On controllability of the real shifted inverse power iteration*

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Abstract

Controllability properties of the inverse power method on projective space are investigated. For complex eigenvalue shifts a simple characterization of the reachable sets in terms of invariant subspaces can be obtained. The real case is more complicated and is investigated in this paper. Necessary and sufficient conditions for complete controllability are obtained in terms of the solvability of a matrix equation. Partial results on conditions for the solvability of this matrix equation are given.

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1 Introduction

Numerical matrix eigenvalue methods such as the QR algorithm or inverse power iterations provide interesting examples of nonlinear discrete dynamical systems defined on Lie groups

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or homogeneous spaces. A typical approach from numerical linear algebra to improve convergence properties of such algorithms is to introduce suitable shift strategies for the eigenvalues. We refer to [2, 3, 5] for papers studying the shifted inverse iteration. In particular, the case of complex shifts is studied in [8].

Such eigenvalue shifts can be viewed as control variables and the resulting algorithms can therefore be analyzed using tools from nonlinear control theory. So far the analysis and design of shift strategies in numerical eigenvalue algorithms has been more a kind of an art rather than being guided by systematic design principles. The situation here is quite similar to that of control theory in the 50's before the introduction of state space methods. The advance made during the past two decades in nonlinear control theory indicates that the time may now be ripe for a more systematic investigation of control theoretic aspects of numerical linear algebra.

In this paper we investigate the controllability properties of the well known inverse power iterations for finding the dominant eigenvector of a matrix using *real* shifts. Shifted versions of the inverse power method lead to nonlinear control systems on projective space, or more generally on Grassmann manifolds. The complex case has been studied in [4].

Let A denote a real $n \times n$-matrix with spectrum $\sigma(A) \subset \mathbb{C}$. The *shifted inverse iteration* in its controlled form is given by

$$x(t+1) = \frac{(A - u_t I)^{-1} x(t)}{\|(A - u_t I)^{-1} x(t)\|}, \quad t \in \mathbb{N}, \tag{1}$$

where $u_t \notin \sigma(A)$. This describes a nonlinear control system on the $(n-1)$-sphere. The trajectory corresponding to a normalized initial condition x_0 and a control sequence $u = (u_0, u_1, \dots)$ is denoted by $\phi(t; x_0, u)$. Via the choice $u_t = x^*(t)Ax(t)$ we obtain from (1) the Rayleigh quotient iteration studied in [2], [3]. Thus the Rayleigh iteration may be interpreted as a *feedback strategy* for the shifted inverse iteration. It is known that in some cases this feedback strategy has undesirable properties, in particular if it is applied to non-Hermitian matrices A [3]. It is the aim of this paper to start a systematic control theoretic investigation of this system class in the hope that this leads to a better understanding of the question when the Rayleigh iteration fails.

In Section 2 we will introduce the shifted inverse power iteration and the associated system on projective space and discuss its forward accessibility properties. In particular there is an easy characterization of the set of universally regular control sequences, that is those sequences with the property, that they steer every point into the interior of its forward orbit. This will be used in Section 3 to give a characterization of complete controllability of the system on projective space in terms of solvability of a matrix equation. In Section 4 we investigate the obtained characterization and interpret it in terms of the characteristic polynomial of A. Some concrete cases in which it is possible to decide based on spectral information whether a matrix leads to complete controllable shifted inverse iteration are presented in Section 5. In Section 6 we discuss the obtained result and present some open problems.

2 The shifted inverse iteration on projective space

We will first motivate the state space on which the analysis will be performed. Given $A \in \mathbb{R}^{n\times n}$ it is easy to see that if the initial condition x_0 for system (1) lies in an invariant subspace of A then the same holds true for the entire trajectory $\phi(t; x_0, u)$, regardless of the control sequence u. In order to understand the controllability properties from x_0 it would then suffice to study the system in the corresponding invariant subspace. Therefore we may restrict our attention to those points not lying in a nontrivial invariant subspace of A, i.e. those $x \in \mathbb{R}^n$ such that $\{x, Ax, \dots, A^{n-1}x\}$ is a basis of $\mathbb{R}^n$. Vectors with this property are called *cyclic* and a matrix A is called cyclic if it has a cyclic vector. In the following we assume that A is cyclic. To keep notation short let us introduce the union of A-invariant subspaces

$$\mathcal{V}(A) := \bigcup_{AV\subset V, 0<dim V<n} V\,.$$

Using the fact that the interesting dynamics of (1) are on the unit sphere and identifying opposite points (which give no further information) we then define our state space of interest to be

$$M := \mathbb{RP}^{n-1} \setminus \mathcal{V}(A)\,, \tag{2}$$

where $\mathbb{RP}^{n-1}$ denotes the real projective space of dimension $n-1$. The natural projection from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{RP}^{n-1}$ will be denoted by $\mathbb{P}$. Thus M consists of the 1-dimensional linear subspaces of $\mathbb{R}^n$, defined by the cyclic vectors of A. Since a cyclic matrix has only a finite number of invariant subspaces, $\mathcal{V}(A)$ is a closed algebraic subset of $\mathbb{R}^n$. Moreover, M is an open and dense subset of $\mathbb{RP}^{n-1}$. The system on M is now given by

$$\begin{aligned} \xi(t+1) &= (A - u_t I)^{-1}\xi(t)\,, \quad t \in \mathbb{N} \\ \xi(0) &= \xi_0 \in M\,, \end{aligned} \tag{3}$$

where $u_t \in U := \mathbb{R} \setminus \sigma(A)$ (the set of admissible control values). We denote the space of finite and infinite admissible control sequences by U^t and $U^{\mathbb{N}}$, respectively. The solution of (3) corresponding to the initial value ξ_0 and a control sequence $u \in U^{\mathbb{N}}$ is denoted by $\varphi(t; \xi_0, u)$. The forward orbit of a point $\xi \in M$ is then given by

$$\mathcal{O}^+(\xi) := \{\eta \in M \mid \exists t \in \mathbb{N}, u \in U^t \text{ such that } \eta = \varphi(t; \xi, u)\}\,.$$

Similarly, the set of points reachable exactly in time t is denoted by $\mathcal{O}_t^+(\xi)$. System (3) is called *forward accessible* [1], if the forward orbit $\mathcal{O}^+(\xi)$ of every point $\xi \in \mathbb{RP}^{n-1}$ has nonempty interior and *uniformly forward accessible (in time t)* if there is a $t \in \mathbb{N}$ such that $\operatorname{int} \mathcal{O}_t^+(\xi) \neq \emptyset$ for all $\xi \in M$. Note that $\operatorname{int} \mathcal{O}^+(\xi) \neq \emptyset$ holds iff there is a $t \in \mathbb{N}$ such that $\operatorname{int} \mathcal{O}_t^+(\xi) \neq \emptyset$. Sard's theorem implies then the existence of a control $u \in U^t$ such that

$$\operatorname{rk} \frac{\partial \varphi(t; \xi, u)}{\partial u} = n - 1\,.$$

A pair $(\xi, u) \in M \times U^t$ is called *regular* if this rank condition holds. The control sequence $u \in U^t$ is called *universally regular* if (ξ, u) is a regular pair for every $\xi \in M$. By [7, Corollaries 3.2 & 3.3] forward accessibility is equivalent to the fact that the set of universally regular control sequences U^t_{reg} is open and dense in U^t for all t large enough. (For a precise statement we refer to [7].)

