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Unconditional stability of the Foschini-Miljanic algorithm

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Abstract

In this note we prove the unconditional stability of the Foschini-Miljanic algorithm. Our results show that the Foschini-Miljanic algorithm is unconditionally stable (convergent) even in the presence of bounded time-varying communication delays, and in the presence of topology changes. The implication of our results may be important for the design of Code Division Multiple Access (CDMA) based wireless networks.

Key words: power control; changing topology; time-varying delays; stability; positive systems

1 Introduction

This study concentrates on distributed power control in *Code Division Multiple Access* (CDMA) based wireless communication networks. Some CDMA based power control algorithms aim to assign power to wireless nodes in a distributed fashion, while guaranteeing a certain *Quality of Service* (QoS). In real communication systems, especially ad-hoc networks, distributed algorithms require communication among the nodes. But processing time (coding and decoding), propagation delays and waiting for availability of channels for transmission all introduce delays into the network. Additionally, the nodes may be mobile, entering or leaving the network, causing the network topology to change constantly. Hence, any stability analysis of distributed algorithms for such realistic situations should consider time-delays in the network and changing network topologies.

The authors in [3] proposed a power control algorithm, the now well known Foschini-Miljanic (FM) algorithm, that provides distributed on-line power control of wireless networks with user-specific *Signal-to-Interference-and-Noise-Ratio* (SINR) requirements. Furthermore, this algorithm yields the minimum transmitter powers that satisfy these requirements.

It was shown in [1] this algorithm is globally asymptotically stable for arbitrarily large but *constant* time-delays. On the other hand, the effects of changing network topologies were studied in [2], where it was derived that the stability condition is purely deterministic and equivalent to the joint spectral radius of the set of switching matrices being less than one. In this work, making use of recent advancements in positive linear systems [7,8,9], we consider both the effects of time-varying delays and changing network topologies. For that we provide a new theoretical result concerning the stability of such systems, which we then use to show that the Foschini-Miljanic algorithm is globally asymptotically stable even under those harder, more realistic conditions. Our results are of practical importance when designing wireless networks in changing environments, as is typically the case for CDMA networks.

The remainder of the paper is structured as follows: After some notational conventions, we provide in Section 2 some mathematical preliminaries including our main theoretical contribution (Theorem 1). The next two sections describe the channel model used as well as the FM algorithm. In Section 5, Theorem 1 is used to derive two stability conditions (Theorems 3 and 4) for the FM algorithm. Finally, two examples as well as concluding remarks are given in Sections 6 and 7, respectively.

Notation The set of real numbers is denoted by $\mathbb{R}$ and the positive orthant of the n-dimensional Euclidean space by $\mathbb{R}^n_+$. The zero vector or matrix (of appropriate dimension) is denoted by $\mathbf{0}$; the identity matrix by $\mathbf{I}$. A

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matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is called *Hurwitz* if its spectrum lies in the left open half-plane; *Metzler* if all its off-diagonal elements are non-negative; its spectral radius is denoted by $\rho(\boldsymbol{A})$. $\boldsymbol{A} \preceq \boldsymbol{B}$ ($\boldsymbol{A} \prec \boldsymbol{B}$) is the element-wise (strict) inequality between matrices or vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. $\boldsymbol{A}$ is non-negative (positive) if and only if $\boldsymbol{A} \succeq \boldsymbol{0}$ ($\boldsymbol{A} \succ \boldsymbol{0}$).

2 Mathematical preliminaries

In what follows, we give some useful results on positive systems that are needed to prove our later results. First, consider the following linear system with m different delayed states whose time-delays are time-varying:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \textstyle\sum_{k=1}^{m} \boldsymbol{B}_k \boldsymbol{x}(t-\tau_k(t)), \qquad t \geq 0 \quad \text{(1a)}$$
$$\boldsymbol{x}(t) = \boldsymbol{\varphi}(t) \succeq \boldsymbol{0}, \qquad t \in [-\bar{\tau}, 0] \quad \text{(1b)}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{x}(t) \succeq \boldsymbol{0}$, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is a Metzler matrix, $\boldsymbol{B}_k \in \mathbb{R}^{n\times n}$ are non-negative matrices for all $k = 1,\ldots,m$, $\boldsymbol{\varphi}(\cdot)$ is a locally Lebesgue integrable vector function and the delays $\tau_k(t)$ are assumed to satisfy:

Assumption 1 *The time-varying time-delays given by $\tau_k(t)$ are Lebesgue measurable in t and bounded, satisfying $0 \leq \tau_k(t) \leq \bar{\tau}_k \leq \bar{\tau}$ for all $t \geq 0$ and each k, and where $\bar{\tau} = \max\{\bar{\tau}_k\}$.*

A dynamical system is said to be *positive* if its state trajectories remain in the positive orthant for all $t \geq 0$ (provided that the initial condition is positive). Thanks to $\boldsymbol{A}$ being Metzler and the $\boldsymbol{B}_k$ being non-negative, the system above is indeed positive, see [10].

