}^+$. That is, conditioned on $X_{n,n} \leq 1/3$, the sample paths of the order statistics $\{X_n(\cdot)\}$ satisfy a weak law of large numbers at $\hat{\chi}$, the path where the samples are uniformly distributed in $[0, 1/3]$.

Example II: rate function for the Beta distribution. Assume that X_1 is distributed as a Beta(α, β) distribution, so that X_1 takes values in $[0, 1]$ with a strictly increasing continuous distribution function $F(x; \alpha, \beta)$ with density

$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1},$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ and $\alpha, \beta > 0$. By Corollary 4, $\{X_n(\cdot)\}$ satisfies the LDP in $\mathcal{V}_{0,1}^+$ with rate function

$$J^{\mathrm{Beta}(\alpha,\beta)}(\chi) = -\log\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right) - (\alpha-1)\int_0^1 \log(\chi(t))dt - (\beta-1)\int_0^1 \log(1-\chi(t))dt \\ - \int_0^1 \log(\dot{\chi}^{(a)}(t))dt,$$

for any $\chi \in \mathcal{V}_{0,1}^+$. Considering $J^{\mathrm{Beta}(\alpha,\beta)}(\hat{\chi})$ where $\hat{\chi}(t) := t$ for $t \in [0,1]$, we are evaluating the large deviations rate of seeing the quantile function of a uniform law given that the underlying distribution is actually a Beta(α, β) distribution. We obtain

$$J^{\mathrm{Beta}(\alpha,\beta)}(\hat{\chi}) = -\log\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right) + \alpha + \beta - 2.$$

This function has its minimum, $J^{\mathrm{Beta}(\alpha,\beta)}(\hat{\chi}) = 0$, when the random variables $\{X_n\}$ have a uniform distribution, $\alpha = \beta = 1$. Note that if $\alpha = 1$, then as $\Gamma(1+\beta) = \beta\Gamma(\beta)$, $J^{\mathrm{Beta}(\alpha,\beta)}(\hat{\chi}) = \beta - \log(\beta) - 1$. This is the rate function evaluated at β for the partial sums $\{n^{-1}\sum_{i=1}^n Y_i\}$ of i.i.d. exponentially distributed random variables $\{Y_i\}$ with mean 1. By symmetry, the same result holds if $\beta = 1$ and α is varied.

Example III: rate function for the Exponential distribution. If $F(x) = 1 - \exp(-\lambda x)$ for all $x \geq 0$ so that $a = 0$ and $b = +\infty$, then $F^{-1}(u) = -\log(1-u)/\lambda$. Thus by Corollary 4

$$J^{\mathrm{Exp}(\lambda)}(\chi) = -\log(\lambda) + \lambda\int_0^1 \chi(t)\,dt - \int_0^1 \log(\dot{\chi}^{(a)}(t))\,dt, \tag{10}$$

which can be readily seen to be strictly convex. For example, if $\hat{\chi}(t) = F^{-1}(t) = -\log(1-t)/\lambda$, then $J^{\mathrm{Exp}(\lambda)}(\hat{\chi}) = 0$. That is, if the sample path is the quantile function of an exponential distribution with rate λ, then the rate function is 0. If, for some $K > 0$, $\hat{\chi}_K(t) = Kt$ for $t \in [0,1)$ and $\hat{\chi}_K(1) = \infty$, then $J^{\mathrm{Exp}(\lambda)}(\hat{\chi}_K) = -\log(\lambda) + \lambda K/2 - \log(K)$. Thus the most likely λ to give rise to the quantile function of a uniform law on $[0, K)$ is when $\lambda_K = 2/K$ and the mean of the exponential distribution corresponds to the mean of the corresponding uniform distribution. For $\lambda_K = 2/K$, $J^{\mathrm{Exp}(\lambda_K)}(\hat{\chi}_K) = -\log(2) + 1 \approx 0.307$, irrespective of K.

Example IV: rate function for the Pareto distribution. If $F(x) = 1 - x^{-\alpha}$ for $\alpha > 0$ so that $a = 1$ and $b = +\infty$, then $F^{-1}(u) = (1-u)^{-1/\alpha}$. Thus by Corollary 4

$$J^{\mathrm{Pareto}(\alpha)}(\chi) = -\log(\alpha) + (\alpha+1)\int_0^1 \log(\chi(t))\,dt - \int_0^1 \log(\dot{\chi}^{(a)}(t))\,dt, \tag{11}$$

which is an example of a non-convex rate function. To see this consider, for $k = 1, 2$ and $\rho \in (0, 1)$, the functions:

$$\chi_{k,\varepsilon}(t) = \begin{cases} t + k & \text{for } t \in [0, 1-\varepsilon) \\ \exp((1-t)^{-\rho}) & \text{for } t \in [1-\varepsilon, 1) \end{cases},$$

then for any $\gamma \in (0, 1)$

$$\begin{aligned}
\lim_{\varepsilon\to 0} J^{\mathrm{Pareto}(\alpha)}(\gamma\chi_{1,\varepsilon} + (1-\gamma)\chi_{2,\varepsilon}) &= -\log(\alpha) + (\alpha+1)\int_0^1 \log(\gamma(t+1) + (1-\gamma)(t+2))dt \\
&> -\log(\alpha) + (\alpha+1)\int_0^1 \left(\gamma\log(t+1) + (1-\gamma)\log(t+2)\right)dt, \\
&= \lim_{\varepsilon\to 0}\left(\gamma J^{\mathrm{Pareto}(\alpha)}(\chi_{1,\varepsilon}) + (1-\gamma)J^{\mathrm{Pareto}(\alpha)}(\chi_{2,\varepsilon})\right),
\end{aligned}$$

by the strict concavity of $x \mapsto \log(x)$. Thus for any ε sufficiently small, the lack of convexity of $J^{\mathrm{Pareto}(\alpha)}(\cdot)$ is demonstrated.