Remark 2.1 The following statements hold for arbitrary nonlinear discrete-time systems $x(t+1) = f(x(t), u(t))$ having the property that the partial maps $f(\cdot, u)$ are diffeomorphisms of the state space.
a) If $u' \in U^t$ is universally regular, the for any $u'' \in U$ the concatenated sequence (u', u'') is universally regular. This follows from an application of the chain rule, because

$$\frac{\partial \varphi(t; x, u)}{\partial u''} = \frac{\partial}{\partial x} f(\varphi(t; x, u''), u') \frac{\partial}{\partial u''} \varphi(t; x, u''), \qquad (4)$$

and in particular that the ranks of the Jacobians with respect to u coincide, as f is a diffeomorphism. This of course extends to the concatenation of further elements.
b) If $u = (u_0, \dots, u_{t-1})$ is universally regular then the reversed sequence given by $\hat{u} := (u_{t-1}, \dots, u_0)$ is universally regular for the time-reversed system $x(t+1) = f^{-1}(x(t), u(t))$. This follows from

$$0 = \frac{\partial}{\partial u} \varphi(-t; \varphi(t; x, u), \hat{u}) = \frac{\partial}{\partial u} \varphi(-t; x(t), \hat{u}) + \frac{\partial}{\partial x} \varphi(-t; x(t), \hat{u}) \frac{\partial}{\partial u} \varphi(t; x, u) .$$

Thus $(\frac{\partial}{\partial x} \varphi(-t; x(t), \hat{u}))^{-1} \frac{\partial}{\partial u} \varphi(-t; x(t), \hat{u}) = -\frac{\partial}{\partial u} \varphi(t; x, u)$ and using the fact that $\varphi(t; \cdot, u)$ is a diffeomorphism, we see that the ranks of the Jacobians with respect to u coincide.

The following result shows forward accessibility for (3).

Lemma 2.2 *System (3) is uniformly forward accessible in time $n-1$. A control sequence $u \in U^t$ is universally regular if and only if there are $n-1$ pairwise different values in the sequence $u_0, \dots, u_{t-1}$.*

Proof. The first claim is a consequence of the second, as we will show the existence of universally regular control sequences in U^{n-1}. By Remark 2.1 (ii) it suffices to prove the assertion for the time-reversed system

$$\xi(t+1) = (A - u_t I) \xi(t) , \qquad (5)$$

which we will examine from now on. Starting with the initial condition $x_0 \in \mathbb{R}^n$ and given $u \in U^{n-1}$ we have that

$$\begin{aligned} \Phi(n-1, u) x_0 \; &:= \; \prod_{s=0}^{n-2} (A - u_s I) \, x_0 \qquad (6) \\ &= \; \left(A^{n-1} - \sum_{s=0}^{n-2} u_s A^{n-2} + \ldots + (-1)^{n-1} \prod_{s=0}^{n-2} u_s I \right) x_0 . \end{aligned}$$

As in Proposition 3.6 of [9] we may see that (ξ, u) is a regular pair for (5) iff the following rank condition is satisfied for $x \in \mathbb{R}^n \setminus \{0\}$ projecting to ξ:

$$\operatorname{rk}\left[\Phi(n-1,u)x \,\vdots\, \frac{\partial}{\partial u}\Phi(n-1,u)\,x\right] = n\,. \tag{7}$$

As x_0 is cyclic vector we may analyze the system with respect to the basis $(e_1, \ldots, e_n) := (x_0, Ax_0, \ldots, A^{n-1}x_0)$. In these coordinates we have from (6) that

$$\Phi(n-1,u)e_1 = e_n + \sum_{j=1}^{n-1}(-1)^j\sigma_j(u_0,\ldots,u_{n-2})e_j\,,$$

where $\sigma_j(u_0, \ldots, u_{n-2})$ denotes the j-th elementary symmetric polynomial of the matrix $\operatorname{diag}(u_0, \ldots, u_{n-2})$. Inserting this expression into (7), we have to check the rank of the following matrix for regularity of the pair $(\mathbb{P}x_0, u)$

$$\begin{pmatrix} * & (-1)^{n-1}\prod_{i\neq 0}u_i & & \cdots & (-1)^{n-1}\prod_{i\neq n-1}u_i \\ * & \vdots & & & \vdots \\ * & \sum_{i\neq 0}u_i & \sum_{i\neq 1}u_i & \cdots & \sum_{i\neq n-1}u_i \\ * & -1 & -1 & \cdots & -1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and therefore it suffices to calculate the determinant of the upper right $(n-1)\times(n-1)$-block to check, whether the rank of this matrix is n. By a well-known property of elementary polynomials, this block is known to be invertible, if and only if $u_0, \ldots, u_{n-2}$ are pairwise distinct. Alternatively, an induction argument shows that

$$\left|\det\begin{pmatrix} (-1)^{n-1}\prod_{i\neq 0}u_i & & \cdots & (-1)^{n-1}\prod_{i\neq n-1}u_i \\ \vdots & & & \vdots \\ \sum_{i\neq 0}u_i & \sum_{i\neq 1}u_i & \cdots & \sum_{i\neq n-1}u_i \\ -1 & -1 & \cdots & -1 \end{pmatrix}\right| = \prod_{i\neq j}|u_i - u_j| \neq 0\,,$$

if and only if $u_0, \ldots, u_{n-2}$ are pairwise distinct. Thus $(\xi, u) \in M \times U^{n-1}$ is a regular pair iff the entries u_i are pairwise different and regardless of ξ. This shows the assertion for sequences of length $n-1$.

Clearly for dimensionality reasons sequences of length less than $n-1$ cannot be universally regular. Let $u \in U^t$, $t > n-1$, be a sequence with at least $n-1$ pairwise different elements. By commutativity the rank of the Jacobian $\frac{\partial}{\partial u}\varphi(t;\xi,u)$ does not change if we reorder the entries of u. Thus let $u = (u', u'')$ with $u' \in U^{n-1}$ consisting of $n-1$ distinct entries. Since each u_s is not an eigenvalue of A we see that $A - u_sI$ is an isomorphism of M and therefore the result follows from Remark 2.1 (i).

To conclude the proof consider the case when $u \in U^t, t \geq n-1$ has less than $n-1$ distinct elements. Using again the characterization (7), after an appropriate reordering, we have to consider expressions of the form

$$\frac{\partial}{\partial u}\prod_{s=0}^{k}(A - u_sI)^{l_s}\,,$$

where $l_s > 0, u_s \in U$ and $0 < k < n-1$. It is easy to see that

$$\operatorname{rk} \frac{\partial}{\partial u} (A - uI)^l x = 1 \,,$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ and all $l > 0$. Then the assertion follows from an application of the chain rule. □

3 Controllability of the projected system

By the results of the previous section we know that every point in M has a forward orbit with interior points and it is reasonable to wonder about controllability properties of system (3). As usual, we will call a point $\xi \in M$ controllable to $\eta \in M$ if $\eta \in \mathcal{O}^+(\xi)$. System (3) is said to be completely controllable on a subset $N \subset M$ if for all $\xi \in N$ we have $N \subset \mathcal{O}^+(\xi)$.
In order to analyze the controllability properties of (3) we introduce the following definition of what can be thought of as regions of approximate controllability in $\mathbb{RP}^{n-1}$. A *control set* of system (3) is a set $D \subset M$ satisfying

(i) $D \subset \operatorname{cl} \mathcal{O}^+(\xi)$ for all $\xi \in D$.

