Robustness of nonlinear systems and their domains of attraction

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1 Introduction

In this paper we consider the problem of analyzing the robustness of stability of nonlinear systems with time-varying perturbations. The key idea is to define a stability radius for the perturbed nonlinear system, and then to examine the related stability radii for the linearized system. Following the approach outlined in [3, 5, 21] we assume there exists a fixed point x^* of the nonlinear system, and that it is singular with respect to the perturbations, *i.e.* not perturbed under the perturbation class considered. For this fixed point we define the exponential stability radius. It was shown in [5] that lower and upper bounds of this stability radius can be obtained by studying the linearization in x^*. We show that generically the stability radii of the nonlinear system and its linearization coincide. A brief introduction to a method for the calculation of the linear stability radius is presented.

Having thus obtained some understanding of the local problem, we go on to consider the problem of determining a robust domain of attraction for the fixed point. For nonlinear systems one basic question is that of the determination of domains of attraction of asymptotically stable fixed points. This question has received considerable attention over the last decade, see *e.g.* [23], [2], [17]. We discuss some topological properties of the robust domain of attraction and present an approximation scheme for its determination.

In both cases, the study of the local problem, and the study of the robust domain of attraction, it is seen that the tools of optimal control theory may be applied to yield methods of calculating both the local stability radius, and the robust domain of attraction.

We proceed as follows. In Section 2 we introduce a stability radius for nonlinear systems with time varying perturbations. The concept of a *robust domain of attraction* is introduced, and some remarks on the ways in which the robust domain of attraction may shrink as the admissible size of perturbations increases are made. In Section 3 we develop a local robustness theory based on the linearization of the system. It is shown that generically the linear stability radii coincide, demonstrating that generically one need only consider the linearization in order to determine the nonlinear robustness properties of the system. In Section 4 a method of calculating the nonlinear stability radius based on the methods of discounted optimal control is presented.

In Section 5 we introduce the concept of the robust domain of attraction and a few properties are discussed. In the following section we then analyze the linearization of the nonlinear systems, finding a ball of initial conditions yielding trajectories which robustly converge to the origin. The determination of the domain of attraction, however, is clearly a nonlinear problem, thus in Section 6 we characterize the robust domain of attraction in terms of an optimal control problem, and present approximations to this problems whose value functions are computable as viscosity solutions of Hamilton-Jacobi-Bellman equations. In order to improve these approximations, we suggest how to use the information provided by the linearization. From this we obtain an algorithm for which we prove convergence.

In Section 7 we summarize the results and give a short outlook on remaining problems.

2 Preliminaries

In this paper we study nonlinear systems of the form

$$\begin{aligned} \dot{x}(t) &= f_0(x(t)), \quad t \in \mathbb{R} \\ x(0) &= x_0 \in \mathbb{R}^n, \end{aligned} \tag{1}$$

which are exponentially stable at a fixed point which we take to be 0. By this we mean that there exists a neighborhood U of 0 and constants $c > 1, \beta < 0$ such that the solutions $\varphi(t; x, 0)$ of (1) satisfy $\|\varphi(t; x, 0)\| \leq ce^{\beta t}$ for all $x \in U$. Under the assumptions of local exponential stability it is of interest to know the domain of attraction of 0, defined by

$$\mathcal{A}(0) := \{x \in \mathbb{R}^n \mid \varphi(t; x, 0) \to 0, t \to \infty\} .$$

Assume that (1) is subject to perturbations of the form

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} d_i(t) f_i(x(t)) =: F(x(t), d(t)), \quad t \in \mathbb{R}, \tag{2}$$

where the perturbation functions f_i leave the fixed point invariant, i.e. $f_i(0) = 0, i = 0, 1, \ldots, m$. We assume that the vector fields f_i are locally Lipschitz continuous and continuously differentiable in 0. The *unknown* perturbation function d is assumed to take values in $D \subset \mathbb{R}^m$, where D is compact, convex, with nonempty interior, and $0 \in \operatorname{int} D$. Denote $\mathcal{D} := \{d \in \ell^\infty(\mathbb{R}, \mathbb{R}^m) \mid d(t) \in D \text{ a. a.}\}$. Solutions to the initial value problem (2) with $x(0) = x_0$ for a particular d will be denoted $\varphi(t; x_0, d)$.

It is our aim to analyze two robustness problems related to this setup. The first one is local in nature, as we study the corresponding time-varying stability radius at 0, defined by

$$\begin{aligned} r_{tv}(f_0, (f_i)) := \inf\{\alpha > 0 \mid\ & \exists\, d \in \alpha\mathcal{D} \text{ such that (2)} \\ & \text{is not exponentially stable at } 0\} . \end{aligned}$$

For time-invariant perturbations this problem has been studied in [21].

The second robustness problem considered is that of the domain of attraction of the unperturbed system. Given $d \in \mathcal{D}$, the domain of attraction of 0 at time $t_0 = 0$ for (2) is

$$\mathcal{A}_d(0) := \{x \in \mathbb{R}^n \mid \varphi(t; x, d) \to 0, t \to \infty\} .$$

A robust domain of attraction may now be defined. In the definition of the robust domain of attraction we do assume that the perturbed system is locally exponentially stable for all perturbations, i.e. $r_{tv}(f_0, (f_i)) > 1$.

Definition 2.1 (D-robust Domain of Attraction) *Let $D \subset \mathbb{R}^m$ as before and assume that $r_{tv}(f_0, (f_i)) > 1$. The D-robust domain of attraction of the equilibrium 0 of (2), is*

$$\begin{aligned}\mathcal{A}_D(0) &:= \{x \in \mathbb{R}^n \mid \forall d \in \mathcal{D}, \varphi(t; x, d) \to 0, t \to \infty\} \\ &= \bigcap_{d \in \mathcal{D}} A_d(0)\,.\end{aligned}$$

When studying the robustness properties of the domain of attraction the two main problems of interest are:

1. Given $D \subset \mathbb{R}^m$, determine $\mathcal{A}_D(0)$.
2. Given $A \subset \mathcal{A}(0)$ and a perturbation set $D \subset \mathbb{R}^m$, determine the largest α such that $A \subset \mathcal{A}_{\alpha D}(0)$.