Lemma 1 *Let $\boldsymbol{x}_a(t)$ resp. $\boldsymbol{x}_b(t)$, $t \geq 0$, be the solution trajectories of the system (1) under the initial conditions $\boldsymbol{\varphi}_a(t)$ resp. $\boldsymbol{\varphi}_b(t)$ for $t \in [-\bar{\tau}, 0]$. Then $\boldsymbol{\varphi}_a(t) \preceq \boldsymbol{\varphi}_b(t)$ implies $\boldsymbol{x}_a(t) \preceq \boldsymbol{x}_b(t)$, $t \geq 0$.*

Proof This can be shown by considering $\boldsymbol{\varphi}(t) := \boldsymbol{\varphi}_a(t) - \boldsymbol{\varphi}_b(t) \succeq \boldsymbol{0}$ as an initial condition of (1) and utilising its linearity and the positivity properties. □

Now, closely related to (1) is the following system based on constant delays, namely the upper bounds $\bar{\tau}_k$:

$$\dot{\boldsymbol{y}}(t) = \boldsymbol{A}\boldsymbol{y}(t) + \textstyle\sum_{k=1}^{m} \boldsymbol{B}_k \boldsymbol{y}(t-\bar{\tau}_k), \qquad t \geq 0 \quad \text{(2a)}$$
$$\boldsymbol{y}(t) = \boldsymbol{\psi}(t) \succeq \boldsymbol{0}, \qquad t \in [-\bar{\tau}, 0] \quad \text{(2b)}$$

Lemma 2 *[4] System (2) is asymptotically stable if and only if there exists a vector $\boldsymbol{c} \succ \boldsymbol{0}$ satisfying*

$$\boldsymbol{c}^{\mathsf{T}} \left(\boldsymbol{A} + \textstyle\sum_{k=1}^{m} \boldsymbol{B}_k\right) \prec \boldsymbol{0} \qquad (3)$$

It is well known that condition (3) holds if and only if the matrix $(\boldsymbol{A} + \sum_{k=1}^{m} \boldsymbol{B}_k)$ is Hurwitz, see [6]. The next lemma gives a useful monotonicity property of (2) if its initial condition is constant and a multiple of a vector $\boldsymbol{c} \succ \boldsymbol{0}$ satisfying (3):

Lemma 3 *[9] Consider system (2). Suppose that there exists a positive vector $\boldsymbol{c} \succ \boldsymbol{0}$ satisfying (3) and that the initial condition is the constant function $\boldsymbol{\psi}(t) \equiv \boldsymbol{c}$ for $t \in [-\bar{\tau}, 0]$. Then the solution $\boldsymbol{y}(t)$ to (2) is strictly monotonically decreasing, that is $\dot{\boldsymbol{y}}(t) \prec \boldsymbol{0}$ for all $t \geq 0$, and it converges asymptotically to zero.*

The last lemma guarantees an ordering between the solutions of systems (1) and (2), provided they use the same matrices $\boldsymbol{A}$ and $\boldsymbol{B}_k$ and the same constant initial condition based on a vector $\boldsymbol{c} \succ \boldsymbol{0}$ satisfying (3).

Lemma 4 *[10] Suppose that there exists a vector $\boldsymbol{c} \succ \boldsymbol{0}$ satisfying (3), and that the initial conditions for system (1) resp. (2) are $\boldsymbol{\varphi}(t) \equiv \boldsymbol{c}$ resp. $\boldsymbol{\psi}(t) \equiv \boldsymbol{c}$, $t \in [-\bar{\tau}, 0]$. Then, for all $t \geq 0$, it holds that $\boldsymbol{x}(t) \preceq \boldsymbol{y}(t)$, where $\boldsymbol{x}(t)$ resp. $\boldsymbol{y}(t)$ are solutions to (1) resp. (2).*

We can now present a useful result on switched positive systems with time-varying time-delays. Given M constituent subsystems or *modes*, we make the common assumption that the switching instants are defined in all the real time axes and that $\inf_k(t_{k+1} - t_k) > 0$, where t_{k+1} and t_k are two consecutive switching instants, so that the switching rule has no accumulation points.

The following theorem states that the existence of a common linear co-positive Lyapunov function $v(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$, $\boldsymbol{c} \succ \boldsymbol{0}$, for all un-delayed modes of the system is sufficient to guarantee the asymptotic stability of the system for bounded time-varying delays and arbitrary switching.

Theorem 1 *Consider the switched positive system with time-varying time-delays for $t \geq 0$*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_{\sigma(t)}\boldsymbol{x}(t) + \textstyle\sum_{k=1}^{m} \boldsymbol{B}_{k,\sigma(t)} \boldsymbol{x}(t-\tau_k(t)) \quad \text{(4a)}$$
$$\boldsymbol{x}(t) = \boldsymbol{\varphi}(t) \succeq \boldsymbol{0}, \qquad t \in [-\bar{\tau}, 0] \quad \text{(4b)}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n_+$, $\boldsymbol{x}(t) \succeq \boldsymbol{0}$, $\sigma : \mathbb{R} \to \{1,\ldots,M\}$ is some (piecewise constant and left-continuous) switching signal (defined in all the real time axes and with $\inf_k(t_{k+1} - t_k) > 0$, where t_{k+1} and t_k are two consecutive switching instants), $\boldsymbol{A}_i \in \mathbb{R}^{n\times n}$ are Metzler and $\boldsymbol{B}_{k,i} \in \mathbb{R}^{n\times n}$ are non-negative matrices, $i = 1,\ldots,M$, and the delays $\tau_k(t)$ are assumed to satisfy Assumption 1. If there exists a vector $\boldsymbol{c} \succ \boldsymbol{0}$ such that

$$\boldsymbol{c}^{\mathsf{T}} \left(\boldsymbol{A}_i + \textstyle\sum_{k=1}^{m} \boldsymbol{B}_{k,i}\right) \prec \boldsymbol{0}, \quad \forall i = 1,\ldots,M \qquad (5)$$

then system (4) is asymptotically stable.