(ii) For every $\xi \in D$ there exists a $u \in U$ such that $\varphi(1; x, u) \in D$.

(iii) D is a maximal set (with respect to inclusion) satisfying (i).

An important subset of a control set D is its *core* defined by

$$\operatorname{core}(D) := \{\xi \in D \mid \operatorname{int} \hat{\mathcal{O}}^-(\xi) \cap D \neq \emptyset \text{ and } \operatorname{int} \hat{\mathcal{O}}^+(\xi) \cap D \neq \emptyset\} \,.$$

Here $\hat{\mathcal{O}}^-(\xi)$ denotes the points $\eta \in \mathbb{RP}^{n-1}$ such that there exist $t \in \mathbb{N}$, $u_0 \in \operatorname{int} U^t$ such that $\varphi(t; \eta, u_0) = \xi$ and (η, u_0) is a regular pair. By this assumption it is evident that on the core of a control set the system is completely controllable.
We are now in a position to state our first result characterizing controllability of (3).

Theorem 3.1 *Let $A \in \mathbb{R}^{n \times n}$ be cyclic. Consider the system* (3) *on M. The following statements are equivalent:*

(i) There exists a $\xi \in M$ such that $\mathcal{O}^+(\xi)$ is dense in M.

(ii) There exists a control set $D \subset M$ with $\operatorname{int} D \neq \emptyset$.

(iii) M is a control set of system (3).

(iv) System (3) *is completely controllable on M.*

(v) There exists a universally regular control sequence $u \in U^t$ such that

$$\prod_{s=0}^{t-1} (A - u_s I)^{-1} \in \mathbb{R}^* I \,. \tag{8}$$

Proof. The implications (iv) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (i) are obvious.
"(v) $\Rightarrow$ (iii)" As (8) is a representation of a multiple of the identity any $\xi \in M$ satisfies $\xi = \varphi(t; \xi, u)$ and therefore ξ is a periodic point under a universally regular control. This implies that every $\xi \in M$ is an element in the core of a control set, [10, Prop. 13]. Thus by [10, Prop. 10] every connected component of M is contained in the core of a control set and it remains to show that it is possible to steer from any connected component of M into any other. Then it follows from maximality of control sets that M is a control set.
We have to show that for every two connected components $Z_1, Z_2 \subset M$ there exist $\xi_i \in Z_i, i = 1, 2$ and a control $u \in U^t$ such that $\xi_2 = \varphi(t; \xi_1, u)$. Note that different connected components of M are separated by the $n-1$-dimensional A-invariant subspaces. Let the (real) Jordan form of A be given by

$$TAT^{-1} = \begin{pmatrix} \mathrm{diag}(J(\lambda_1), \ldots, J(\lambda_k)) & 0 \\ 0 & B \end{pmatrix},$$

where the $J(\lambda_i)$ are Jordan blocks to real eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_k$ and B has only complex eigenvalues. For $i = 1, \ldots, k$ let e_{l_i} denote the unit cyclic vector of $J(\lambda_i)$, then the $n-1$-dimensional A-invariant subspaces are given exactly by the sets $T^{-1}\{x \in \mathbb{R}^n \mid x_{l_i} = 0\}, i = 1, \ldots, k$. The connected components of M can then be described by index sets I, where $I \subset \{1, \ldots, k\}$, and are of the form

$$Z(I) := \mathbb{P}\{x \in \mathbb{R}^n \setminus \mathcal{V}(A) \mid x_j > 0\,,\, j \in I, x_j < 0\,,\, j \notin I\}\,.$$

(Note that by multiplication by -1 there are now for each connected component two representations of the above kind. But this is irrelevant for our purposes.) For a control value $\lambda_j > u > \lambda_{j+1}$ it is obvious that $(\lambda_i - u)^{-1}$ is positive if $i \leq j$ and negative otherwise. Hence the connected component given by an index set I is mapped by $(A - uI)^{-1}$ onto the connected component given by the index sets $\tilde{I}$, where $\tilde{I} = \{i \in I \mid i \leq j\} \cup \{i \notin I \mid i > j\}$. If we are now given two connected components with index sets I_1, I_2 let j be the smallest index for which the sets differ, i.e. $I_1 \cap \{1, \ldots, j-1\} = I_2 \cap \{1, \ldots, j-1\}$ and j belongs either to I_1 or to I_2. If we choose $\lambda_j > u > \lambda_{j+1}$ and $\xi \in I_1$ then $(A - uI)^{-1}\xi$ belongs to a connected component I' satisfying $I' \cap \{1, \ldots, j\} = I_2 \cap \{1, \ldots, j\}$, because by the previously explained rule I' coincides with I_1 (and thus with I_2) for the indices below j and does not coincide with I_1 (and thus with I_2) as regards the index j. We have thus steered into a connected component whose index set coincides with the one of I_2 in at least one more index and it follows that after a finite number of such steps we can steer into the index set I_2.
"(ii) $\Rightarrow$ (v)" Let $D \subset M$ be a control set. By Theorem 15 in [10] for every open set $W \subset \mathrm{core}(D)$ there exist $\xi \in W$, $t \in \mathbb{N}$ and $u \in U^t_{reg}$ such that $\xi = \varphi(t; \xi, u)$. Choose W such that for all $\eta \in W$ the representation $\eta = \mathbb{P}\sum_{i=1}^n \alpha_i x_i$ in terms of a basis given by (generalized) eigenvectors of A implies $\alpha_i \neq 0, i = 1, \ldots, n$. For such an η we now have a representation

$$\eta = \prod_{s=0}^{t-1}(A - u_s I)^{-1}\eta\,.$$

That is in the basis $\{x_1, \dots, x_n\}$ (with an associated change of basis T) we have

$$\alpha = c \prod_{s=0}^{t-1} (TAT^{-1} - u_s I)^{-1} \alpha \,,$$

for a suitable constant $c \neq 0$ and $\alpha = (\alpha_1, \dots, \alpha_n)$. In particular, for each Jordan block $J(\lambda)$ of A corresponding to the indices $i_{\lambda,1}, \dots, i_{\lambda,k(\lambda)}$ it holds that

$$\begin{bmatrix} \alpha_{i_{\lambda,1}} \\ \vdots \\ \alpha_{i_{\lambda,k(\lambda)}} \end{bmatrix} = c \prod_{s=0}^{t-1} (J(\lambda) - u_s I)^{-1} \begin{bmatrix} \alpha_{i_{\lambda,1}} \\ \vdots \\ \alpha_{i_{\lambda,k(\lambda)}} \end{bmatrix} = c \prod_{s=0}^{t-1} J(\lambda - u_s)^{-1} \begin{bmatrix} \alpha_{i_{\lambda,1}} \\ \vdots \\ \alpha_{i_{\lambda,k(\lambda)}} \end{bmatrix} .$$