In the first case we are most interested in determining the robust domain of attraction, while in the second case we consider a variant of a stability radius problem. Here the focus is on the mechanism by which stability is lost. Note that if the allowable perturbations are increased there are three different scenarios with which the the property $A \subset \mathcal{A}_{\alpha D}(0)$ is lost at some minimal α_0.

1. Loss of stability at 0: *i.e.* $A \subset \mathcal{A}_{\alpha D}(0)$ for $\alpha < \alpha_0$ and on the other hand $\operatorname{dist}(A, \partial\mathcal{A}_{\alpha D}(0)) > \delta > 0$ for all $0 < \alpha < \alpha_0$. This is the case if linear systems are considered.
2. Contraction of the domain of attraction: As $\alpha \to \alpha_0$ it holds that $\operatorname{dist}(A, \partial\mathcal{A}_{\alpha D}(0)) \to 0$.
3. Birth of an attractor in $\operatorname{int} A$: while $\operatorname{dist}(A, \partial\mathcal{A}_{\alpha D}(0)) > \delta > 0$ for all $0 < \alpha < \alpha_0$ it holds that $A \cap \partial\mathcal{A}_{\alpha_0 D}(0) \neq \emptyset$.

An example for the last scenario is given in the following example.

Example 2.2 *Consider the following perturbed system on $\mathbb{R}$:*

$$\dot{x} = -x + d(t)x\sin(x)$$

with $D = [-1, 1]$. Then, $\mathcal{A}_D(0) = (-\pi/2, \pi/2)$, while for $0 < \alpha < 1$ we have $\mathcal{A}_{\alpha D}(0) = \mathbb{R}$.

In this paper we concentrate on the first question, that is determining the robust domain of attraction, and of obtaining estimates of the robust domain of attraction, *i.e.* determining sets which are guaranteed to lie within $\mathcal{A}_{\alpha D}(0)$. We begin with the analysis of the local problem, for which it is to be expected that linearization will play a vital role. This is developed in the following section.

3 Linearization Theory

As the functions f_i are continuously differentiable we may study the linearization in 0 associated with the nonlinear system (2) given by

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} d_i(t) A_i x(t)\,, \quad t \in \mathbb{R}, \tag{3}$$

where A_i denotes the Jacobian of f_i in 0, $i = 0, \dots, m$. We abbreviate $A(d) = A_0 + \sum d_i A_i$. Solutions of (3) are denoted by $\psi(t; x, d)$.

In the analysis of the problem where constant perturbations were considered, it was seen that an examination of the movement of the eigenvalues as the system was perturbed was key in understanding the problem. It was then possible to show that generically the stability radius for the nonlinear system could be determined by considering the linearization. In the time-varying case we are considering here it is necessary to consider the *Bohl exponent* (or equivalently the maximal Lyapunov exponent), see [15], [3], [4]. For $d \in \mathcal{D}$, let $\Phi_d(t,s)$ denote the evolution operator of the time-varying linear system $\dot{x} = A(d(t))x(t)$. The Bohl exponent of system (3) given $d \in \mathcal{D}$ is defined by

$$\beta(d) := \inf\{\beta \in \mathbb{R} \,|\, \exists M : \|\Phi_d(t,s)\| \le M e^{\beta(t-s)}\, \forall t \ge s \ge 0\}\,,$$

see [8], [15] for an introduction to the properties of the Bohl exponent, in particular for the fact that exponential stability of a linear time-varying system given by $d \in \mathcal{D}$ is equivalent to $\beta(d) < 0$. We denote the maximal Bohl exponent of system (3) by

$$\beta(A_0, \dots, A_m, D) := \max_{d \in \mathcal{D}} \beta(d)\,,$$

where we will suppress the dependence on the matrices $(A_0, \dots, A_m)$ and on the perturbation set D when these are clear from the context. In [1], [3] it is shown that the maximum is indeed attained. Furthermore if $\beta = \beta(A_0, \dots, A_m, D) < 0$ then there exists for every $\varepsilon > 0$ an $M_\varepsilon \ge 1$ such that

$$\|\Phi_d(t,s)\| \le M_\varepsilon e^{(\beta+\varepsilon)(t-s)}\,, \quad \forall t \ge s \ge 0, d \in \mathcal{D}\,. \tag{4}$$

Define the linear stability radii

$$r_{Ly}(A_0, (A_i)) = \inf\{\alpha > 0 \,|\, \beta(A_0, ..., A_m, \alpha D) \ge 0\}\,, \tag{5}$$
$$\bar{r}_{Ly}(A_0, (A_i)) = \inf\{\alpha > 0 \,|\, \beta(A_0, ..., A_m, \alpha D) > 0\}\,. \tag{6}$$

The following lemma from [5] states a basic relationship between the nonlinear and the linear stability radii.

Lemma 3.1

$$r_{Ly}(A_0, (A_i)) \le r_{tv}(f_0, (f_i)) \le \bar{r}_{Ly}(A_0, (A_i))\,.$$

The previous lemma gives rise to the question whether $r_{Ly}(A_0, (A_i))$ and $\bar{r}_{Ly}(A_0, (A_i))$ commonly coincide. A simple example shows that this need not always be the case.

Example 3.2 *Let $n = 2, m = 1$ and*

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly $r_{Ly}(A_0, A_1) = 0$ and $\bar{r}_{Ly}(A_0, A_1) = 1$.

This example is trivial in the sense, that it shows a difference for the two linear stability radii only in the case when the unperturbed linear system is not asymptotically stable. In the discrete-time case it can be shown that only such trivial examples exist. In the continuous time case it is an open question whether there are nontrivial examples. Here we will show a genericity result.