If $\hat{\chi}$ corresponds to the quantile function of the Pareto(α), $\hat{\chi}(t) = (1-t)^{-1/\alpha}$, then $J^{\mathrm{Pareto}(\alpha)}(\hat{\chi}) = 0$. If $K > 0$ and $\hat{\chi}(t) = 1 + Kt$ for $t < 1$ and $\hat{\chi}(1) = \infty$, corresponding to the quantile function of a uniform distribution on $[1, 1+K)$, then $J^{\mathrm{Pareto}(\alpha)}(\hat{\chi}) = -\log(\alpha K) + (\alpha+1)((K+1)K^{-1}\log(K+1)-1)$. The minimum over α is attained at $\alpha_K = K/((K+1)\log(K+1)-K)$ for which $J^{\mathrm{Pareto}(\alpha_K)}(\hat{\chi}) = -2\log(K) + \log((K+1)\log(K+1)-K) + (K+1)K^{-1}\log(K+1)$. This has its infimum as $K \to 0$, so that $\alpha_K \to \infty$ and $J^{\mathrm{Pareto}(\alpha_K)}(\hat{\chi})$ tends to $-\log(2) + 1 \approx 0.307$. If $\hat{\chi}(t) = (1-t)^{-1/\beta}$, for any $\beta > 0$, then

$$J^{\mathrm{Pareto}(\alpha)}(\hat{\chi}) = \frac{\alpha}{\beta} - \log\left(\frac{\alpha}{\beta}\right) - 1.$$

The order statistics path $\hat{\phi}$ of the uniformly distributed random variables on $[0, 1]$ that attains this is $\hat{\phi}(t) = F \circ \hat{\chi}(t) = 1 - (1-t)^{\alpha/\beta}$. Thus it is possible on the scale of large deviations for the sample path of the order statistics of any Pareto law to emulate that of any other.

Example V: Exponential and Pareto distribution. If $\hat{\chi}(t) = -\log(1-t)/\lambda + 1$ corresponding to a quantile function of an Exponential law on $[1, \infty)$, then $J^{\mathrm{Pareto}(\alpha)}(\hat{\chi}) = \log(\lambda/\alpha) - 1 - \exp(\lambda)(\alpha+1)\mathrm{Ei}(-\lambda)$, where $\mathrm{Ei}(-\lambda) = -\int_\lambda^\infty \exp(-t)/t\,dt$. Thus, in the large deviations limit with a finite rate, the order statistics of any i.i.d. Pareto distributed random variables can emulate the quantile function of any i.i.d Exponentially distributed random variables. On the other hand, if $\hat{\chi}(t) = (1-t)^{-1/\alpha} - 1$, corresponding to the quantile function of a Pareto distribution on $[0, \infty)$, then $J^{\mathrm{Exp}(\lambda)}(\hat{\chi}) = +\infty$ if $\alpha \le 1$ and, if $\alpha > 1$, $J^{\mathrm{Exp}(\lambda)}(\hat{\chi}) = \log(\alpha/\lambda) + (\lambda\alpha + 1 - \alpha^2)/(\alpha(\alpha-1)) < \infty$. That is, in the large deviations limit, the order statistics of exponentially distributed random variables *cannot* emulate the order statistics of Pareto distributed random variables with infinite mean. We return to this point in Section 4.2, Example VII.

3.2 Comment on Sanov's Theorem

Sanov's Theorem (e.g [7] Section 6.2) considers the empirical laws of a process of i.i.d. random variables. With the laws considered as random elements of the space of probability measures equipped with either the topology of weak convergence or the τ topology, Sanov's Theorem proves the empirical laws satisfy the LDP with relative entropy as the rate function. For some modern developments, see for example [15] and references therein.

As stated in the Introduction, the empirical laws and the sample paths of order statistics are closely related. The following corollary shows that a version of Sanov's Theorem for the empirical distribution functions can be recovered from the sample path LDP for the order statistics. For the sequence $\{X_n\}$, the empirical distribution functions $F_n \in D[0,1]$ are defined by

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \leq x\}},$$

for all $n \geq 1$.

For $-\infty \leq a < b \leq \infty$ and each $\chi \in \mathcal{V}^+_{a,b}$ define the right inverse χ^{-1} by $\chi^{-1}(t) := \inf\{s : \chi(s) > t\}$ for all $t \in [a,b)$, $\chi^{-1}(1) := b$. Thus χ^{-1} is an element of $\mathcal{V}^+_{0,1}[a,b]$, the set of non-decreasing elements in the space of càdlàg functions on $[a,b]$ with $\chi^{-1}(a) \geq 0$ and $\chi^{-1}(b) = 1$. For our purposes, it suffices to equip $\mathcal{V}^+_{0,1}[a,b]$ with the Skorohod M_1' topology, which is finer than the M_1 topology and so than the topology of weak convergence.