An easy calculation shows that this implies $c \prod_{s=0}^{t-1} (J(\lambda) - u_s I)^{-1} = I$ independent of $\lambda \in \sigma(A)$.
"(i) $\Rightarrow$ (ii)" Let $\xi \in M$ be such that $\mathcal{O}^+(\xi)$ is dense in M, then also the set of points that can be reached from ξ applying universally regular controls is dense in M. Because of forward accessibility and invertibility of the system also $\hat{\mathcal{O}}^-(\xi)$ has nonempty interior [1]. Thus we can steer with a universally regular control from ξ into its backward orbit and then back to ξ. The concatenation of these controls is also universally regular by Remark 2.1 (i). Thus ξ is a periodic point under a universally regular control and therefore contained in the interior of a control set by [10, Prop. 13].
"(iii) $\Rightarrow$ (iv)" The assumption implies that $\mathcal{O}^+(\xi)$ is dense in M for any $\xi \in M$. Now we can simply apply the argument of the previous part to any two points $\xi, \eta \in M$ to show that it is possible to steer from ξ to η. □
The unusual fact about the system we are studying is thus that by the universally regular representation of one element of the system's semigroup we can immediately conclude that the system is completely controllable. Furthermore, already the fact that there is a control set of the system implies complete controllability on the whole state space M. On the other hand it is worth pointing out, that if the conditions of the above theorem are not met, then no forward orbit of (3) is dense in M.
For brevity we will call a cyclic matrix A *II-controllable* (for inverse iteration controllable), if A satisfies any of the equivalent conditions of Theorem 3.1.
We will reformulate the results of the previous theorem in terms of companion matrices as this will be the necessary tool for analysis of the interpretation of this controllability result in terms of the characteristic polynomial of A in the ensuing section. Recall that if A is cyclic, then it is similar to the companion matrix

$$C_A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & \alpha_0 \\ 1 & 0 & 0 & \dots & 0 & \alpha_1 \\ 0 & 1 & 0 & \dots & 0 & \alpha_2 \\ & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \alpha_{n-2} \\ 0 & 0 & \dots & 0 & 1 & \alpha_{n-1} \end{bmatrix}$$

associated with its characteristic polynomial $q_A(z) = z^n - \sum_{i=0}^{n-1} \alpha_i z^i$. Thus via coordinate transformation we obtain from (1) the equivalent system

$$x(t+1) = \frac{(C_A - u_t I)^{-1} x(t)}{\|(C_A - u_t I)^{-1} x(t)\|}, \quad t \in \mathbb{N}, \tag{9}$$

and from (3) the projected system

$$\begin{aligned} \xi(t+1) &= (C_A - u_t I)^{-1} \xi(t), \quad t \in \mathbb{N} \\ \xi(0) &= \xi_0 \in \tilde{M}, \end{aligned} \tag{10}$$

where $u_t \in U$ as before, and $\tilde{M} = \mathbb{RP}^{n-1} \setminus \mathcal{V}(C_A)$.
This representation is closely linked to the *polynomial models* introduced by Fuhrmann, see e.g. [6]. Let $X_q := \mathbb{R}[z]/q_A\mathbb{R}[z]$ with the associated vector space isomorphism

$$\mathbb{R}^n \to X_q, \quad x \mapsto \sum_{j=0}^{n-1} x_j z^j \bmod q_A.$$

On X_q we consider the linear operator S_q given by

$$S_q(p) = zp \bmod q_A.$$

With these definitions it is known that the following diagram commutes

$$\begin{array}{ccc} \mathbb{R}^n & \stackrel{C_A}{\longrightarrow} & \mathbb{R}^n \\ \downarrow & & \downarrow \\ X_q & \underset{S_q}{\longrightarrow} & X_q \end{array}. \tag{11}$$

Thus (9) is equivalent to the associated system on the polynomial model space X_q. In order to keep track of the invariant subspaces of our linear operators let us point out that the invariant subspaces of the operator S_q in X_q are simply the spaces of polynomials which have a fixed common divisor with the characteristic polynomial q_A. This may be seen by noting that if h is a common divisor of p and q_A then from

$$zp(z) = S_q(p)(z) + q_A(z)r(z)$$

it follows that h is also a factor of $S_q(p)$, see [4] for details.
We have another simple characterization of II-controllability in terms of the existence of a universally regular periodic orbit through the first unit vector of the system in companion form. This may come as a surprise. It is helpful to remember here, that by construction e_1 is a cyclic vector for C_A, the matrix in companion form.

Corollary 3.2 *Let $A \in \mathbb{R}^{n\times n}$ be cyclic with characteristic polynomial q_A. Consider the system (3) on M. The following statements are equivalent:*

(i) the matrix A is II-controllable.

(ii) $\tilde{M}$ *is a control set of system* (10).

(iii) There exist $t \in \mathbb{N}$, $u \in U^t_{reg}$ *such that* $\mathbb{P}e_1$ *is a periodic point for system* (10) *under the control sequence* u.

Proof. The equivalence of (i) and (ii) is obvious by similarity transformation and Theorem 3.1. In order to see that assertion (i) implies (iii) it is sufficient to apply (8). So we now have to prove that (iii) implies the existence of a universally regular control sequence u such that (8) is satisfied. By (11) we have for every $u \in U^t$ the following commutative diagram

$$\begin{array}{ccc} \mathbb{R}^n & \overset{\prod_{s=0}^{t-1}(C_A - u_s I)^{-1}}{\longrightarrow} & \mathbb{R}^n \\ \downarrow & & \downarrow \\ X_q & \underset{\prod_{s=0}^{t-1}(S_q - u_s I)^{-1}}{\longrightarrow} & X_q \end{array} \tag{12}$$

Thus using the universally regular u satisfying (iii) the following diagram commutes

$$\begin{array}{ccc} e_1 & \overset{\prod_{s=0}^{t-1}(C_A - u_s I)^{-1}}{\longrightarrow} & \prod_{s=0}^{t-1}(C_A - u_s I)^{-1} e_1 = \alpha e_1 \\ \downarrow & & \downarrow \\ 1 & \underset{\prod_{s=0}^{t-1}(z - u_s)^{-1}}{\longrightarrow} & \alpha 1 \end{array} \tag{13}$$

The lower row of this diagram implies

$$\prod_{s=0}^{t-1}(z - u_s)^{-1} = \alpha 1 \bmod q_A \,,$$

and insertion of C_A yields

$$\prod_{s=0}^{t-1}(C_A - u_s I)^{-1} = \alpha I + r(A)q(A) = \alpha I \,.$$

Now (8) follows via similarity of A and C_A. □

4 Polynomial characterizations of II-controllability

As has already become evident in the last result of the previous section the question of II-controllability is closely linked to properties of real polynomials. We will now further investigate this relationship. Here we follow the ideas for the complex case in [4] and derive comparable results for the real case.
In the following theorem we use the notation $p \wedge q = 1$ to denote the fact that the two polynomials $p, q \in \mathbb{R}[z]$ are coprime.