We now further develop the properties of r_{Ly} and $\bar{r}_{Ly}$, when considered as functions of $(A_0, (A_i))$. Recall the following definitions of semi-continuity. A function $f : \mathbb{R}^n \to \mathbb{R}$ is upper (lower) semi-continuous when, given $f(x_0) > c$ (resp. $f(x_0) < c$) for some $x_0 \in \mathbb{R}^n$ there exists a neighborhood $U \subset \mathbb{R}^n$ of x_0 such that $\forall x \in U,\ f(x) > c$ (resp. $f(x) < c$).

It is shown in [1], that the maximal Lyapunov exponent is a continuous function from the set of compact subsets of $\mathbb{R}^{n\times n}$ endowed with the Hausdorff Metric to $\mathbb{R}$. Here the Lyapunov exponent corresponding to a compact set $M \subset \mathbb{R}^{n\times n}$ is defined as the one given by the differential inclusion

$$\dot{x} \in \{Ax \mid A \in M\}.$$

We may now prove the following semi-continuity properties for r_{Ly} and $\bar{r}_{Ly}$, generalizing the results for time-invariant perturbations.

Lemma 3.3 *(i) $r_{Ly}(A_0; (A_i))$ is an upper semi-continuous function of $(A_0; (A_i))$.*

(ii) $\bar{r}_{Ly}(A_0; (A_i))$ is a lower semi-continuous function of $(A_0; (A_i))$.

Proof. Suppose that $\alpha_0 := r_{Ly}(A_0; (A_i)) > c$. Then $\forall d \in c\mathcal{D}$, $\beta(d) \le \varepsilon < 0$. By upper semi-continuity of $\beta(\cdot)$ (see [8]), there exists a neighborhood U of $(A_0; (A_i))$ such that for all $(B_0; (B_i)) \in U$, $\beta(B_0, ..., B_m, cD) \le \varepsilon/2 < 0$, so that $r_{Ly}(B_0; (B_i)) > c$. Thus r_{Ly} is upper semi-continuous.

A similar argument establishes lower semi-continuity of $\bar{r}_{Ly}$. □

In some situations it may be interesting to consider an extended version of the stability radius for the linearized system.

$$r^c_{Ly}(A_0, (A_i)) := \inf\{\alpha > 0 \,|\, \beta(A_0, ..., A_m, \alpha D) \ge c\} \tag{7}$$
$$\bar{r}^c_{Ly}(A_0, (A_i)) := \inf\{\alpha > 0 \,|\, \beta(A_0, ..., A_m, \alpha D) > c\} \tag{8}$$

This allows measurement of the robustness of the system with respect to a guaranteed level of exponential convergence or divergence. It is straightforward to show that these new stability radii may be linked to those of (5) as follows.

Lemma 3.4 *Let $(A_0, \dots, A_m) \in \mathbb{R}^{(n\times n)\times(m+1)}$, then*

$$r^c_{Ly}(A_0, (A_i)) = r_{Ly}(A_0 - cI; (A_i)), \tag{9}$$
$$\bar{r}^c_{Ly}(A_0, (A_i)) = \bar{r}_{Ly}(A_0 - cI; (A_i)). \tag{10}$$

The following lemma is needed in the proof of the main result of this section.

Lemma 3.5 *Let $m, n \in \mathbb{N}$ and $(A_0, \dots, A_m) \in \mathbb{R}^{(n\times n)\times m}$ be fixed. For the maps*

$$g : c \mapsto r_{Ly}(A_0 - cI, (A_i)), \quad \overline{g} : c \mapsto \overline{r}_{Ly}(A_0 - cI, (A_i)),$$

the following statements hold:

(i) g is upper semi-continuous, $\overline{g}$ is lower semi-continuous.

(ii) g is discontinuous at c_0 iff $\overline{g}$ is discontinuous at c_0 iff $g(c_0) < \overline{g}(c_0)$.

(iii) $g, \overline{g}$ have at most countably many discontinuities.

(iv) $g(c) = \overline{g}(c)$ for all $c \in \mathbb{R}$ with the exception of at most countably many points.

Proof.

(i) Semi-continuity is an immediate consequence of Lemma 3.3.

(ii) Let $g(c_0) < \overline{g}(c_0)$ then by definition and continuity of the maximal Lyapunov exponent $g(c_0 + \varepsilon) \geq \overline{g}(c_0)$ for any $\varepsilon > 0$. Thus the assumption implies discontinuity of g at c_0. Discontinuity of $\overline{g}$ at c_0 follows from $g(c_0) \geq \overline{g}(c_0 - \varepsilon)$. Conversely, let g be discontinuous at c_0. By semi-continuity this implies that for any $\varepsilon > 0$ and a suitable constant C we have $g(c_0) < C < g(c_0 + \varepsilon) \leq \overline{g}(c_0 + \varepsilon)$. Now the right hand term tends to $\overline{g}(c_0)$ as $\varepsilon \to 0$, proving $g(c_0) < C \leq \overline{g}(c_0)$. A similar argument works for $\overline{g}$.

(iii) This follows from the monotonicity of $g, \overline{g}$.

(iv) This follows from (ii) and (iii).

□

With the help of the previous results, it is possible to prove the following genericity result, which is the main result of this section.

Theorem 3.6 *(i) For fixed $m \geq 1$ the set $\mathcal{L}$ given by*

$$\{(A_0, \dots, A_m) \,|\, r_{Ly}(A_0, (A_i)) = \overline{r}_{Ly}(A_0, (A_i))\}$$

is a countable intersection of open and dense sets. Furthermore, the Lebesgue measure of the complement $\mathcal{L}^c$ is 0.

(ii) For fixed $m \geq 1$ the set $\mathcal{N}$ of maps $(f_0, \dots, f_m)$ satisfying

$$r_{Ly}(A_0, (A_i)) = \overline{r}_{Ly}(A_0, (A_i)) = r_{tv}(f_0, (f_i))$$

contains a countable intersection of open and dense sets with respect to the C^1-topology on the space of C^1-maps $(f_0, \dots, f_m)$ satisfying $f_0(x^*) = x^*, f_i(x^*) = 0$, $i = 1, \dots, m$.