Corollary 5 (Sanov's Theorem) *Assume that $F(x) = \int_a^x f(y)\,dy$ where $f(y) > 0$ almost everywhere and $-\infty \leq a < b \leq \infty$. Then $\{F_n\}$ satisfies the LDP in $\mathcal{V}^+_{0,1}[a,b]$ equipped with the Skorohod M_1' topology with rate function $H^F(\eta) = \infty$ unless η is absolutely continuous, in which case*

$$H^F(\eta) = \int_a^b \dot{\eta}(s) \log\left(\frac{\dot{\eta}(s)}{f(s)}\right) ds. \tag{12}$$

Proof: For any $x \in (a,b]$,

$$\begin{aligned}
F_n(x) &= \frac{1}{n}\left(\inf\{m \in \{0,1,\ldots,n+1\} : X_{m,n} > x\} - 1\right)\\
&= \frac{1}{n}\left((n+1)\inf\left\{r \in \left\{0, \frac{1}{n+1}, \ldots, \frac{n+1}{n+1}\right\} : X_{(n+1)r,n} > x\right\} - 1\right)\\
&= \left(1+\frac{1}{n}\right)\inf\left\{s \in [0,1] : X_{[(n+1)s],n} > x\right\} - \frac{1}{n},\\
&= \left(1+\frac{1}{n}\right)X_n^{-1}(x) - \frac{1}{n},
\end{aligned}$$

where we have used the definition of pseudo-inverse above in our identification of $X_n^{-1}(x)$. Hence $\sup_{x\in[a,b]}|F_n(x)-X_n^{-1}(x)|\le 2/n$ and the sequences $\{F_n\}$ and $\{X_n^{-1}(\cdot)\}$ are exponentially equivalent (e.g. Definition 4.2.10 [7]) in the uniform topology. Thus to prove $\{F_n\}$ satisfies the LDP, it suffices to show that $\{X_n^{-1}(\cdot)\}$ does. By Theorem 13.6.2 [37] (with straight-forward modifications if $b<\infty$), the function $\mathcal{V}_{a,b}^{+}\mapsto\mathcal{V}_{0,1}^{+}[a,b]$ such that $\phi\mapsto\phi^{-1}$ can be seen to be continuous from the M_1 to M_1' topologies at all strictly increasing ϕ. It is necessary to move to the M_1' topology as, in general, inversion is not continuous in the M_1 topology. As $J^F(\phi)=\infty$ unless ϕ is strictly increasing, the LDP for $\{X_n^{-1}(\cdot)\}$ follows from an application of Theorem 2. Take $\eta\in\mathcal{V}_{0,1}^{+}[a,b]$ so that η^{-1} is strictly increasing (i.e. η is absolutely continuous) and let $\ell=\ell_1+\ell_2$ be the Lebesgue decomposition of the Lebesgue measure ℓ with respect to the measure $\ell\circ\eta$, i.e. the image measure of ℓ under η^{-1}. By Lemma 3.6 in [22] we have

$$\dot{(\eta^{-1})}^{(a)}(t)=\frac{d\ell_1}{d\eta}(\eta^{-1}(t)),\quad \ell\text{-almost everywhere.}$$

So, by Corollary 4 we deduce

$$\begin{aligned}
H^F(\eta)=J^F(\eta^{-1}) &= -\int_0^1\left(\log(f(\eta^{-1}(t)))+\log\dot{(\eta^{-1})}^{(a)}(t)\right)dt\\
&= -\int_0^1\log\left(f(\eta^{-1}(t)))\frac{d\ell_1}{d\eta}(\eta^{-1}(t))\right)dt\\
&= -\int_{\eta^{-1}(0)}^{b}\log\left(f(s)\frac{d\ell_1}{d\eta}(s)\right)\dot\eta(s)\,ds\\
&= -\int_{\eta^{-1}(0)}^{b}\log\left(f(s)\frac{d\ell}{d\eta}(s)\right)\dot\eta(s)\,ds \qquad (13)\\
&= \int_a^b\dot\eta(s)\log\left(\frac{\dot\eta(s)}{f(s)}\right)ds; \qquad (14)
\end{aligned}$$

indeed, the equality (13) follows by the absolute continuity of η (which implies $\ell=\ell_1$); (14) follows as $\dot\eta(s)=0$ for $s\in[a,\eta^{-1}(0))$ and $0\log(0):=0$.

∎

4 Applications

4.1 Sample path large deviations for L-statistics

Let $K:[0,1]\mapsto\mathbb{R}$ and consider the sequence of random variables called L-statistics:

$$T_n:=\frac{1}{n+1}\sum_{i=1}^{n}K\left(\frac{i}{n+1}\right)X_{i,n}\text{ for }n\ge 1.$$

Much is known regarding the large n behavior of the sequence of random variables $\{T_n\}$ and, in particular, its central limit behavior, e.g. Stigler [32], Vandemaele and Veraverbeke [33], Callaert, Vandemaele and Veraverbeke [5] and Aleshkyavichene [2]. Large deviation results for L-statistics can be found in Groeneboom, Oosterhoff and Ruymgaart [10], Groeneboom and Shorack [11], and Boistard [4].

Consider the measure on $[0,1]$ defined by

$$\mu_n(ds) := \frac{1}{n+1}\sum_{i=1}^{n} \delta_{i/(n+1)}(ds),$$

where δ_x is the Dirac delta measure at x, and set

$$V_n(t) := \int_0^t K(s) X_{[(n+1)s],n}\, \mu_n(ds), \quad t \in [0,1].$$

Clearly $T_n = V_n(1)$. For large values of n, approximating the empirical measure with the Lebesgue measure on [0,1], we are led to consider the following sample paths

$$T_n(t) := \int_0^t K(s) X_{[(n+1)s],n}\, ds, \quad t \in [0,1].$$

The sample paths $\{V_n(\cdot)\}$ and $\{T_n(\cdot)\}$ record the shape of the L-statistics for the complete range of quantile values, while T_n records it only over the whole range.

4.2 Trimmed means

The function K can be chosen to remove outliers. When

$$K(u) = \begin{cases} 1/(1-2\gamma) & \text{if } u \in [\gamma, 1-\gamma] \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

for $\gamma \in (0, 1/2)$, T_n is called the trimmed mean and is a robust statistic. It provides an average value of the observations after the exclusion of large and small observations. It is used, for example, in scoring at the Olympic Games. In the case where $\gamma = 1/4$, it is called the interquartile mean.