Theorem 4.1 *Let* $A \in \mathbb{R}^{n \times n}$ *be cyclic with characteristic polynomial* q. *Consider the system* (3) *on* M. *The following statements are equivalent:*

(i) the matrix A is II-controllable.

(ii) For every $B \in \Gamma_A := \{p(A) \mid p \in \mathbb{R}[z], p \wedge q = 1\}$ there exist $t \in \mathbb{N}$, $u \in U^t_{reg}$, $\alpha \in \mathbb{R}^*$ such that

$$B = \alpha \prod_{s=0}^{t-1} (A - u_s I)\,,$$

i.e. $\Gamma_A = \Gamma_A^{\mathbb{R}} := \{p(A) \mid p(z) = \alpha \prod_{s=0}^{t-1} (z - u_s), u_s \in \mathbb{R}, p \wedge q = 1, \alpha \in \mathbb{R}^*\}$.

(iii) For every $p \in \mathbb{R}[z], p \wedge q = 1$ there exist $t \in \mathbb{N}$, $u \in U^t_{reg}$, $\alpha \in \mathbb{R}^*$ such that

$$p(z) = \alpha \prod_{s=0}^{t-1} (z - u_s) \bmod q(z)\,.$$

(iv) There exists a monic polynomial f with only real roots and at least $n-1$ pairwise different roots, $\alpha \in \mathbb{R}^*$ and $r(z) \in \mathbb{R}[z]$ such that

$$f(z) = \alpha + r(z)q(z)\,. \tag{14}$$

Remark 4.2 From (14) it is easy to deduce the following statement: If for a cyclic matrix A with characteristic polynomial q there exists a monic polynomial f with only real roots that are *all* pairwise distinct such that (14) is satisfied, then there is a neighborhood of A consisting of II-controllable matrices. The reason for this is that, keeping α and $r(z)$ fixed, small changes in the coefficients of q will only lead to small changes in the coefficients of f, and the assumption guarantees that all polynomials in a neighborhood of f have simple real roots.

Proof. By Theorem 3.1 the II-controllability of A is equivalent to the existence of a universally regular control sequence u such that $\prod_{s=0}^{t-1}(A - u_s I)^{-1} \in \mathbb{R}^* I$ which implies $\prod_{s=0}^{t-1}(A - u_s) = \alpha I$ for some $\alpha \neq 0$. It is an immediate consequence of the Cayley-Hamilton theorem that this is equivalent to (iv). Likewise the equivalence of (ii) and (iii) follows from Cayley-Hamilton. The assertion (iv) is a special case of (iii) for the polynomial $p(z) = \alpha$ so that we have to show that (iv) implies (iii). By Corollary 3.2 assumption (iv) implies that the system in companion form has $\tilde{M}$ as a control set. As system (10) is forward accessible it follows in particular that $\text{core}(\tilde{M}) = \tilde{M}$. Hence, for every $x \in \mathbb{R}^n \setminus \mathcal{V}(C_A)$ there exist a $t \in \mathbb{N}$ and a universally regular $u \in U^t$ such that

$$\prod_{s=0}^{t-1} (C_A - u_s I)^{-1} x = \alpha e_1$$

for some $\alpha \neq 0$. Using again the commutative diagram (12) and the remark that the invariant subspaces of S_q are given by polynomials which have a common factor with q, this implies that for every $p \in X_q, p \wedge q = 1$ there is $u \in U^t_{reg}$ such that

$$p(z) = \alpha \prod_{s=0}^{t-1} (z - u_s) 1 \bmod q(z)\,.$$

This completes the proof. □

As an immediate consequence of Theorem 3.1 we obtain a complete characterization of the reachable sets of the inverse power iteration given by

$$\xi(t+1) = (A - u_t I)^{-1}\xi(t)\,, \quad t \in \mathbb{N}\,, \qquad \xi(0) = \xi_0 \in \mathbb{RP}^{n-1}\,, \tag{15}$$

for II-controllable matrices $A \in \mathbb{R}^{n\times n}$. This extends a result in [4] for real matrices.

Corollary 4.3 *Let A be II-controllable with characteristic polynomial q_A, then*
(i) for each $\xi = \mathbb{P}x \in \mathbb{RP}^{n-1}$ we have

$$\begin{aligned} \mathcal{O}^+(\xi) &= \mathbb{P} \bigcap_{x\in V, AV\subset V} V \setminus \bigcup_{x\notin V, AV\subset V} V\,, \\ \operatorname{cl}\mathcal{O}^+(\xi) &= \mathbb{P} \bigcap_{x\in V, AV\subset V} V = \mathbb{P}\operatorname{span}\{x, Ax, A^2x, \dots, A^{n-1}x\}\,. \end{aligned}$$

(ii) There is a one-to-one correspondence between

a) The forward orbits of system (15).

b) The closures of the forward orbits of system (15).

c) The A-invariant subspaces of $\mathbb{R}^n$.

d) The factors of $q_A(z)$ over the polynomial ring $\mathbb{R}[z]$.

Proof. (i) For $x \neq 0$ let $W_x := \operatorname{span}\{x, Ax, A^2x, \dots, A^{n-1}x\}$ denote the A-invariant subspace generated by x. By construction this subspace is invariant under $(A-uI)^{-1}$ so that it follows that for $\xi = \mathbb{P}x$ we have $\mathcal{O}^+(\xi) \subset \mathbb{P}W_x = \mathbb{P}\bigcap_{x\in V, AV\subset V} V$. As A is cyclic so is the restriction of A to the A-invariant subspace W_x which we denote by A_x. Furthermore, A_x is II-controllable because the existence of a universally regular representation of a multiple of the identity as in (8) is an immediate consequence of the II-controllability of A. This shows that the orbit of x is the set M of cyclic vectors of A_x by Theorem 3.1 which is equivalent to $\mathcal{O}^+(\xi) = \mathbb{P}W_x \setminus \bigcup_{x\notin V, AV\subset V} V$, which shows the assertion. The statement about the closure $\operatorname{cl}\mathcal{O}^+(\xi)$ is an immediate consequence.
(ii) Given an A-invariant subspace V we see from (i) that $\mathbb{P}V$ is the closure of the forward orbit for all $\xi = \mathbb{P}x$ such that x generates V. For such $\xi = \mathbb{P}x$ the forward orbit is $\mathbb{P}V$ without the projection of the A-invariant subspaces contained in V but not containing x. Also this characterization is independent of the generating element x of V. This shows the one-to-one correspondence between the orbits, their closures and A-invariant subspaces of $\mathbb{R}^n$. On the other hand the relation between the A-invariant subspaces and the factors of q_A over $\mathbb{R}[z]$ is well known, see [4]. □

5 Conditions for II controllability

The result of the previous section raises the question for which cyclic matrices A admit a representation of the form (8) or equivalently when (14) is possible. With respect to this question we have the following preliminary results.

Proposition 5.1 *Let $A \in \mathbb{R}^{n\times n}$ be cyclic with characteristic polynomial q_A.*

(i) A is not II-controllable, if it satisfies one of the following conditions

(a) A has a nonreal eigenvalue of multiplicity $\mu > 1$.

(b) A has a real eigenvalue of multiplicity $\mu > 2$.