Proof. (i): We introduce the set

$$T_0 := \{(A_0, \dots, A_m, \alpha) \in \mathbb{R}^{(n\times n)\times(m+1)} \times \mathbb{R} |\ \beta(A_0, \dots, A_m, \alpha D) = 0\},$$

which is clearly closed by the continuity of the maximal Lyapunov exponent.

Note that $(A_0, \dots, A_m, r_{Ly}) \in T_0$, and $(A_0, \dots, A_m, \bar{r}_{Ly}) \in T_0$ again by continuity of the maximal Lyapunov exponent. Thus $(A_0, \dots, A_m) \in \mathcal{L}^c$ iff $\exists\ a, b \geq 0,\ a \neq b$ such that $(A_0, \dots, A_m, a)$, $(A_0, \dots, A_m, b) \in T_0$. Under this condition if follows for all $a < c < b$ that $(A_0, \dots, A_m, c) \in T_0$. For $k \geq 1$ we denote

$$T_{0,k} := \left\{(A_0, \dots, A_m, \alpha) \in \mathbb{R}^{(n\times n)\times(m+1)} \times \mathbb{R} \;\middle|\; \left(A_0, \dots, A_m, \alpha + \frac{1}{k}\right) \in T_0\right\}.$$

Thus $\mathcal{L}^c = \bigcup_{k=1}^{\infty} Q_k$ where

$$Q_k := \{(A_0, \dots, A_m) \,|\exists\, a \geq 0 \text{ such that } (A_0, \dots, A_m, a) \in T_0 \cap T_{0,k}\}.$$

The Q_k are projections of $T_0 \cap T_{0,k}$ onto $\mathbb{R}^{(n\times n)\times(m+1)}$. As $T_0 \cap T_{0,k}$ is closed also Q_k is closed for every $k \geq 1$. Therefore we now need to show that all of the sets Q_k that compose $\mathcal{L}^c$ are nowhere dense in $\mathbb{R}^{n\times n\times(m+1)}$. For this it is sufficient that in every neighborhood of any point in $\mathcal{L}^c$ there exists a point that does not belong to $\mathcal{L}^c$, as any closed set either has interior points or is nowhere dense. As any affine subspace of the form $\{(A_0 - cI, \dots, A_m) \mid c \in \mathbb{R}\}$ intersects $\mathcal{L}^c$ in at most countably many points by Corollary 3.5 (iv) the assertion is proved.

In particular, this shows that $\mathcal{L}^c$ is Lebesgue measurable, and an easy application of Fubini's theorem in conjunction with Corollary 3.5 (iv) shows that the Lebesgue measure of $\mathcal{L}^c$ is zero.

(ii): Note that for $(f_0, \dots, f_m) \in \mathcal{N}$ it is sufficient that for the linearized system $(A_0, \dots, A_m) \in \mathcal{L}$. It is thus sufficient to show that the preimage of an open and dense set under the continuous, linear map

$$\{f_0, f_1, \dots, f_m\} \mapsto \{\frac{\partial f_0}{\partial x}(x^*), \dots, \frac{\partial f_m}{\partial x}(x^*)\}$$

is open and dense. This, however, is clear by definition of the C^1-topology. □

A consequence of the previous theorem is that other stability radii which might be defined for the nonlinear system, e.g. with respect to Lyapunov, or asymptotic stability generically coincide with the exponential stability radius.

4 Calculating the local stability radius

In this section we will briefly present a method for the calculation of the linear stability radii. It is based on an idea for the calculation of Lyapunov exponents presented in [13] and in the discrete-time case in [25]. We define the Lyapunov exponent corresponding to an initial condition $x_0 \neq 0$ and a disturbance $d \in \mathcal{D}$ to be

$$\lambda(x_0, d) = \limsup_{t\to\infty} \frac{1}{t} \log \|\psi(t; x_0, d)\|. \tag{11}$$

By [3]

$$\beta(A_0,\dots,A_m,D) = \max\{\lambda(x_0,d) \mid x_0 \neq 0, d \in \mathcal{D}\},$$

that is $\beta(A_0,...,A_m,D)$ is equal to the maximal Lyapunov exponent of the family of time-varying systems given by (3) and the condition $d \in \mathcal{D}$. This is the quantity studied in [1] and [3]. We now briefly present a way for the calculation of approximations of Lyapunov exponents. Via projection onto the projective space $\mathbb{P}^{n-1}$ we obtain (in local coordinates, which we take to be vectors of unit length) from system (3) the system

$$\dot{s}(t) = \big(A(d(t)) - s(t)^T A(d(t)) s(t) \cdot \mathrm{Id}\big)\, s(t)\,, \tag{12}$$

$$s(0) = s_0 = x_0/\|x_0\| \in \mathbb{P}^{n-1}.$$

Defining the function $q : \mathbb{P}^{n-1} \times D \to \mathbb{R}$, $q(s,d) = s^T A(d) s$ it is an easy calculation (see [3]) that for $s_0 \in \mathbb{P}^{n-1}, d \in \mathcal{D}$ we have

$$||\Phi_d(t,0)s_0|| = \exp\left(\int_0^t q(\eta(s;x_0,d),d(s))ds\right). \tag{13}$$

Thus the Lyapunov exponent is of the form on an average yield along a trajectory of system (12), i.e.