In this case the sample paths $T_n(\cdot)$ are given by

$$T_n(t) = (1-2\gamma)^{-1} \int_\gamma^t X_{[(n+1)s],n}\, ds \text{ for } t \in [\gamma, 1-\gamma] \tag{16}$$

Theorem 6 (Trimmed means) *Assume that F is strictly increasing. If K is of the form in equation* (15) *with $\gamma \in (0, 1/2)$, then $\{T_n(\cdot)\}$ satisfies the LDP in $C[\gamma, 1-\gamma]$, the space of* *continuous functions on* $[\gamma, 1-\gamma]$ *with* $\phi(\gamma) = 0$*, equipped with the topology induced by the uniform norm* $||\phi|| := \sup_{t\in[\gamma,1-\gamma]} |\phi(t)|$ *and with the rate function*

$$H^\gamma(\chi) = \inf_{\phi \in \mathcal{V}^+_{a,b}} \left\{ J^F(\phi) : (1-2\gamma)^{-1} \int_\gamma^t \phi(s) ds = \chi(t) \textit{ for all } t \in [\gamma, 1-\gamma] \right\},$$

where $J^F(\cdot)$ *is defined in equation* (6)*. Note that* $H^\gamma(\chi) = 0$ *if* $\chi(t) = (1-2\gamma)^{-1} \int_\gamma^t F^{-1}(u)\, du$ *for all* $t \in [\gamma, 1-\gamma]$*.*

Proof: Consider the map $g_\gamma : D[0,1] \mapsto C[\gamma, 1-\gamma]$ defined by

$$g_\gamma(\phi)(t) = \frac{1}{1-2\gamma} \int_\gamma^t \phi(s)\, ds \text{ for } t \in [\gamma, 1-\gamma]$$

and note that $g_\gamma(X_{[(n+1)\cdot],n}) = T_n(\cdot)$. As Theorem 3 proves that $\{X_n(\cdot)\}$ satisfies the LDP in $\mathcal{V}^+_{a,b}$, in order to deduce the LDP by invoking the extended contraction principle, Theorem 2, we must check that g_γ is continuous at all ϕ such that $J^F(\phi) < \infty$. When $C[\gamma, 1-\gamma]$ is equipped with the topology of uniform convergence, g_γ is continuous (e.g. Theorem 11.5.1 [37]) at all ϕ taking values in $\mathbb{R}$. Thus concern only arises if $a = -\infty$ or $b = +\infty$. This causes no difficulty as $J^F(\phi) < \infty$ only if $a < \phi(\gamma) < \phi(1-\gamma) < b$. To see this, consider (for example) $\phi(1-\gamma) = b$. Then, using equation (6),

$$\begin{aligned} J^F(\phi) &\geq \inf_{\psi \in \mathcal{V}^+_{0,1}} \left\{ -\int_0^1 \log(\dot{\psi}^{(a)}(t))\, dt : F^{-1}(\psi(1-\gamma)) = \phi(1-\gamma) \right\} \\ &\geq \inf_{\psi \in \mathcal{V}^+_{0,1}} \left\{ -\int_0^1 \log(\dot{\psi}^{(a)}(t))\, dt : \psi(1-\gamma) = 1 \right\}. \end{aligned}$$

As $\psi(1-\gamma) = 1$, $\psi(1) = 1$ and ψ is non-decreasing, $\dot{\psi}^{(a)}(t) = 0$ for $t \in (1-\gamma, 1]$, $-\log(0) = \infty$ and thus $J^F(\phi) = \infty$. Hence the LDP follows from Theorem 3 and an application of Theorem 2.

■

In contrast to Example V, Section 3.1, the following example shows that for trimmed means, Exponential Laws *can* emulate Pareto Laws with finite rate, even Pareto Laws with infinite mean.

Example VIII: Exponential emulating Pareto. If $F(x) = 1 - \exp(-\lambda x)$ for all $x \geq 0$ so that $a = 0$ and $b = +\infty$, then $F^{-1}(u) = -\log(1-u)/\lambda$ and $J^{\mathrm{Exp}(\lambda)}(\chi)$ is given in equation (10). Consider the test function $\chi(t) = (1-2\gamma)^{-1} \int_\gamma^t ((1-s)^{-1/\alpha} - 1)\, ds$ for $t \in [\gamma, 1-\gamma]$

corresponding to the trimmed mean of a Pareto distribution on $[0,\infty)$ with parameter α (see Section 3.1, Example V). Using the expression in Theorem 3,

$$\begin{aligned}
H^\gamma(\chi) &= \inf_{\phi\in\mathcal{V}^+_{0,\infty}} \left\{ J^{\mathrm{Exp}(\lambda)}(\phi) : \phi(t) = (1-t)^{-1/\alpha} - 1 \text{ for all } t\in[\gamma,1-\gamma] \right\} \\
&= \inf_{\psi\in\mathcal{V}^+_{0,1}} \left\{ J^U(\psi) : \psi(t) = 1 - e^{-\lambda((1-t)^{-1/\alpha}-1)} \text{ for all } t\in[\gamma,1-\gamma] \right\} \\
&= \inf_{\psi\in\mathcal{V}^+_{0,1}} \left\{ -\int_0^1 \log(\dot{\psi}^{(a)}(s))ds : \psi(\gamma) = 1 - e^{-\lambda((1-\gamma)^{-1/\alpha}-1)}, \right. \\
&\qquad \psi(1-\gamma) = 1 - e^{-\lambda(\gamma^{-1/\alpha}-1)}, \\
&\qquad \left. \dot{\psi}(t) = \frac{\lambda}{\alpha}(1-t)^{-\frac{(1+\alpha)}{\alpha}} e^{-\lambda((1-t)^{-1/\alpha}-1)} \text{ for all } t\in[\gamma,1-\gamma] \right\}.
\end{aligned}$$