Proof Consider system (4) with its initial condition $\boldsymbol{\varphi}(t)$. If there exists a positive vector $\boldsymbol{c} \succ \boldsymbol{0}$ such that (5) is satisfied, it is always possible to pick a positive constant $\gamma > 0$ so that $\boldsymbol{\varphi}(t) \prec \gamma\boldsymbol{c}$ for all $t \in [-\bar{\tau}, 0]$. Naturally, $\gamma\boldsymbol{c}$ will still satisfy condition (5).

Denote by $\boldsymbol{x}^{(\boldsymbol{\varphi}(t))}(t)$ the solution of system (4) with initial condition $\boldsymbol{\varphi}(t)$, $t \in [-\bar{\tau}, 0]$ with whichever initial mode $\sigma(0) \in \{1, \ldots, M\}$ active. Further, let $\boldsymbol{y}^{(\boldsymbol{\psi}(t))}$ be the solution of system (2) starting from the constant initial condition $\boldsymbol{\psi}(t) \equiv \gamma \boldsymbol{c}$, $t \in [-\bar{\tau}, 0]$.

Since, by construction, $\boldsymbol{\varphi}(t) \prec \boldsymbol{\psi}(t)$, we can apply Lemma 1 to find that before the first switching instant $\boldsymbol{x}^{(\boldsymbol{\varphi}(t))}(t) \preceq \boldsymbol{x}^{(\boldsymbol{\psi}(t))}(t)$. But using Lemma 4, we also get $\boldsymbol{x}^{(\boldsymbol{\psi}(t))}(t) \preceq \boldsymbol{y}^{(\boldsymbol{\psi}(t))}(t)$ so that, in the end

$$\boldsymbol{x}^{(\boldsymbol{\varphi}(t))}(t) \preceq \boldsymbol{y}^{(\boldsymbol{\psi}(t))}(t) \tag{6}$$

Furthermore, we note that $\boldsymbol{y}^{(\boldsymbol{\psi}(t))}(t)$ is strictly monotonically decreasing, thanks to Lemma 3.

Now, let t_k be the switching instants, so that t_1 is the time instant when the first switching occurs (and where the system switches to mode $\sigma(t_1) \in \{1, \ldots, M\}$). We know by the monotonicity argument above that $\boldsymbol{x}(t) \prec \gamma \boldsymbol{c}$ for $t \in [t_1 - \bar{\tau}, t_1]$, hence there exists some $0 < \mu_1 < 1$ such that $\boldsymbol{x}(t) \preceq \mu_1 \gamma \boldsymbol{c}$ for $t \in [t_1 - \bar{\tau}, t_1]$. Since μ_1 is positive, $\mu_1 \gamma \boldsymbol{c}$ will still satisfy condition (5).

Consider now this new initial condition for system (4): $\boldsymbol{\varphi}_1(t) = \boldsymbol{x}(t)$ for $t \in [t_1 - \bar{\tau}, t_1]$. Using an argument as above it follows again that $\boldsymbol{x}^{(\boldsymbol{\varphi}_1(t))}(t) \preceq \boldsymbol{x}^{(\mu_1 \gamma \boldsymbol{c})}(t)$, $t \in [t_1 - \bar{\tau}, t_1]$, and also that $\boldsymbol{x}^{(\mu_1 \gamma \boldsymbol{c})}(t) \preceq \boldsymbol{y}^{(\mu_1 \gamma \boldsymbol{c})}(t)$, where $\boldsymbol{y}^{(\mu_1 \gamma \boldsymbol{c})}(t)$ is strictly monotonically decreasing. Hence, at the next switching instant t_2, we can define again a new parameter $0 < \mu_2 < 1$ such that $\boldsymbol{x}(t) \preceq \mu_2 \mu_1 \gamma \boldsymbol{c}$ for $t \in [t_2 - \bar{\tau}, t_2]$, and so on.

Therefore, whenever system (4) switches, the new active mode starts from an initial condition that can be upper bounded by a constant initial condition which satisfies condition (5). So the real solution can always be upper bounded by a strictly decreasing function starting from that constant initial condition.

At the k^{th} switching time we can thus write

$$\boldsymbol{x}(t) \preceq \gamma\, \boldsymbol{c} \textstyle\prod_{i=1}^{k-1} \mu_i, \qquad t \in [t_k - \bar{\tau}, t_k] \tag{7}$$

where $0 < \mu_i < 1$ for all $i = 1 \ldots, k-1$. To finish the proof, we now just need to show that $\prod_{i=1}^{k-1} \mu_i \longrightarrow 0$ as k grows, since this implies that the solution of system (4) must also converge to zero.