$$\lambda(x_0,d) = \limsup_{t\to\infty} \frac{1}{t}\int_0^t q(\eta(s;x_0,d),d(s))ds\,,$$

where $\eta(s;x_0,d)$ denotes the trajectory of (12). Interpreting this expression as an average yield optimal control problem on $\mathbb{P}^{n-1}$, we introduce the following approximating functional for $\delta > 0$

$$J_\delta(x_0,d) := \int_0^\infty \delta e^{-\delta s} q(\eta(s;x_0,d),d(s))ds\,, \tag{14}$$

with associated value function $V_{\delta,D}(x) := \sup_{d\in\mathcal{D}} J_\delta(x_0,d)$. For these optimal control problems it is known [11] that

$$\kappa_\delta(D) := \max_{x\in\mathbb{P}^{n-1}} V_{\delta,D}(x) \geq \beta(A_0,\dots,A_m,D)\,,$$

and $\kappa_\delta(D) \to \beta(A_0,\dots,A_m,D)$ as $\delta \to 0$, but here we need more details about the corresponding rate of convergence. Recall that a set of matrices $M \subset \mathbb{R}^{n\times n}$ is called irreducible, if only the trivial subspaces $\{0\}$ and $\mathbb{R}^n$ are invariant under all $A \in M$. Recall further, that an invariant control set C of system (12) is a set with the properties

(i) For all $x \in C$ it holds that
$\mathrm{cl}\, O^+(x) := \{y \in \mathbb{P}^{n-1} \mid \exists d \in D, t \geq 0 : y = \eta(t,x,d)\} = \mathrm{cl}\, C$.

(ii) C is a maximal set with property (i).

We have the following relation between these concepts.

Lemma 4.1 *The following statements are equivalent*

(i) The set $A(D)$ is irreducible.

(ii) Every invariant control set C of (12) *contains a basis of $\mathbb{R}^n$.*

Proof. Assume that C is an invariant control set and let $x_1, \dots, x_l \in C$ be a basis of $V := \operatorname{span} C$. Let $x \in V$ and $d \in \mathcal{D}$ be arbitrary. If $x = \sum_{k=1}^{l} \gamma_k x_k$ then it follows

$$\Phi_d(t,0)x = \sum_{k=1}^{l} \gamma_k \Phi_d(t,0) x_k \,,$$

Now by invariance of C the projection of the trajectories $\Phi_d(t,0)x_k$ onto the projective space remain in C. This shows that any trajectory of (3) starting in V remains there for all times. Hence for all $x \in V$, $d \in D$ we have $A(d)x \in V$, so that C spans a subspace invariant under all $A(d), d \in D$. Now the assertion follows. □

If $A(D)$ is not irreducible then there exists a coordinate transform T such that for any $A \in A(D)$ we have

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} & \dots & \dots & A_{1k} \\ 0 & A_{22} & A_{23} & \dots & A_{2k} \\ 0 & 0 & A_{33} & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \dots & 0 & A_{kk} \end{bmatrix},$$

where each of the sets $\mathcal{A}(\mathcal{D})_j := \{A(d)_{jj}; d \in D\}, i = 1 \dots d$ is irreducible. It is then easy to see that

$$r_{Ly}(A_0, (A_i)) = \min_{j=1,\dots,k} r_{Ly}(A_{0,j}, (A_{i,j}))\,,$$

thus we can constrict all considerations to the irreducible case.

The following result is shown in [13] for the case that the projected system is locally accessible, but the proof easily carries over to the following slightly more general assumption.

(A) The set $A(\alpha D)$ is irreducible and there exists a closed maximal integral manifold of (12) that contains exactly one invariant control set.

We present it here because we need a further detail on parameter dependence that is not immediate from [13] (although it is not hard to obtain the result from the proofs presented in that reference). Assumption (A) is satisfied in particular if (12) is locally accessible. However, it is sufficient that in addition to irreducibility there exists a single matrix $A \in A(D)$ with a simple eigenvalue λ_{max} satisfying $\Re\lambda_{max} > \Re\lambda, \lambda \in \sigma(A) \setminus \{\lambda_{max}\}$. This may be seen as follows: The existence of a closed maximal integral manifold N is clear as the orbits of system (12) are orbits of a Lie group. On the other hand if $\hat{x} \in \mathbb{P}^{n-1}$ is an eigenvector corresponding to the eigenvalue λ_{max} then it follows from standard arguments [7] that $\hat{x} \in \mathrm{cl}\, C$ for any invariant control set in $\mathbb{P}^{n-1}$. As the system is forward accessible on N and N is closed this implies $\hat{x} \in C$ for any invariant control set contained in N. By maximality of control sets this implies the existence of a unique invariant control set in N. The existence of at least one invariant control set follows by compactness of N. Furthermore the relative interior $\mathrm{int}_N\, C$ is nonempty, as the system restricted to maximal integral manifolds is locally accessible, see [7].

Lemma 4.2 *Consider system (3) and two constants* $0 < \underline{\alpha} \leq \overline{\alpha} < \infty$. *Assume that (A) holds for system* (12) *with control range* $\underline{\alpha}D$ *then there exists a constant* $M > 0$ *such that for all* $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ *we have*

$$\kappa_\delta(\alpha D) \in [\beta(\alpha D), \beta(\alpha D) + \delta M]\,.$$

Proof. Let N be the closed maximal integral manifold that contains exactly one invariant control set of (12) with control range $\underline{\alpha}D$. We denote this invariant control set by $C_{\underline{\alpha}}$. Then for every $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ there exists an invariant control set $C_\alpha \supset C_{\underline{\alpha}}$ of (12) with control range αD [7]. By Lemma 4.1 we may fix a basis $x_1, \dots, x_n \in \mathrm{int}_N\, C_{\underline{\alpha}}$. Define

$$A := \max_{x \in \mathbb{P}^{d-1}, d \in \overline{\alpha} D} |q(x,d)|\,.$$

As $C_{\underline{\alpha}}$ is the only invariant control set in the compact manifold N it follows that for all $x \in N$ we have $O^+(x) \cap C \neq \emptyset$ and T defined as follows is finite

$$T := \max_{k=1,\dots,n} \sup_{x \in N} \inf\{t \geq 0 \mid \exists d \in \underline{\alpha}\mathcal{D} \text{ such that } \eta(t,x,d) = x_k\}\,.$$