Our aim is to show that the right hand side of this expression is finite for all $\alpha > 0$, demonstrating that the trimmed means of exponentially distributed random variables can emulate those of Pareto distributed random variables with infinite mean. Consider the following function that is defined by its derivative:

$$\dot{\psi}(t) = \begin{cases} \frac{1-\exp\left(-\lambda((1-\gamma)^{-1/\alpha}-1)\right)}{\gamma} & \text{if } t\in[0,\gamma) \\ \frac{\lambda}{\alpha}(1-t)^{-\frac{(1+\alpha)}{\alpha}} \exp\left(-\lambda((1-t)^{-1/\alpha}-1)\right) & \text{if } t\in[\gamma,1-\gamma) \\ \frac{\exp\left(-\lambda(\gamma^{-1/\alpha}-1)\right)}{\gamma} & \text{if } t\in[1-\gamma,1] \end{cases}.$$

As ψ meets the constraints in the infimum, we can upper bound $H^\gamma(\chi)$ by evaluating $J^U(\psi)$. If $\alpha\neq 1$,

$$\begin{aligned}
J^U(\psi) &= -\int_0^1 \log(\dot{\psi}(t))\,dt \\
&= 2\gamma\log\gamma - \gamma\log(1-\exp(-\lambda((1-\gamma)^{-1/\alpha}-1))) + \lambda\gamma(\gamma^{-1/\alpha}-1) \\
&\quad - (1-2\gamma)\log\left(\frac{\lambda}{\alpha}\right) + \frac{\alpha+1}{\alpha}(2\gamma-1+(1-\gamma)\log(1-\gamma)-\gamma\log\gamma) \\
&\quad - \lambda(1-2\gamma) + \frac{\lambda\alpha}{\alpha-1}\left((1-\gamma)^{(\alpha-1)/\alpha} - \gamma^{(\alpha-1)/\alpha}\right).
\end{aligned}$$

If $\alpha = 1$, the last term in is replaced with $\lambda(\log(1-\gamma)-\log(\gamma))$. For any $\lambda > 0$ and any $\gamma\in(0,1/2)$, this expression is finite for $\alpha\in(0,1)$, although growing quickly as $\alpha\to 0$. Thus the paths of trimmed means of exponential distributions can mimic those of a Pareto distribution with infinite mean, even though Example V in Section 3.1 shows that this is *not* possible for untrimmed means.

4.3 Hill Plots

Consider a sequence of i.i.d. random variables $\{X_n\}$ with common distribution function F supported on $[1,\infty)$. Given a sample set of order statistics $X_{1,n},\dots,X_{n,n}$ we wish to determine if the original distribution is ultimately a Pareto(α) law on $[1,\infty)$, i.e. if the distribution function is $F(x)=1-x^{-\alpha}$ for x sufficiently large. Hill's [13] widely-used methodology to answer this question employs the following empirical quantities: for each $1\le k\le n$, define

$$H_{k,n} := \frac{1}{k+1}\sum_{i=n-k+1}^{n}\log(X_{i,n}) - \frac{k}{k+1}\log(X_{n-k,n}). \tag{17}$$

The approach is based on the following observation: if it was indeed the case that X_i ultimately behaves as a Pareto(α) law, then $\log(X_i)$ is ultimately an exponentially distributed random variable with mean $1/\alpha$. Thus determining that the tail of F is a Pareto distribution function is equivalent to determining that the tail of the distribution function of $\log(X_i)$ is exponential. In Hill's methodology, one creates a "Hill Plot": for a contiguous range of small k, e.g $k\in\{[n/100],[n/100]+1,\dots,[n/10]\}$, one plots $H_{k,n}$ versus k. If the resulting plot is almost a straight line, one deduces that the tail of the distribution function of $\log(X_1)$ is exponential and thus F ultimately coincides with a Pareto law with a parameter given by one over the height of the line.

Due to its practical importance, much is known about properties of Hill's estimator (e.g. de Hann and Resnick [6], Resnick and Stărică [27], Drees, de Hann and Resnick [8], Segers [29], Haeusler and Segers [12] and references therein). Here, as an application of Theorem 3, by considering the sample paths of Hill's estimator, we prove the LDP for Hill Plots and use it to estimate the likelihood that a non-Pareto distribution is misidentified as being a Pareto distribution.

Consider the sample paths of Hill's estimator defined by $H_n(0):=0$ and, for $t\in(0,1]$,

$$\begin{aligned}
H_n(t) &:= \frac{1}{t}\int_{1-t}^{1}\log(X_n(s))\,ds - \log(X_n(1-t))\\
&= \frac{1}{(n+1)t}\sum_{i=[(n+1)(1-t)]+1}^{n}\log(X_{i,n})\\
&\quad - \frac{1}{(n+1)t}\,(n-[(n+1)(1-t)])\log(X_{[(n+1)(1-t)],n}).
\end{aligned}$$

For $k\in\{1,\dots,n\}$ we have

$$H_n((k+1)/(n+1)) = \frac{1}{k+1}\sum_{i=n-k+1}^{n}\log(X_{i,n}) - \frac{k}{k+1}\log(X_{n-k,n}) = H_{k,n},$$

so that $H_n(\cdot)$ is, indeed, the sample path of Hill's estimator with sample size n. That is, $H_n(\cdot)$ is the Hill Plot with a sample of size n.

The following theorem proves that Hill Plots satisfy the large deviation principle for i.i.d. random variables with a continuous increasing distribution function F that have bounded support or satisfy tail conditions. After the Theorem, we will show that these conditions are verified, for example, for any Weibull law, including those with heavier than exponential tails.