Consider system (2) as describing the switched system between two consecutive switching instants, during which it is in some mode $q \in \{1, \ldots, M\}$. According to Lemma 3, system (2) it is monotonically decreasing if its initial condition is constant and satisfies (3). Since this is indeed the case here (independently of the mode q the system is in) we know that $\dot{\boldsymbol{y}}(t) \prec \boldsymbol{0}$ for all $t \geq 0$. In fact, $\boldsymbol{y}(t)$ is exponentially decreasing: To see this, consider $\tilde{\boldsymbol{y}}(t) = \mathrm{e}^{\varepsilon_q t} \boldsymbol{y}(t)$ for some sufficiently small $\varepsilon_q > 0$. Differentiating with respect to t yields

$$\dot{\tilde{\boldsymbol{y}}}(t) = \mathrm{e}^{\varepsilon_q t}\big(\varepsilon_q \boldsymbol{y}(t) + \dot{\boldsymbol{y}}(t)\big) \tag{8}$$

$$= (\varepsilon_q \mathbf{I} + \boldsymbol{A})\tilde{\boldsymbol{y}}(t) + \textstyle\sum_{k=1}^{m} \boldsymbol{B}_k \mathrm{e}^{\varepsilon_q \bar{\tau}_k} \tilde{\boldsymbol{y}}(t - \bar{\tau}_k) \tag{9}$$

If there exists $\boldsymbol{c} \succ \boldsymbol{0}$ such that $\boldsymbol{c}^{\mathsf{T}}(\boldsymbol{A} + \sum_{k=1}^{m} \boldsymbol{B}_k) \prec \boldsymbol{0}$, then one can use a limiting argument to show that, for ε_q small enough, $\boldsymbol{c}^{\mathsf{T}}(\varepsilon_q \mathbf{I} + \boldsymbol{A} + \sum_{k=1}^{m} \boldsymbol{B}_k \mathrm{e}^{\varepsilon_q \bar{\tau}_k})$ will still be negative. Recalling Lemma 2 this implies that $\dot{\tilde{\boldsymbol{y}}}(t)$ is asymptotically stable, and thus $\boldsymbol{y}(t) = \mathrm{e}^{-\varepsilon_q t} \tilde{\boldsymbol{y}}(t)$ must be exponentially decreasing.

Hence, the solution of (4) is upper bounded, between each two consecutive switching instants, by one of M functions that decrease exponentially with a rate of at least $-\varepsilon_q$ where $q \in \{1, \ldots, M\}$. Now, consider the first switching instant t_1: we have $\boldsymbol{x}^{(\boldsymbol{\varphi}(t_0))}(t) \preceq \mu_1 \gamma \boldsymbol{c}$, $t \in [t_1 - \bar{\tau}, t_1]$, and, since $\boldsymbol{y}(t)$ is exponentially decreasing, $\boldsymbol{y}^{(\gamma \boldsymbol{c})}(t) = e^{-\varepsilon_q (t_1 - t_0)} \gamma \boldsymbol{c}$, with $q \in \{1, \ldots, M\}$, $t \in [t_1 - \bar{\tau}, t_1]$. Moreover, as already said, $\boldsymbol{x}^{(\boldsymbol{\varphi}(t_0))}(t) \prec \boldsymbol{y}^{(\gamma \boldsymbol{c})}(t)$, hence there always exists a $t_x \in (t_0, t_1]$, such that $\mu_1 \gamma \boldsymbol{c} \preceq e^{-\varepsilon_q (t_x - t_0)} \gamma \boldsymbol{c}$. Calling $\lambda_q = \varepsilon_q \frac{t_x - t_0}{t_1 - t_0}$, we can say that $\mu_1 \gamma \boldsymbol{c} \preceq e^{-\lambda_q (t_1 - t_0)} \gamma \boldsymbol{c}$, with $0 < \lambda_q \leq \varepsilon_q$. Finally, we can write $\mu_1 \preceq e^{-\lambda_q (t_1 - t_0)}$. Defining $\Delta t_i := t_{i+1} - t_i$ for each $i = 0, 1, 2, \ldots$ as the time between two consecutive switching instants, then

$$\textstyle\prod_{i=1}^{k} \mu_i \leq \prod_{i=1}^{k} \mathrm{e}^{-\lambda_{\sigma(t_i)} \Delta t_i}, \tag{10}$$

or, applying the natural logarithm to both sides and taking the limit (as t grows),

$$\lim_{k \to \infty} \ln \textstyle\prod_{i=1}^{k} \mu_i \leq \lim_{k \to \infty} \sum_{i=1}^{k} -\lambda_{\sigma(t_i)} \Delta t_i = -\infty$$

since the Δt_i are uniformly lower bounded away from zero, and there are only finitely many positive $\lambda_{\sigma(t_i)}$. This implies $\prod_{j=1}^{k} \mu_j \longrightarrow 0$ as k grows (since $\mu_j > 0$ for all $j \geq 0$). Asymptotic stability of the switched positive system now follows from (7), which completes the proof. □

Comment At this point we have shown that, roughly speaking, if a common linear co-positive Lyapunov for the switched system (4) exists, then the system will be stable for any (bounded) time-varying time-delays, and independently of the switching sequence. The question now would be how to check for the existence of such Lyapunov function, and this will be addressed now.