(ii) A is II-controllable, if $\sigma(A) \subset \mathbb{R}$ and no eigenvalue has multiplicity $\mu > 2$.

Proof. (i) (a) This is an immediate consequence of (14). Differentiating this equality we obtain

$$f'(z) = r'(z)q_A(z) + r(z)q_A'(z)\,. \tag{16}$$

If q_A has multiple roots with nonzero imaginary part, the previous equation implies that these are roots of f'. This implies that f does not have only real roots as the roots of f' are contained in the convex hull of the roots of f.
(i) (b) If a polynomial p has only real roots then it is easy to see that all the roots of p' that are not roots of p are simple and real. From (16) we see that real eigenvalues of A of multiplicity greater than two are at least double roots of f'. Also, by construction of f they are not roots of f, which yields a contradiction.
(ii) Let $\Lambda := \{\lambda_1, \dots, \lambda_k\}$ be the set of simple eigenvalues and $\mu_1 < \dots < \mu_l$ be the double eigenvalues of A. If $q_A''(\mu_j)$ has the same sign for $j = 1, \dots, l$, then for some $\alpha > 0$ small enough one of the polynomials

$$\alpha + q_A(z)\,, \quad -\alpha + q_A(z)$$

has only real roots, all of which are distinct and we are done. Otherwise, suppose $q_A''(\mu_j) > 0$ for $j = 1, \dots, j_0 - 1$ and $q_A''(\mu_{j_0}) < 0$. Choose $\mu_{j_0-1} < c < \mu_{j_0}$, $c \notin \Lambda$, then the polynomial

$$g_1(z) := (c - z)q_A(z)$$

has the same set of double roots as q_A and we have $g_1''(\mu_j) = (c - \mu_j)q_A''(\mu_j)$. From the choice of c it follows that $g_1''(\mu_j) > 0$ for $j = 1, \dots, j_0$ and we see that after the multiplication of q_A with a finite number t of factors of the form $c_s - z$ we obtain a polynomial where all double roots are local minima. Hence the polynomial

$$\alpha + q_A(z)\prod_{s=0}^{t-1}(c_s - z)$$

has only simple real roots for all $\alpha > 0$ small enough. □
We state the following lemma, which despite its triviality provides a way to construct further cases in which representations of the form (14) do not exist.

Lemma 5.2 *Let $A \in \mathbb{R}^{n\times n}$ be arbitrary.*

(i) If for two eigenvalues $\lambda_1, \lambda_2 \in \sigma(A)$ and all $u \in \mathbb{R}$ we have

$$|\lambda_1 - u| < |\lambda_2 - u|,$$

then A does not satisfy (8) for any $t \in \mathbb{N}$ and any sequence $(u_0, ..., u_{t-1}) \in U^t$. In particular, if A is cyclic then A is not II-controllable.

(ii) If A is cyclic and the spectrum $\sigma(A)$ is symmetric with respect to rotation by a root of unity, i.e. $\sigma(A) = \exp(2\pi i/m)\sigma(A)$ (taking into account multiplicities), then the existence of a representation of the form (14) implies that there exists a universally regular control sequence $u \in U^t$ such that

$$\prod_{s=0}^{t-1}(A^m - u_s^m) \in \mathbb{R}^* I. \tag{17}$$

Proof. (i) Assuming (8) is satisfied for $u \in U^t$ we have $\alpha I = \prod_{s=0}^{t-1}(A - u_s I)$. In particular, this implies $\alpha = \prod_{s=0}^{t-1}(\lambda_1 - u_s) = \prod_{s=0}^{t-1}(\lambda_2 - u_s)$. However, the assumption implies that $\prod_{s=0}^{t-1}|\lambda_1 - u_s| < \prod_{s=0}^{t-1}|\lambda_2 - u_s|$, a contradiction.
(ii) Again we assume that a polynomial $f(z) = \prod_{s=0}^{t-1}(z - u_s)$ satisfying (14) exists, then for each $\lambda \in \sigma(A)$ we have

$$f(\lambda) = \prod_{s=0}^{t-1}(\lambda - u_s) = \alpha. \tag{18}$$

Multiplication with $\exp(k2\pi i/m)^t$ (the k-th root of unity of order m raised to the t) yields

$$\prod_{s=0}^{t-1}(\exp(k2\pi i/m)\lambda - \exp(k2\pi i/m)u_s) = \exp(k2\pi i/m)^t \alpha,$$

for all $\lambda \in \sigma(A)$. As $\sigma(A) = \exp(k2\pi i/m)\sigma(A)$ by assumption this implies that

$$\prod_{s=0}^{t-1}(\lambda - \exp(k2\pi i/m)u_s) = \exp(k2\pi i/m)^t \alpha,$$

for all $\lambda \in \sigma(A)$, $k = 1, \ldots, m$. Multiplying these equations for $k = 1, ..., m$ we obtain that

$$\prod_{s=0}^{t-1}(\lambda^m - u_s^m) = \prod_{s=0}^{t-1}\prod_{k=1}^{m}(\lambda - \exp(k2\pi i/m)u_s) = \prod_{k=1}^{m}\exp(k2\pi i/m)^t \alpha,$$

and the expression on the right hand side is equal to $(-1)^{m+1}\alpha^m$. For the case $m > 2$ the II-controllability of A implies that all nonreal eigenvalues of A are simple by Proposition 5.1 (i). Then the real eigenvalues of A are also simple by the assumed symmetry.

Hence all the eigenvalues of A^m have algebraic and geometric multiplicity equal to m and the above calculation shows that the polynomial $\prod_{s=0}^{t-1}(z - u_s^m)$ evaluated in A^m is a nonzero multiple of the identity.
For the case $m = 2$ and real eigenvalues $\lambda, -\lambda$ it is possible that the multiplicity of λ is two. In this case it remains to show that $\frac{d}{dz}\prod_{s=0}^{t-1}(z^2 - u_i^2)_{z=\lambda} = 0$. Note that we have in addition to (18) that $f'(\lambda) = 0 = f'(-\lambda)$. Now $g(z) := \prod_{s=0}^{t-1}(z^2 - u_i^2) = -f(z)f(-z)$. So that $g'(\lambda) = -f'(\lambda)f(-\lambda) - f(\lambda)f'(-\lambda) = 0$ as desired. □

We can exploit the previous result in the following immediate fashion.

Corollary 5.3 *Let $A \in \mathbb{R}^{n\times n}$ be cyclic.*

(i) If for two eigenvalues $\lambda_1, \lambda_2 \in \sigma(A)$ we have $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2, |\lambda_1| \neq |\lambda_2|$ then A is not II-controllable.

(ii) If the spectrum $\sigma(A)$ is symmetric with respect to rotation by a root of unity, i.e. $\sigma(A) = \exp(2\pi i/m)\sigma(A)$ (taking into account multiplicities) and two eigenvalues of A^m satisfy the condition of (i) then A is not II-controllable.

If, furthermore, m is even, then it is sufficient that for two eigenvalues of A^m we have

$$|\lambda_1 - u| < |\lambda_2 - u| ,$$

for all $u > 0$ in order that A is not II-controllable.