Theorem 7 (LDP for Hill Plots) *Assume the same hypotheses of Theorem 3 with $a \geq 1$. In addition, suppose $b < \infty$, or alternatively, $F(x)$ differentiable for all x sufficiently large, there exists $\beta \in (0,1)$ such that*

$$\lim_{\varepsilon \to 0} \varepsilon \log \left(1 - F(\exp(\varepsilon^{-\beta}))\right) = -\infty \tag{18}$$

and, defining the function $x_F(t) := F(\exp((1-t)^{-\beta}) + 1)$, for all t sufficiently close to 1, $x_F(t) > t$ and

$$\frac{d}{dt} x_F(t) \geq \frac{x_F(t)(1 - x_F(t))}{x_F(t) - t} \log \left(\frac{(1-t)x_F(t)}{t(1 - x_F(t))} \right). \tag{19}$$

Then $\{H_n(\cdot)\}$ satisfies the LDP in $D[0,1]$, equipped with the Skorohod M_1 topology with the rate function:

$$L^F(\chi) = \inf_{\phi \in \mathcal{V}^+_{a,b}} \left\{ J^F(\phi) : \frac{1}{t} \int_{1-t}^1 \log(\phi(s)) ds - \log(\phi(1-t)) = \chi(t) \text{ for all } t \in (0,1] \right\}.$$

Note that $L^F(\chi) = 0$ if $\chi(t) = t^{-1} \int_{1-t}^1 \log(F^{-1}(s)) ds - \log(F^{-1}(1-t))$.

Proof: Define the function $h : \mathcal{V}^+_{a,b} \mapsto D[0,1]$ by $h(\chi)(0) := 0$ and

$$h(\chi)(t) = \frac{1}{t} \int_{1-t}^1 \log(\chi(s))\, ds - \log(\chi(1-t)).$$

To prove the LDP for $\{h(X_n)(\cdot)\}$ we apply extensions of the contraction principle with h after noting the following. The function h can be written as $h = h_4 \circ h_3 \circ h_2 \circ h_1$, where

$$h_1(\chi)(t) = (\chi(t), \chi(1-t)),\ h_2(\chi, \psi)(t) = (\log \chi(t), \log \psi(t)),$$
$$h_3(\chi, \psi)(t) = \left(\int_{1-t}^1 \chi(s) ds, \psi(t) \right) \text{ and } h_4(\chi, \psi)(t) = \frac{1}{t}\chi(t) - \psi(t).$$

The function h_1 is continuous by arguments analogous to those in Theorem 8.1 [36], while h_2, using the continuity of $\log(\cdot)$, is continuous by Theorem 13.2.3 [37]. For continuous χ, the function h_4 is continuous by Corollary 12.7.1 [37]. If $b < \infty$, then we can appeal to Theorem 11.5.1 [37] to deduce the continuity of h_3 and the result follows from an application of the contraction principle. However, if $b = +\infty$, the function h_3 is not continuous. In this case, if the second set of additional conditions in the statement of the theorem holds, then we will show that an approximate version of the contraction principle can be employed.

Consider the first component of the function $h_3 \circ h_2 \circ h_1$. That is, g defined by

$$g(\chi)(t) = \int_{1-t}^{1} \log \chi(s) ds.$$

If $\chi(1) = +\infty$, then we cannot appeal to Theorem 11.5.1 [37] to deduce the continuity of g, as this theorem holds only if χ is real valued. Instead we consider the family of functions $\{g_\varepsilon : \varepsilon > 0\}$ defined by

$$g_\varepsilon(\chi)(t) = \int_{1-t}^{1-\varepsilon} \log \chi(s) ds$$

that approximate the behavior of $g(\chi)$. By similar logic to that in Theorem 6, for any $\varepsilon > 0$ the function g_ε is continuous at all χ such that $J^F(\chi) < \infty$, so that Theorem 2 can be applied, obtaining the LDP for $\{g_\varepsilon(X_n)\}$.

Thus we will show that $\{g(X_n)\}$ satisfies the LDP by applying the approximate contraction principle Theorem 4.2.23 [7]. This approach requires $\{g_\varepsilon(X_n)\}$ to be exponentially good approximations of $\{g(X_n)\}$,

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P\left(\int_{1-\varepsilon}^{1} \log(X_n(s))\, ds \geq \delta\right) = -\infty, \tag{20}$$

as well as the verification of equation (4.2.24) [7] for which it suffices to prove that

$$\limsup_{\varepsilon\to 0} \sup_{\{\chi\in\mathcal{V}^+_{a,b}: J^F(\chi)\leq\alpha\}} \int_{1-\varepsilon}^{1} \log(\chi(s))\, ds = 0, \text{ for every } \alpha \in (0,\infty). \tag{21}$$

Given $\delta > 0$, recalling that $\beta \in (0,1)$, choose $\varepsilon_\delta > 0$ such that $\varepsilon^{1-\beta}/(1-\beta) + \varepsilon < \delta$ for all $0 < \varepsilon \leq \varepsilon_\delta$. Then with $\chi \in \mathcal{V}^+_{a,b}$,

$$\begin{aligned}
\left\{\chi : \int_{1-\varepsilon}^{1} \log(\chi(s))\, ds \geq \delta\right\} &\subset \left\{\chi : \int_{1-\varepsilon}^{1} \log(\chi(s))\, ds > \frac{\varepsilon^{1-\beta}}{1-\beta} + \varepsilon\right\} \\
&= \left\{\chi : \int_{1-\varepsilon}^{1} \log(\chi(s))\, ds > \int_{1-\varepsilon}^{1} \left(\log\left(\exp((1-s)^{-\beta})\right) + 1\right) ds\right\} \\
&\subset \left\{\chi : \int_{1-\varepsilon}^{1} \log\left(\frac{\chi(s)}{\exp((1-s)^{-\beta}) + 1}\right) ds > 0\right\} \\
&\subset \left\{\chi : \sup_{t\in[1-\varepsilon,1]} \left(\chi(t) - \exp((1-t)^{-\beta})\right) \geq 1\right\} \\
&=: A_\varepsilon.
\end{aligned}$$