We claim that for any $k = 1, \dots, n$, $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ and $d \in \alpha\mathcal{D}$ we have for all $t > 0$

$$\frac{1}{t}\int_0^t q(\eta(s;x_k,d), d(s))ds \leq \beta(\alpha D) + \frac{2AT}{t}\,.$$

Assume to the contrary the existence of $d \in \alpha D$ and a $t > 0$ such that

$$\frac{1}{t}\int_0^t q(\eta(s;x_k,d), d(s))ds > \beta(\alpha D) + \frac{2AT}{t}\,,$$

and denote $y = \eta(t, x_k, d)$, then there exists a control $d_1 \in \alpha\mathcal{D}$ such that $x_k = \eta(t_1, y, d_1)$ for some $t_1 \leq T$. Denoting the concatenation of the controls $d(\cdot)|_{[0,t)}$ and $d_1(\cdot)|_{[0,t_2)}$ by d_2 and extending d_2 periodically to $\mathbb{R}$ we obtain with $t_2 = t + t_1$

$$\lambda(x_k, d_2) = \frac{1}{t_2}\int_0^{t_2} q(\eta(s;x_k,d_2), d_2(s))ds > \frac{t}{t_2}\beta(\alpha D) + \frac{2AT}{t_2} - \frac{At_1}{t_2} \geq \beta(\alpha D)\,.$$

This contradicts the definition of β. Now the assertion follows as in Section 5 of [13]. □

As $\beta(\alpha D)$ is monotonically increasing in α and $\beta(r_{Ly}(A_0,(A_i))D) = 0$ it follows that

$$c(D) := \liminf_{h \downarrow 0} -\frac{\beta((r_{Ly}(A_0,(A_i)) - h)D)}{h} \geq 0\,.$$

The number $c(D)$ may be interpreted as the supremum of the gradients of those linear functions that have their zero in $r_{tv}(A_0,(A_i))$ and are larger than β on some interval of the form $[a, r_{Ly}(A_0,(A_i))]$, where $a < r_{Ly}(A_0,(A_i))$.

Theorem 4.3 *Consider system (3) and assume that (A) holds for some $\alpha < r_{Ly}(A_0, (A_i))$, then the following properties hold.*

(i) For all $\delta > 0$ it holds that

$$r_{Ly}(A_0, (A_i)) \geq r_\delta(A_0, \mathcal{D}) := \inf\{\alpha > 0 \mid \kappa_\delta(\alpha D) \geq 0\}\,. \tag{15}$$

(ii) $$r_{Ly}(A_0, (A_i)) = \lim_{\delta \to 0} r_\delta(A_0, \mathcal{D}).$$

(iii) If $c(D) > 0$ then there exist $\bar{\delta} > 0$ and a constant $M > 0$ such that for all $0 < \delta < \bar{\delta}$

$$r_{Ly}(A_0, (A_i)) - r_\delta(A_0, \mathcal{D}) \leq \delta M\,.$$

Proof.

(i) If $\alpha > r_{Ly}(A_0, (A_i))$, then $0 \leq \beta(\alpha D) \leq \kappa_\delta(\alpha D)$ by Lemma 4.2. Thus $\alpha \geq r_\delta(A_0, \mathcal{D})$.

(ii) If $\alpha < r_{Ly}(A_0, (A_i))$, then $\beta(\alpha D) < 0$ and by Lemma 4.2 there exists a δ_α such that for all $0 < \delta < \delta_\alpha$ it holds that $\kappa_\delta(\alpha D) < 0$, and therefore for $0 < \delta < \delta_\alpha$ it follows that $\alpha \leq r_\delta(A_0, \mathcal{D}) \leq r_{Ly}(A_0, (A_i))$. Letting α tend to $r_{Ly}(A_0, (A_i))$ from below shows the assertion.

(iii) Choose $\varepsilon > 0$ small enough such that $c := c(D) - \varepsilon > 0$. Then there exists an $\eta > 0$ such that for all $\alpha \in [r_{Ly}(A_0, (A_i)) - \eta, r_{Ly}(A_0, (A_i))]$ we have

$$\beta(\alpha D) < c(\alpha - r_{Ly}(A_0, (A_i))).$$

By Lemma 4.2 for every $\alpha \in [r_{Ly}(A_0, (A_i)) - \eta, r_{Ly}(A_0, (A_i))]$ there exists an $M_\alpha > 0$ such that

$$\kappa_\delta(\alpha D) \leq c(\alpha - r_{Ly}(A_0, (A_i))) + M_\alpha \delta\,.$$

Let $M := \sup\{M_\alpha \mid \alpha \in [r_{Ly}(A_0, (A_i)) - \eta, r_{Ly}(A_0, (A_i))]\}$ and denote the zero of the right hand side in the above equation by

$$\tilde{r}_\delta := r_{Ly}(A_0, (A_i)) - \frac{M}{c(D) - \varepsilon}\delta \leq r_\delta(A_0, \mathcal{D})\,.$$

Then for all $0 < \delta < \delta'$ small enough so that $M\delta c^{-1} < \eta$ we obtain

$$r_{Ly}(A_0, (A_i)) - r_\delta(A_0, \mathcal{D}) \leq r_{Ly}(A_0, (A_i)) - \tilde{r}_\delta = \frac{M}{c}\delta\,.$$

□

The time varying stability radius may thus be calculated by applying Theorem 4.3. A description of the actual mathematical background for the calculation of the objects defined in this section can be found in [9, 10, 12] and references therein.

5 Robust Domains of Attraction

We now turn our attention to a study of the robust domain of attraction. After determining some properties of such sets, we study the problem of determining a ball about the origin which is guaranteed to be in the robust domain of attraction.

We now collect some properties of the robust domain of attraction $\mathcal{A}_D(0)$. It is maybe surprising, that these resemble closely the properties of domains of attraction of fixed points of unperturbed systems, [14]. For our proofs, we need the following slight extensions of the results in [14] and [24].