Proof. The assumption of (i) on the eigenvalues λ_1, λ_2 implies that of Lemma 5.2 (i). To prove (ii) suppose A is II-controllable. Then, under the symmetry constraint, A^m satisfies (17). Thus if two eigenvalues of A^m satisfy condition (i) the same arguments as for Lemma 5.2 (i) lead to a contradiction. This shows the first part of (ii). For the second part note that if m is even, then the term u^m in (17) is nonnegative. So the argument can be reduced to the nonnegative shifts for A^m. As the construction of a universally regular control requires the application of $n - 1$ different shifts, a single shift can be disregarded. So that, omitting $u = 0$, it is sufficient that only nonnegative shifts fulfill the absolute value inequality.

□

Using this corollary it is easy to construct examples of matrices that are not II-controllable. Such are e.g. the companion matrices of the polynomial $p(z) = z(z^2 + 1)$ and the 7-th degree polynomial whose roots are 0, the three cubic roots of i and their respective complex conjugates. Using the last statement, one sees that the matrix corresponding to $p(z) = (z^2 - 1)(z^2 + 1)$ is not II-controllable. Many more examples like this can be constructed, as we see in the following result. This list is, of course, far from complete and can be extended easily for matrices of higher dimensions and more intricate spectral patterns. The patterns discussed in the following corollary are depicted in Figures 1,2 for the special case, that the spectrum is symmetric to the imaginary axis. The shaded regions show those areas in which 4 centrally symmetric eigenvalues exclude II-controllability if the other eigenvalue(s) are fixed.

Corollary 5.4 *If one of the following cases is satisfied, then the matrix A is not II-controllable.*

(i) Let $A \in \mathbb{R}^{n\times n}, n \geq 4$ and assume there exist $\lambda_i \in \sigma(A), i = 1,\dots,4$ with $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, $\lambda_3 = \overline{\lambda_4}$, $\operatorname{Re}\lambda_3 = \frac{1}{2}(\lambda_1+\lambda_2)$ and $\operatorname{Im}\lambda_3 \geq \frac{1}{2}|\lambda_1 - \lambda_2|$.

(ii) Let $A \in \mathbb{R}^{n\times n}, n \geq 5$ and assume there exist $\lambda_i \in \sigma(A), i = 1,\dots,5$ with $\lambda_1 \in \mathbb{R}$, $\lambda_2 = \overline{\lambda_3}, \lambda_4 = \overline{\lambda_5}$, $\lambda_1 = \frac{1}{4}\sum_{j=2}^{5}\lambda_j$, $\operatorname{Im}\lambda_2 = \operatorname{Im}\lambda_4 \geq |\operatorname{Re}\lambda_1 - \operatorname{Re}\lambda_2| > 0$.

(iii) Let $A \in \mathbb{R}^{n\times n}, n \geq 6$ and assume there exist $\lambda_i \in \sigma(A), i = 1,\dots,6$ with $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, $\lambda_3 = \overline{\lambda_4}$, $\lambda_5 = \overline{\lambda_6}$, $\frac{1}{2}(\lambda_1+\lambda_2) = \frac{1}{4}\sum_{j=3}^{6}\lambda_j$ with the properties $|\lambda_j - \frac{1}{2}(\lambda_1+\lambda_2)| \geq \frac{1}{2}|\lambda_1-\lambda_2|, j = 3,\dots,6$ and $\operatorname{Re}((\lambda_j - \frac{1}{2}(\lambda_1+\lambda_2))^2) < \frac{1}{2}|\lambda_1-\lambda_2|$ for $j = 3,\dots,6$.

(iv) Let $A \in \mathbb{R}^{n\times n}, n \geq 6$ and assume there exist $\lambda_i \in \sigma(A), i = 1,\dots,6$ with $\lambda_1 = \overline{\lambda_2}$, $\lambda_3 = \overline{\lambda_4}$, $\lambda_5 = \overline{\lambda_6}$, $\operatorname{Im}\lambda_3 = \operatorname{Im}\lambda_5$,$\operatorname{Re}\lambda_1 = \frac{1}{4}\sum_{j=3}^{6}\lambda_j$ with the property $\operatorname{Re}((\lambda_3 - \frac{1}{2}(\lambda_1+\lambda_2))^2) < (\operatorname{Im}\lambda_1)^2$.

Proof. In all cases we will just check that the assumptions of Corollary 5.3 (ii) for the case $m = 2$ are satisfied.

(i) Note that after a shift we may assume that $\operatorname{Re}\lambda_3 = 0$. Then the assumption imply that $\sigma(A) = -\sigma(A)$. The eigenvalues of A^2 are $\lambda_1^2 > 0$ and $\lambda_3^2 < 0$, which satisfy by assumption, that $\lambda_1^2 \leq |\lambda_3^2|$ and thus also

$$|\lambda_1^2 - u| < \lambda_1^2 + u \leq |\lambda_3^2| + u = |\lambda_3^2 - u|\,, \text{ for all } u > 0.$$

(ii) After a shift we may assume that $\lambda_1 = 0$ and the remaining four eigenvalues are centrally symmetric to zero, so that we have $\sigma(A) = -\sigma(A)$. The eigenvalues of A^2 are $0, \lambda_2^2 = \overline{\lambda_4^2}$. Now the assumptions imply that $\operatorname{Re}(\lambda_2^2) < 0$ and hence for all $u > 0$ we have that

$$|\lambda_1^2 - u| = u < |\operatorname{Re}(\lambda_2^2) - u| < |\lambda_2^2 - u|\,.$$

(iii) After a shift we may assume that $-\lambda_1 = \lambda_2$ and the remaining four eigenvalues are centrally symmetric to zero, so that we have $\sigma(A) = -\sigma(A)$. The eigenvalues of A^2 given by $\lambda_1^2, \lambda_3^2 = \overline{\lambda_4^2}$ all have multiplicity 2. The asumption implies that $|\lambda_3| \geq |\lambda_1|$ and $\operatorname{Re}(\lambda_3^2) < |\lambda_1|$ and hence for all $u > 0$ we have that

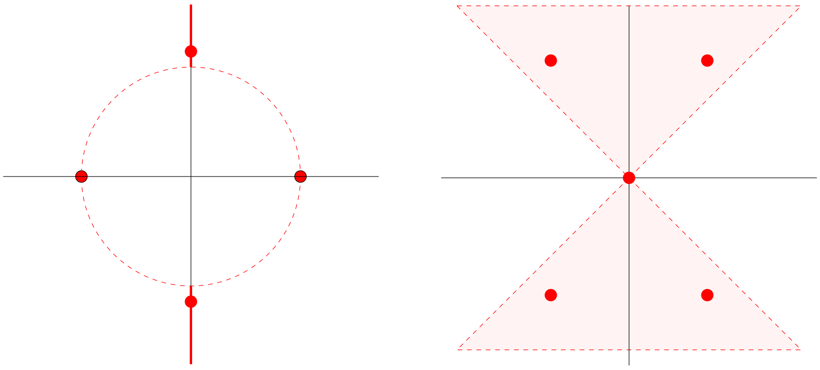
$$|\lambda_1^2 - u|^2 = \lambda_1^4 - 2u\lambda_1^2 + u^2 < |\lambda_3|^2 - 2u\operatorname{Re}(\lambda_3^2) + u^2 = |\lambda_3^2 - u|^2\,,$$

and the assertion follows upon taking square roots.