We can apply the large deviations upper bound on the closure of A_ε, $\bar{A}_\varepsilon$, to obtain, for any $\rho > \varepsilon$,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log P\left(\int_{1-\varepsilon}^{1} \log(X_n(s))\, ds \geq \delta\right) &\leq \limsup_{n\to\infty} \frac{1}{n} \log P\left(X_n \in A_\varepsilon\right) \\
&\leq - \inf_{\chi\in\mathcal{V}_{a,b}^{+}} \{J^F(\chi) : \chi \in \bar{A}_\varepsilon\} \\
&\leq - \inf_{t\in[1-\rho,1]} \inf_{\chi\in\mathcal{V}_{a,b}^{+}} \left\{J^F(\chi) : \chi(t) \geq \exp((1-t)^{-\beta}) + 1\right\} \\
&= - \inf_{t\in[1-\rho,1]} \inf_{\phi\in\mathcal{V}_{0,1}^{+}} \left\{- \int_0^1 \log(\dot{\phi}^{(a)}(s))ds : \phi(t) \geq x_F(t)\right\}.
\end{aligned}$$

Using Jensen's inequality, for $x \geq t$, we have

$$\inf_{\phi\in\mathcal{V}_{0,1}^{+}} \left\{- \int_0^1 \log(\dot{\phi}^{(a)}(s))\, ds : \phi(t) \geq x\right\} = -t \log\left(\frac{x}{t}\right) - (1-t)\log\left(\frac{1-x}{1-t}\right),$$

and this infimum is attained at $\phi(s) := sx/t$ if $s < t$ and $\phi(s) := x + (s-t)(1-x)/(1-t)$ if $s \in [t,1]$. Let $\rho \in (0,1)$ be sufficiently small so that, for all $t \in [1-\rho, 1]$, $x_F(t) > t$ and inequality (19) holds. Then the function

$$t \to -t \log\left(\frac{x_F(t)}{t}\right) - (1-t)\log\left(\frac{1-x_F(t)}{1-t}\right), \quad t \in [1-\rho, 1]$$

is increasing. Thus, we have

$$\begin{aligned}
\inf\{J^F(\chi) : \chi \in \bar{A}_\varepsilon\} &\geq \inf_{t\in[1-\rho,1]} \inf_{\phi\in\mathcal{V}_{0,1}^{+}} \left\{- \int_0^1 \log(\dot{\phi}^{(a)}(s))ds : \phi(t) \geq x_F(t)\right\} \\
&= \inf_{t\in[1-\rho,1]} \left(-t\log\left(\frac{x_F(t)}{t}\right) - (1-t)\log\left(\frac{1-x_F(t)}{1-t}\right)\right) \\
&\geq -(1-\rho)\log\left(\frac{F(\exp(\rho^{-\beta})+1)}{1-\rho}\right) - \rho\log\left(\frac{1-F(\exp(\rho^{-\beta})+1)}{\rho}\right),
\end{aligned}$$

and this latter term tends to $+\infty$ as $\rho \to 0$ by assumption (18). So equation (20) is satisfied and the sequences $\{g_\varepsilon(X_n)\}$ are exponentially good approximations of $\{g(X_n)\}$.

To establish (21), reasoning by contradiction, assume that there exists $\delta > 0$ and a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$ and

$$\sup_{\{\chi\in\mathcal{V}_{a,b}^{+} : J^F(\chi)\leq\alpha\}} \int_{1-\varepsilon_n}^{1} \log(\chi(s))\, ds \geq \delta.$$

The function $\chi \to \int_{1-\varepsilon}^{1} \log(\chi(s))\, ds$ is continuous for any $\varepsilon > 0$. Therefore, by the goodness of J^F, there exist $\chi_{\varepsilon_n,\alpha}$ which attains the supremum and $J^F(\chi_{\varepsilon_n,\alpha}) \leq \alpha$. Thus we have a contradiction because

$$\alpha \geq J^F(\chi_{\varepsilon_n,\alpha}) \geq -(1-\varepsilon_n)\log\left(\frac{F(\exp(\varepsilon_n^{-\beta})+1)}{1-\varepsilon_n}\right) - \varepsilon_n \log\left(\frac{1-F(\exp(\varepsilon_n^{-\beta})+1)}{\varepsilon_n}\right)$$

and, using the hypothesis in equation (18), this final term tends to $+\infty$ as $n \to \infty$.

■

Example IX: Every Law that is ultimately Weibull satisfies the conditions of Theorem 7. Consider a law that is ultimately Weibull, $F(x) = 1 - e^{-x^\alpha}$ for some $\alpha > 0$ and for all x sufficiently large. For any $\beta \in (0,1)$,

$$\lim_{\varepsilon \to 0} \varepsilon \log\left(1 - F(\exp(\varepsilon^{-\beta}))\right) = -\lim_{\varepsilon \to 0} \varepsilon \exp\left(\alpha \varepsilon^{-\beta}\right) = -\infty.$$

and thus equation (18) is satisfied. Define

$$x_F(t) = F(l(t))$$

where $l(t) = \exp((1-t)^{-\beta}) + 1$ for some $\beta \in (0,1)$. It is easy to check that $x_F(t) > t$ for all t sufficiently close to 1. Equation (19) is equivalent to

$$\alpha\beta(1-t)^{-(\beta+1)}(l(t)-1)(l(t))^{\alpha-1} \geq \frac{1-e^{-(l(t))^\alpha}}{1-e^{-(l(t))^\alpha}-t} \log\left(\frac{1-t}{t}\frac{1-e^{-(l(t))^\alpha}}{e^{-(l(t))^\alpha}}\right) \tag{22}$$

for all t sufficiently close to 1. Equation (22) holds if we can show that

$$\lim_{t \to 1} \frac{\alpha\beta(1-t)^{-(\beta+1)}(l(t)-1)(l(t))^{\alpha-1}(1-e^{-(l(t))^\alpha}-t)}{(1-e^{-(l(t))^\alpha})\log\left(\frac{1-t}{t}\frac{1-e^{-(l(t))^\alpha}}{e^{-(l(t))^\alpha}}\right)} = +\infty.$$