Theorem 4 in [7] provides a (necessary and sufficient) test for the existence of a common linear co-positive Lyapunov function. For that, define $\mathcal{S}_{n,M}$ for positive integers n and M as the set containing all possible mappings $s : \{1, \ldots, n\} \to \{1, \ldots, M\}$. Given M matrices $\boldsymbol{A}_j$, these mappings are used to construct matrices $\boldsymbol{A}_s(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_M)$ of the form $\boldsymbol{A}_s(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_M) := \big[\, \boldsymbol{A}_{s(1)}^{(1)} \ \ldots \ \boldsymbol{A}_{s(n)}^{(n)} \big]$ where the ith column $\boldsymbol{A}_s^{(i)}$ of $\boldsymbol{A}_s$ is the ith column of one of the $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_M$ matrices, depending on $s \in \mathcal{S}_{n,M}$.

Theorem 2 *[7] Given M Metzler matrices $\boldsymbol{A}_i$ and $m \cdot M$ non-negative matrices $\boldsymbol{B}_{k,i}$, then there exists a positive vector $\boldsymbol{c} \succ \boldsymbol{0}$ such that $\boldsymbol{c}^\mathsf{T}\left(\boldsymbol{A}_i + \sum_{k=1}^m \boldsymbol{B}_{k,i}\right) =: \boldsymbol{c}^\mathsf{T}\tilde{\boldsymbol{A}}_i \prec \boldsymbol{0}\ \forall i = 1, \ldots, M$ if and only if $\tilde{\boldsymbol{A}}_s(\tilde{\boldsymbol{A}}_1, \ldots, \tilde{\boldsymbol{A}}_M)$ is Hurwitz for all $s \in \mathcal{S}_{n,M}$.*

3 Channel Model

We consider a network in which the links are unidirectional and each node is supported by an omnidirectional antenna. The link quality is measured by the Signal-to-Interference-and-Noise-Ratio (SINR). Let $\mathcal{N}$ and $\mathcal{R}$ denote all transmitters and receivers in the network, respectively. In a network with n communication pairs ($n = |\mathcal{N}| = |\mathcal{R}|$), the channel gain on the link between transmitter $i \in \mathcal{N}$ and receiver $j \in \mathcal{R}$ is denoted by g_{ij} and incorporates the mean path-loss as a function of distance, shadowing and fading, as well as cross-correlations between signature sequences. All the g_{ij} are positive (since all nodes are equipped with omnidirectional antennae) and can take values in the range $(0, 1]$. Without loss of generality, we assume that the intended receiver of transmitter i is also indexed by i. The power level used by transmitter i is denoted by p_i, and ν_i denotes the variance of thermal noise at the receiver i, which is assumed to be an additive Gaussian noise.

The interference power at the ith receiver consists of both the interference caused by other transmitters in the network $\sum_{j\in\mathcal{N}_{-i}} g_{ji}p_j$ (where $\mathcal{N}_{-i}$ denotes all the other nodes different from i that interfere with node i's communications), and the thermal noise ν_i in node i's receiver. That means the SINR at the receiver i is

$$\Gamma_i = \frac{g_{ii}p_i}{\sum_{j\in\mathcal{N}_{-i}} g_{ji}p_j + \nu_i} \tag{11}$$

Due to the unreliability of the wireless links, it is necessary to ensure Quality of Service (QoS) in terms of SINR in wireless networks. Hence a transmission from transmitter i to its corresponding receiver is successful (error-free) if the SINR of the receiver is greater than or equal to the *capture ratio* γ_i, which depends on the modulation and coding characteristics of the radio. In other words, it is required that

$$\frac{g_{ii}p_i}{\sum_{j\in\mathcal{N}_{-i}} g_{ji}p_j + \nu_i} \geq \gamma_i \tag{12}$$

Inequality (12) describes the QoS requirement of a communication pair (i, i) while a transmission takes place. After manipulation, (12) becomes

$$p_i \geq \gamma_i \left(\sum_{j\in\mathcal{N}_{-i}} \tfrac{g_{ji}}{g_{ii}} p_j + \tfrac{\nu_i}{g_{ii}}\right) \tag{13}$$

In matrix form, for a network consisting of n communication pairs, this can be written as $\boldsymbol{p} \succeq \boldsymbol{\Gamma}\boldsymbol{G}\boldsymbol{p} + \boldsymbol{\eta}$ where we define $\boldsymbol{\Gamma} = \operatorname{diag}(\gamma_i)$, $\boldsymbol{p}^\mathsf{T} = \begin{pmatrix} p_1 & \ldots & p_n \end{pmatrix}$, $\boldsymbol{\eta}^\mathsf{T} = \begin{pmatrix} \eta_1 & \ldots & \eta_n \end{pmatrix}$ with $\eta_i = \gamma_i\nu_i/g_{ii}$, and $(\boldsymbol{G})_{ij} = g_{ji}/g_{ii}$ if $i \neq j$, zero otherwise. Finally, with $\boldsymbol{C} := \boldsymbol{\Gamma}\boldsymbol{G}$

$$(\mathbf{I} - \boldsymbol{C})\boldsymbol{p} \succeq \boldsymbol{\eta} \tag{14}$$