Lemma 5.1 *Let* 0 *be an asymptotically stable fixed point of the unperturbed system (1). Let* W *be an open subset of* $\mathcal{A}(0)$ *with* $0 \in W$ *that is invariant under (1). Then* W *is connected and contractible to zero. If* f_0 *is of class* C^r*, then there exists a* C^r *diffeomorphism from* W *to* $\mathbb{R}^n$*.*

For the proof of the preceding lemma we recall the following basic lemma from [18].

Lemma 5.2 *Let* M *be a paracompact manifold such that every compact subset is contained in an open set which is diffeomorphic to* $\mathbb{R}^n$*. Then* M *is diffeomorphic to* $\mathbb{R}^n$*.*

Proof. (of Lemma 5.1) Let $V \subset W$ be a connected neighborhood of 0. By definition for each $x \in W$ we have $\varphi(t;x,0) \in V$ for all t large enough. This shows connectedness of W. In the following we denote image of a set V under the flow of (1) by $\varphi(\cdot;V)$. To complete the proof let K be an arbitrary compact subset of W. We choose a relatively compact neighborhood of K that is invariant under (1) as follows: Choose a relatively compact neighborhood V of 0 contained in $\mathcal{A}(0)$ that is invariant under (1). This exists as a sublevel set of a Lyapunov function. Then for some $T > 0$ we have $K \subset \varphi(-T;V)$, and $\varphi(-T;V)$ is relatively compact and invariant. Hence $V_2 := W \cap \varphi(-T;V)$ is a relatively compact open neighborhood of K. For $\varepsilon > 0$ small enough the set $\varphi(\varepsilon;V_2)$ is still an open neighborhood of K with the property that $\mathrm{cl}\,\varphi(\varepsilon;V_2) \subset W$. Now choose a C^r function $\alpha : \mathbb{R}^n \to \mathbb{R}_+$ with the property $\alpha_{|\,\mathrm{cl}\,\varphi(\varepsilon;V_2)} \equiv 1$ and $\alpha_{|\mathbb{R}^n \setminus W} \equiv 0$ and consider the system

$$\dot{x} = \alpha(x) f_0(x)\,, \tag{16}$$

and denote the corresponding flow by $\tilde{\varphi}$. Choose $\varepsilon > 0$ such that $B(0,\varepsilon) \subset W$. Then for some $T > 0$ we have $K \subset \tilde{\varphi}(-T;B(0,\varepsilon))$. On the other hand by construction of α it follows that $\tilde{\varphi}(-T;B(0,\varepsilon)) \subset W$ and $\tilde{\varphi}(-T;B(0,\varepsilon))$ is diffeomorphic to $\mathbb{R}^n$ via the diffeomorphism induced by the flow of (16). This completes the proof. □

Let us briefly recall that system (2) with a perturbation set D is called locally uniformly asymptotically stable, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x\| < \delta$ implies $\|\varphi(t;x,d)\| < \varepsilon$ for all $t > 0, d \in \mathcal{D}$ and if there exists a neighborhood U of 0 such that for all $x \in U$ we have $\varphi(t;x,d) \to 0$ uniformly in d as $t \to \infty$.

Proposition 5.3 *Consider system (2) and assume that* 0 *is locally uniformly asymptotically stable for the perturbation set* D*, then*

- *(i)* $x_0 \in \mathcal{A}_D(0) \Leftrightarrow \lim_{t\to\infty} \sup_{d\in\mathcal{D}} \|\varphi(t;x_0,d)\| = 0\,.$
- *(ii)* $\mathcal{A}_D(0)$ *is an open,connected, invariant set.*

(iii) $\mathrm{cl}\,\mathcal{A}_D(0)$ *is an invariant set.*

(iv) $\mathcal{A}_D(0)$ *is contractible to* 0.

(v) If for some $d \in D$ $f(d)$ *is of class* C^r, *then* $\mathcal{A}_D(0)$ *is* C^r*-diffeomorphic to* $\mathbb{R}^n$.

(vi) For every $x \in \partial\mathcal{A}_D(0)$ *there exists* $d \in \mathcal{D}$ *such that* $\varphi(t; x, d) \in \partial\mathcal{A}_D(0)$ *for all* $t \geq 0$.

Proof.

(i) Clearly we need only show "$\Rightarrow$". Assume that $x \in \mathcal{A}_D(0)$ and there exist sequences $\{d_k\} \subset \mathcal{D}$, $T_k \to \infty$ and $\varepsilon > 0$ such that $\|\varphi(T_k, x, d_k)\| > \varepsilon > 0$ for all $k \in \mathbb{N}$. By uniform stability there exists a $\delta > 0$ such that $\|z\| < \delta$ implies $\|\varphi(t, z, d)\| < \varepsilon$ for all $d \in \mathcal{D}, t \geq 0$. Without loss of generality $d_k \to d \in \mathcal{D}$ in the weak-$*$ topology on $\mathcal{D}$. By assumption there exists a t_0 such that $\|\varphi(t_0, x, d)\| < \delta$. As $\varphi(t_0, x, d_k) \to \varphi(t_0, x, d)$ this means for all k large enough $\|\varphi(t, x, d_k)\| < \varepsilon$ for $t \geq t_0$, a contradiction.

(ii) By assumption there is an open neighborhood V of 0 contained in $\mathcal{A}_D(0)$. By definition from each $x \in \mathcal{A}_D(0)$ there exists a trajectory $\varphi(\cdot, x, d)$ entering V. This shows connectedness. To prove invariance assume that for some $x \in \mathcal{A}_D(0), d_1 \in \mathcal{D}$ there exists a $t > 0$ such that $y := \varphi(t; x, d_1) \notin \mathcal{A}_D(0)$. This implies the existence of a $d_2 \in \mathcal{D}$ such that $\varphi(t, y, d_2) \not\to 0$. But then for the concatenation d given by $d_{|[0,t]} \equiv d_1$, $d_{|(t,\infty)} \equiv d_2(\cdot - t)$ it follows that $\varphi(t; x, d) \not\to 0$ contradicting the choice of x. Finally, to prove that $\mathcal{A}_D(0)$ is open, assume the contrary and let $x \in \mathcal{A}_D(0)$ and assume we are given a sequence $x_k \to x$ with $x_k \notin \mathcal{A}_D(0)$. Then there exist controls d_k such that $\varphi(t; x_k, d_k) \not\to 0$. As in (i) this leads to a contradiction.