(iv) After a shift we may assume that $\operatorname{Re}\lambda_1 = 0$ and the remaining four eigenvalues are centrally symmetric to zero, so that we have $\sigma(A) = -\sigma(A)$. The assumptions imply that $\operatorname{Re}(\lambda_3^2) < \lambda_1^2 < 0$ and so for all $u > 0$ we have

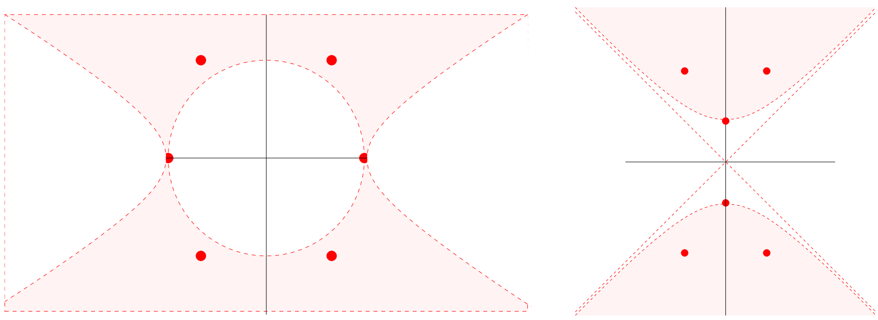
$$|\lambda_1^2 - u| = -\lambda_1^2 + u < |\operatorname{Re}(\lambda_3^2) - u| < |\lambda_3^2 - u|\,.$$

□



(a) 4 eigenvalues (b) 5 eigenvalues

Figure 1: Cases (i) and (ii)



(a) 6 eigenvalues, (iii) (b) 6 eigenvalues, (iv)

Figure 2: Impossible locations of 6 eigenvalues

Finally, for $n \leq 3$ the following complete result can be given.

Proposition 5.5 *Let* $A \in \mathbb{R}^{n \times n}$ *be cyclic.*

(i) If $n = 1, 2$ *then* A *is II-controllable.*

(ii) If $n = 3$ *then* A *is II-controllable if and only if the eigenvalues* $\lambda_1, \lambda_2, \lambda_3$ *of* A *do not have a common real part, i.e. do not satisfy* $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = \operatorname{Re}\lambda_3$.

Proof. (i) In the case $n = 1$ there is nothing to show. If $n = 2$ then note that for any quadratic monic polynomial q we can find a constant c such that $q(z) - c$ has two distinct real roots, so that it is trivial to satisfy condition (14).
(ii) The necessity of the result follows from Proposition 5.1 (i) and Corollary 5.3 (i). Thus it remains to show the sufficiency. If all roots of q are real the assertion is a consequence of Proposition 5.1 (ii). So we have to treat the case of one pair of imaginary roots $\lambda, \bar{\lambda}$ and a

real root μ with $\mu \neq \operatorname{Re}\lambda$. Note that by the transformation $z \mapsto z/\alpha - \beta\ \alpha, \beta \in \mathbb{R}, \alpha \neq 0$ it suffices to prove the assertion for $\lambda = ir$ purely imaginary where $r > 0$ and $\mu = 1$.
We will now construct products of two shifts with the property that the absolute value of the result is the same for μ and λ (and hence also for $\overline{\lambda}$). For any two real numbers a, b we have:

$$(ir - a)(ir - b) = ab - r^2 + ir(a + b)$$

$$(1 - a)(1 - b) = ab + 1 - (a + b)$$

With the notation

$$x = ab, \quad y = a + b \tag{19}$$

the condition

$$|(ir - a)(ir - b)|^2 = |(1 - a)(1 - b)|^2$$

may equivalently be written as

$$(x - r^2)^2 + r^2 y^2 = (x - y + 1)^2 .$$

This leads to the requirement

$$x = \frac{1}{2} \frac{(r^2 - 1)\, y^2 + 2\, y + r^4 - 1}{r^2 + 1 - y}, \; y \neq 1 + r^2 \tag{20}$$

And the condition that $a, b \in \mathbb{R}$ enforces $y^2/4 \geq x$. However,

$$y^2/4 > \frac{1}{2} \frac{(r^2 - 1)\, y^2 + 2\, y + r^4 - 1}{r^2 + 1 - y}$$

may clearly be solved for all $r \in \mathbb{R}$ by choosing $y > 0$ large enough (depending on r). Thus we may conclude as follows: Given $r \in \mathbb{R}$ we may choose $a, b \in \mathbb{R}$ such that the polynomial $f_{a,b}(z) := (z - a)(z - b)$ satisfies

$$|f_{a,b}(ir)| = |f_{a,b}(1)|$$

by choosing x, y such that (20) is satisfied and such that there exist real solutions a, b for (19). With these choices of a, b we can reach from the point ir the set of points

$$x - r^2 + iry = \frac{1}{2} \frac{(r^2 - 1)\, y^2 + 2\, y + r^4 - 1}{r^2 + 1 - y} - \frac{r^4 + r^2 - yr^2}{r^2 + 1 - y} + iry$$

$$= \frac{1}{2} \left(\frac{r^2 y^2}{r^2 + 1 + y} - r^2 + 1 + y \right) + iry$$

(for y large enough) in such a fashion that the absolute value of these points coincides with the absolute values of the points reached from 1 with the application of the same shifts.
By continuity and as the solutions for (20) do not all have the same argument (otherwise they would lie on a line through zero) we may choose x, y subject to the condition that

$$\arg(x - r^2 + iry) \in \pi\mathbb{Q}.$$

And hence for some $m \in \mathbb{N}$ we have $f_{a,b}(ir)^m = f_{a,b}(1)^m$. To obtain a universally regular representation we need of course 3 different zeros of f. This can be easily obtained by choosing a different pair a', b' subject to the same condition. Then we have constructed f as desired. □

6 Conclusions

In this paper we have studied the inverse power iteration with real shifts as a control system on projective space. An algebraic condition for complete controllability on the submanifold M of cyclic vectors has been derived. The characterization of this condition is far from complete. But the results obtained already reveal an intricate structure, that appears to be quite complicated in high dimensions. The easiest case to control is that of a purely real spectrum. This fits in nicely with known results on the analysis of the shifted inverse power iteration which has good convergence properties in the case of Hermitian matrices.
Some fundamental questions about the concept of II-controllability remain unanswered. It is unclear, whether the set of II-controllable cyclic matrices is open, though we know from Remark 4.2 and Proposition 5.1 (ii) that it contains a large open set. Furthermore it would be reasonable to conjecture that the property of II-controllability is generic, but again we have been unable to find a proof beyond dimension 3, although all conditions we have found implying uncontrollability are of course non-generic. The matter would be simplified if it were known that the set of II-controllable matrices is semi-algebraic (as it turns out to be for $n = 1, 2, 3$). If it were known that it is possible to give bounds (only depending on the dimension n) on the necessary length of the universally regular control sequence u in the representation (8), then it is an immediate consequence of the Tarski-Seidenberg principle, that the set of II-controllable matrices is semi-algebraic.

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