We note that $\boldsymbol{C}$ has positive off-diagonal elements which implies that it is irreducible. By the Perron-Frobenius Theorem [5] we then have that the spectral radius of $\boldsymbol{C}$ is a simple eigenvalue, while the corresponding eigenvector is positive elementwise. A necessary and sufficient condition for existence of a non-negative solution to inequality (14) for every positive vector $\boldsymbol{\eta}$ is that $(\mathbf{I}-\boldsymbol{C})^{-1}$ exists and is non-negative. However, $(\mathbf{I} - \boldsymbol{C})^{-1} \succeq \mathbf{0}$ if and only if $\rho(\boldsymbol{C}) < 1$, or, equivalently, $(\boldsymbol{C}-\mathbf{I})$ is Hurwitz (since $(\boldsymbol{C} - \mathbf{I})$ is Metzler), see [6].

4 The Foschini-Miljanic algorithm

The Foschini-Miljanic (FM) algorithm is given by the following distributed power update formula [3]:

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = \kappa_i \left[-p_i(t) + \gamma_i \left(\sum_{j\in\mathcal{N}_{-i}} \frac{g_{ji}}{g_{ii}} p_j(t) + \frac{\nu_i}{g_{ii}}\right)\right] \tag{15}$$

where $\kappa_i > 0$ denote the proportionality constants and γ_i denote the desired SINR. It is assumed that each node i has only knowledge of the interference at its own receiver.

In matrix form, for a given network configuration this yields $\dot{\boldsymbol{p}}(t) = -\boldsymbol{K}(\mathbf{I} - \boldsymbol{C})\boldsymbol{p}(t) + \boldsymbol{\eta}$. Since the transmitter uses measurements from its intended receiver, delays are inevitably introduced into the system for a number of reasons such as processing time (coding/decoding), propagation delays and availability of the channel for transmission. Consequently, a realistic analysis of the algorithm must consider, time-varying delays:

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = \kappa_i \left[-p_i(t) + \gamma_i \left(\sum_{j\in\mathcal{N}_{-i}} \frac{g_{ji}}{g_{ii}} p_j\big(t - \tau_j(t)\big) + \frac{\nu_i}{g_{ii}}\right)\right]$$

where we assume that $\tau_i(t)$ satisfy Assumption 1. In matrix form this can be written as

$$\dot{\boldsymbol{p}}(t) = -\boldsymbol{K}\boldsymbol{p}(t) + \boldsymbol{K}\left(\sum_{k=1}^n \boldsymbol{B}_k\boldsymbol{p}\big(t - \tau_k(t)\big) + \boldsymbol{\eta}\right) \tag{16}$$

where $\boldsymbol{K} = \operatorname{diag}(\kappa_i)$, $(\boldsymbol{B}_k)_{ij}$ is zero if $j = k$ or $i \neq k$, or equal to $\gamma_k g_{ji}/g_{kk}$ otherwise. Note that $\sum_{k=1}^n \boldsymbol{B}_k = \boldsymbol{C}$.

Assuming feasibility of the solution, and defining $\boldsymbol{x}(t) = \boldsymbol{p}^* - \boldsymbol{p}(t)$ to describe the deviation from the desired power levels $\boldsymbol{p}^* = (\mathbf{I}-\boldsymbol{C})^{-1}\boldsymbol{\eta} \succ \mathbf{0}$ in order to satisfy (14), then the stability of (16) is equivalent to and can be assessed by study of the following system:

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{K}\boldsymbol{x}(t) + \sum_{k=1}^n \boldsymbol{K}\boldsymbol{B}_k\boldsymbol{x}\big(t - \tau_k(t)\big) \tag{17}$$

for which it is easy to see that the origin is the equilibrium. If its initial condition is non-negative (which can be guaranteed by starting with all zero power levels) then (17) defines a positive system as the diagonal matrix $-\boldsymbol{K}$ is Metzler and the $\boldsymbol{K}\boldsymbol{B}_k$ are non-negative.

5 Main results

Our main result states the following: Consider among all network topologies the worst-case topology — that is the one with a link assignment that causes maximum interference in all wireless receivers, that is with smallest g_{ii} and largest g_{ji}, maximising the SINRs (11). If the corresponding network regulated by the Foschini-Miljanic algorithm is asymptotically stable then delays in the states or variations in the channel gains cannot destabilise the system. In other words, the Foschini-Miljanic algorithm is unconditionally stable. Moreover, we provide a sufficient condition for its stability under time-varying delays and when the topology changes arbitrarily between M different configurations (modes).

Note that changes in the network configuration may occur for a variety of reasons, such as moving nodes, or change of communication pairs (i.e. a transmitter pairs up with a different receiver). In any case, the nature of the changes as well as the network hardware with finite processing speed will always result in times between switches that are uniformly bounded away from zero.

We now consider two application cases in wireless networks where the existence of a common linear co-positive Lyapunov function is verifiable.