For this note that, for all t close to 1 we have that

$$\begin{aligned}(1-e^{-(l(t))^\alpha})\log\left(\frac{1-t}{t}\frac{1-e^{-(l(t))^\alpha}}{e^{-(l(t))^\alpha}}\right) &\leq (1-e^{-(l(t))^\alpha})\log(e^{(l(t))^\alpha}-1)\\ &\leq (1-e^{-(l(t))^\alpha})(l(t))^\alpha\end{aligned}$$

and so

$$\begin{aligned}&\frac{\alpha\beta(1-t)^{-(\beta+1)}(l(t)-1)(l(t))^{\alpha-1}(1-e^{-(l(t))^\alpha}-t)}{(1-e^{-(l(t))^\alpha})\log\left(\frac{1-t}{t}\frac{1-e^{-(l(t))^\alpha}}{e^{-(l(t))^\alpha}}\right)}\\ &\geq \frac{\alpha\beta(1-t)^{-(\beta+1)}(l(t)-1)(1-e^{-(l(t))^\alpha}-t)}{(1-e^{-(l(t))^\alpha})l(t)}.\end{aligned} \tag{23}$$

The claim follows noticing that the term in (23) goes to $+\infty$ as $t \to 1$ because it is asymptotically equivalent to $\alpha\beta(1-t)^{-\beta}$.

Example X: Truncated-Pareto emulating Pareto. As an application of Theorem 7, we determine estimates on the likelihood that the Hill Plot misclassifies the distribution function F as having Pareto tails when it does not. For certain financial objects it has been suggested that while on short time scales fluctuations in value are large, in the longer term they are not, see e.g. Mantegna and Stanley [18]. Similar observations have been made in ground-water hydrology, atmospheric science and many other fields; for examples see Aban, Meerschaert and Panorska [1] and references therein. This has led to the proposal of, e.g., financial market models based on random walks whose increments have apparent power-tail behavior near the center of their support, but whose tails decay at least as fast as an exponential distribution. Truncated Lévy distributions have been used with either a sudden truncation [18] or a transition to an exponential distribution beyond a given cut-off [14]. Similarly, truncated-Pareto distributions have also been proposed. Consider a truncated Pareto distribution with parameter $\gamma > 0$ supported on $[1, K)$ that changes into an exponential distribution on $[K, \infty)$ with rate λ:

$$F(x) = \begin{cases} 1 - x^{-\gamma} & \text{if } x \in [1, K) \\ 1 - K^{-\gamma} e^{-\lambda(x-K)} & \text{if } x \in [K, \infty) \end{cases}$$

and therefore its quantile function is:

$$F^{-1}(u) = \begin{cases} (1-u)^{-1/\gamma} & \text{if } u \in [0, 1-K^{-\gamma}) \\ K - \lambda^{-1} \log(K^{\gamma}(1-u)) & \text{if } u \in [1-K^{-\gamma}, 1] \end{cases}.$$

By the preceding example, the conditions of Theorem 7 are met.

Consider $L^F(\hat{\chi})$ where $\hat{\chi}$ corresponds to the Hill Plot of the Pareto(α) distribution. Then, for the quantile function $\hat{\phi}(s) = (1-s)^{-1/\alpha}$, we have

$$\hat{\chi}(t) = \frac{1}{t} \int_{1-t}^{1} \log(\hat{\phi}(s))\, ds - \log(\hat{\phi}(1-t)) = \frac{1}{\alpha}.$$

Referring to Corollary 4, the function f is defined by $f(x) := \dot{F}(x) = \gamma x^{-\gamma-1}$ if $x \in [1, K)$ and $f(x) := \dot{F}(x) = \lambda K^{-\gamma} \exp(-\lambda(x-K))$ if $x \in (K, \infty)$. Thus by the expression of L^F in Theorem 7 and (9) we have

$$\begin{aligned} L^F(\hat{\chi}) &\leq J^F(\hat{\phi}) \\ &= -\int_0^1 \left(\log(f(\hat{\phi}(t)) + \log(\dot{\hat{\phi}}(t)) \right) dt = -\int_0^{1-K^{-\alpha}} \log(\gamma((1-t)^{-1/\alpha})^{-\gamma-1})\, dt \\ &\quad - \int_{1-K^{-\alpha}}^1 \log(\lambda K^{-\gamma} \exp(-\lambda((1-t)^{-1/\alpha} - K)))\, dt - \int_0^1 \log\left(\frac{(1-t)^{-1/\alpha - 1}}{\alpha} \right) dt. \end{aligned}$$

The second term in this equation, corresponding to the exponential part of the distribution emulating the quantile function of a Pareto(α) law, leads the integral to be infinite if $\alpha \in (0, 1]$ and finite if $\alpha > 1$. That is, if the real distribution is a Pareto(γ) distribution truncated by an Exponential(λ), then with finite rate one can observe a Pareto(α) Hill Plot so long as $\alpha > 1$.

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