5.1 Worst-case network topology

In some applications network designers can have a reasonable *a priori* estimate of the worst-case scenario, that is the link assignment that causes maximum interference in all wireless receivers. But then, since this worst-case may actually occur, any power control algorithm needs to be designed to be stable even in this hardest of situations. However, if that is the case, as we will show now, the system will automatically be stable for any other network topology with less interference.

Theorem 3 *Consider a set of M different network configurations represented by matrices $\boldsymbol{B}_{k,i}$ and let $\boldsymbol{C}_i = \sum_{k=1}^{n} \boldsymbol{B}_{k,i}$. Additionally, assuming network M corresponds to the worst-case scenario, let $\boldsymbol{C}_{\mathrm{w}} := \boldsymbol{C}_M$ so that $\boldsymbol{C}_{\mathrm{w}} \succeq \boldsymbol{C}_i$ for $i = 1, \ldots, M$. If the system is stable for $\boldsymbol{C}_{\mathrm{w}}$, that is $\boldsymbol{A}_{\mathrm{w}} := \boldsymbol{C}_{\mathrm{w}} - \mathbf{I}$ is a Hurwitz matrix, then the power control algorithm* (16) *is asymptotically stable under arbitrary switching (defined in all the real time axes and with $\inf_k(t_{k+1} - t_k) > 0$, where t_{k+1} and t_k are two consecutive switching instants), for any time-varying delays satisfying Assumption 1, for any initial states $p_i(0) \geq 0$ and for any proportionality constants, $\kappa_i > 0$.*

Proof Suppose that the worst-case matrix $\boldsymbol{A}_{\mathrm{w}}$ is Hurwitz. Then, as we mentioned above, there must exist a vector $\boldsymbol{c} \succ \mathbf{0}$ such that $\boldsymbol{c}^{\mathsf{T}}(\boldsymbol{C}_{\mathrm{w}} - \mathbf{I}) \prec \mathbf{0}$. Hence, for all other matrices satisfying $\boldsymbol{C}_i - \mathbf{I} \preceq \boldsymbol{C}_{\mathrm{w}} - \mathbf{I}$ it follows immediately that $\boldsymbol{c}^{\mathsf{T}}(\boldsymbol{C}_i - \mathbf{I}) \prec \mathbf{0}$ for all i. But since $\boldsymbol{K}$ is a diagonal matrix with positive entries, this also means that $\tilde{\boldsymbol{c}}^{\mathsf{T}}(-\boldsymbol{K} + \sum_{k=1}^{n} \boldsymbol{K}\boldsymbol{B}_{k,i}) \prec \mathbf{0}$ for all i, where $\tilde{\boldsymbol{c}}^{\mathsf{T}} = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{K}^{-1} \succ \mathbf{0}$. This fact together with Theorem 1 provides the assertions of the theorem. □

5.2 General case

In some situations the possible variations in the gain matrix may be known *a priori*, and thus there is a finite number of configurations that characterise the possible configuration of the system. In such situations, the next theorem provides a sufficient condition for stability of the Foschini-Miljanic algorithm under time-varying delays and when the topology changes arbitrarily among M different configurations.

Theorem 4 *Consider a set of M different network configurations described by matrices $\boldsymbol{C}_i = \sum_{k=1}^{n} \boldsymbol{B}_{k,i}$, where $i = 1, \ldots, M$, and let $\boldsymbol{A}_i := \boldsymbol{C}_i - \mathbf{I}$. If the $\boldsymbol{A}_\sigma(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_M)$ are Hurwitz for all $s \in \mathcal{S}_{n,M}$, then the power control algorithm* (16) *is asymptotically stable under arbitrary switching (defined in all the real time axes and with $\inf_k(t_{k+1} - t_k) > 0$, where t_{k+1} and t_k are two consecutive switching instants), for any time-varying delays $\tau_k(t)$ satisfying Assumption 1, for any initial states $p_i(0) \geq 0$, and for any proportionality constants $\kappa_i > 0$.*

Proof By construction, all $\boldsymbol{A}_i$ are Metzler matrices. $\boldsymbol{A}_\sigma(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_M)$ being Hurwitz for all $s \in \mathcal{S}_{n,M}$ is a necessary and sufficient condition, according to Theorem 2 to say that there exists a positive vector $\boldsymbol{c} \succ \mathbf{0}$ such that $\boldsymbol{c}^{\mathsf{T}}(-\mathbf{I} + \sum_{k=1}^{n} \boldsymbol{B}_{k,i}) \prec \mathbf{0}$ for all i. This again also means that since $\boldsymbol{K}$ is a diagonal matrix with positive entries, then $\tilde{\boldsymbol{c}}^{\mathsf{T}}(-\boldsymbol{K} + \sum_{k=1}^{n} \boldsymbol{K}\boldsymbol{B}_{k,i}) \prec \mathbf{0}$ for all i, where $\tilde{\boldsymbol{c}}^{\mathsf{T}} = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{K}^{-1} \succ \mathbf{0}$. By Theorem 1, comparing (17) to (4), this is sufficient to guarantee stability. □

Comment Theorem 4 may also be formulated in terms of feasibility of the linear programming problem to find a vector $\boldsymbol{c} \succ \mathbf{0}$ such that $\boldsymbol{c}^{\mathsf{T}}\left[\,\boldsymbol{A}_1 \;\ldots\; \boldsymbol{A}_M \;\, -\boldsymbol{I}\,\right] \prec \mathbf{0}$, see for instance [11,7].

6 Examples

To illustrate the theoretical results presented in Theorems 3 and 4, we now consider two different models that fulfill the two stability conditions above.

Example for Theorem 3 We first consider three network configurations described by the following matrices:

$$\boldsymbol{C}_1 = \begin{bmatrix} 0 & 0.35 & 0.45 \\ 0.12 & 0 & 0.05 \\ 0.04 & 0.23 & 0 \end{bmatrix}, \quad \boldsymbol{C}_2 = \begin{bmatrix} 0 & 0.15 & 0.15 \\ 0.40 & 0 & 0.20 \\ 0.70 & 0.12 & 0 \end{bmatrix}, \quad \boldsymbol{C}_3 = \begin{bmatrix} 0 & 0.37 & 0.45 \\ 0.40 & 0 & 0.27 \\ 0.70 & 0.23 & 0 \end{bmatrix}$$

From Theorem 3, if the system converges for the worst-case scenario, where the network is represented by matrix $\boldsymbol{C}_{\mathrm{w}}$, then the power control algorithm (16) is asymptotically stable for arbitrary switching of network configurations better than (or the same as) the worst-case. In this example, the spectral radius of $\boldsymbol{C}_{\mathrm{w}} = \boldsymbol{C}_3$ is less than one ($\rho(\boldsymbol{C}_{\mathrm{w}}) \simeq 0.814$) and hence stability of the power control algorithm (16) is guaranteed.

Figure 1 confirms this. It shows the results from a simulation run, plotting the deviations from the desired power levels as a function of time. The system used was based on the above matrices, where the time-varying delays have been simulated with sinusoidal generators, $\tau(t) = 2 + 2\sin(t)$ and the switching sequence has been chosen randomly (and is indicated with a grey line). As suggested earlier, the system was initialised with zero power levels. It can be seen that indeed the deviations disappear asymptotically.

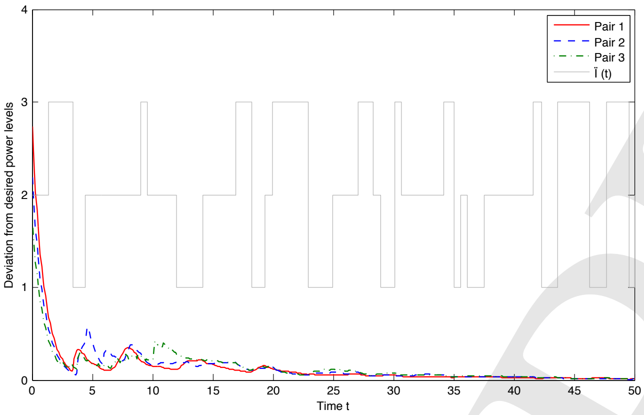

Figure 1. Simulation of the switched network represented by matrices $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ and $\boldsymbol{C}_3$, each consisting of three communication pairs. The plot shows the evolution of the deviation from the desired power levels; the switching sequence $\sigma(t)$ is also shown (that is, $\sigma(t) = 1$ means the network is represented by matrix $\boldsymbol{C}_1$, and so on).

Example for Theorem 4 We now consider three modes such that the stability condition in Theorem 4 is fulfilled, given by the following matrices

$$\boldsymbol{C}_1 = \begin{bmatrix} 0 & 0.35 & 0.45 \\ 0.62 & 0 & 0.05 \\ 0.44 & 0.23 & 0 \end{bmatrix}, \quad \boldsymbol{C}_2 = \begin{bmatrix} 0 & 0.35 & 0.15 \\ 0.40 & 0 & 0.45 \\ 0.37 & 0.53 & 0 \end{bmatrix}, \quad \boldsymbol{C}_3 = \begin{bmatrix} 0 & 0.32 & 0.55 \\ 0.42 & 0 & 0.25 \\ 0.64 & 0.23 & 0 \end{bmatrix}$$

From Theorem 4, if $\boldsymbol{C}_s(\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3)$ have a spectral radius less than one for all $s \in \mathcal{S}_{n,3}$, then the power control algorithm (16) is asymptotically stable under arbitrary switching. In the example here, indeed, $\max\left(\rho(\boldsymbol{C}_s(\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3))\right) \simeq 0.91 < 1$ (corresponding to the permutation $s = (1, 2, 3)$) and therefore, the resulting system would be asymptotically stable under arbitrary switching.

7 Conclusion

We have shown that the Foschini-Miljanic algorithm is asymptotically stable under time-varying delays and changing network topologies. For that, we extended the current theory on positive systems in order to account for switched positive systems with time-varying delays.

Future directions include the study of the conditions for stability of constantly changing network topologies where it is not possible to distinguish the network into different configurations. Further, we intend to compare our results with the stability conditions of the undelayed Foschini-Miljanic algorithm to investigate the impact of delays on the system.

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