(iii) If for some $x \in \mathrm{cl}\,\mathcal{A}_D(0)$ and $d \in \mathcal{D}$ we have $\varphi(t; x, d) \notin \mathrm{cl}\,\mathcal{A}_D(0)$ then by continuous dependence on initial conditions we have that $\mathcal{A}_D(0)$ is not invariant, contradicting (ii).

(iv) This follows from Lemma 5.1.

(v) This is a consequence of (ii) and Lemma 5.1.

(vi) This follows by definition as $\mathcal{A}_D(0)$ is open and $\mathrm{cl}\,\mathcal{A}_D(0)$ is invariant.

$\square$

The first question to consider is under which conditions $\mathcal{A}_D(0)$ contains a neighborhood of 0, that is to give a sufficient condition for local uniform stability of system (2). To examine this question we use the linearization (3) of (2) at 0 with maximal Bohl exponent $\beta(A_0, \ldots, A_m, D)$. By [8, Th. VII.1.3] the Bohl exponent is upper semi-continuous even under nonlinear perturbations. Thus:

Lemma 5.4 *Consider* (2) *with linearization* (3).
(i) If $\beta(A_0, \ldots, A_m, D) < 0$ *then* $\mathcal{A}_D(0)$ *contains an open neighborhood of* 0.
(ii) If $\beta(A_0, \ldots, A_m, D) > 0$ *then* $0 \in \partial\mathcal{A}_D(0)$.

The following example shows that for the case $\beta(A_0, \ldots, A_m, D) = 0$ both situations are possible.

Example 5.5 *Let $A_0, A_1, \dots, A_m \in \mathbb{R}^{n\times n}$ and $D \subset \mathbb{R}^m$ be such that the maximal Bohl exponent $\beta(A_0, A_1, \dots, A_m, D) = 0$ and consider the systems*

$$\dot{x} = -x(t) < x(t), x(t) > + \left(A_0 + \sum_{i=1}^{m} d_i(t) A_i \right) x(t) \tag{17}$$

$$\dot{x} = x(t) < x(t), x(t) > + \left(A_0 + \sum_{i=1}^{m} d_i(t) A_i \right) x(t) \tag{18}$$

By [1] there is a norm v on $\mathbb{R}^n$ that is a Lyapunov function for the linearization

$$\dot{x} = \left(A_0 + \sum_{i=1}^{m} d_i(t) A_i \right) x(t)$$

in the following sense. Denote the dual norm by v^*. Then for any x, y with $v(x) = 1, v^*(y) = 1$ and $< x, y >= 1$ it holds that $< A(d)x, y >\leq 0$ for all $d \in D$ and so that for each x there exists a y and $d \in D$ with $< A(d)x, y >= 0$. By homogeneity of the norm we obtain for the chosen pair x, y that $< -x(t) < x(t), x(t) > +A(d)x, y >< 0$ for system (17). This shows that v is a global Lyapunov function for (17) and hence $\mathcal{A}_D(0) = \mathbb{R}^n$. Similarly, for (18) one obtains that $\mathcal{A}_D(0) = \{0\}$.

By the results of the previous Section 3 the point where the Bohl exponent does not indicate whether $0 \in \mathrm{int}\mathcal{A}_D(0)$ is exactly the perturbation intensity at which the system becomes exponentially unstable.

Furthermore the linearization can be used to obtain a more precise statement on the size of the ball contained in $\mathcal{A}_D(0)$, which is a consequence of [8, Th.VII.1.3]. To this end denote $L(D) := \max_{d\in D} \|A(d)\|$.

Lemma 5.6 *Let $\beta(A_0, \dots, A_m, D) < \beta < 0$ and $M_\beta \geq 1$ such that (4) is satisfied and fix $\beta < \beta' < 0$ and $M > M_\beta$. Let $h > 0,\ q > 0$ be such that*

$$1 - M_\beta e^{-(\beta-\beta')h} > 0 \quad \text{and}$$

$$qhe^{(2L(D)+q+\beta')h} = \min\{M - M_\beta, 1 - M_\beta e^{-(\beta-\beta')h}\}.$$

If $\|F(x,d) - A(d)x\| < q$ for all $x \in B(0,\varepsilon), d \in D$ then

$$\|\varphi(t; x, d)\| \leq M e^{\beta' t}\|x\|\,, \quad \forall x \in B(0, \varepsilon/M), d \in \mathcal{D}\,.$$

In particular, it follows that $B(0, \varepsilon/M) \subset \mathcal{A}_D(0)$.

6 An optimal control characterization of the robust domain of attraction

In this section we present an optimal strategy for the approximation of the robust domain of attraction. This is motivated as follows. By definition we have

$$x_0 \in \mathcal{A}_D(0) \Leftrightarrow \forall d \in \mathcal{D} \lim_{t\to\infty} \|\varphi(t; x_0, d)\| = 0\,,$$

Assuming that $\beta(A_0, \dots, A_m, D) < 0$ and applying Lemma 5.4 we obtain immediately that

$$x_0 \in \mathcal{A}_D(0) \Leftrightarrow \forall d \in \mathcal{D}, \quad \limsup_{t \to \infty} \|\varphi(t; x_0, d)\| = 0 \tag{19}$$

Motivated by Proposition 5.3 (i), the domain of attraction can thus be characterized via the following optimization problem. Define

$$J_0(x, d) := \limsup_{t \to \infty} \|\varphi(t; x, d)\|$$

and the corresponding value function

$$v_0(x) = \sup_{d \in D} J_0(x, d) ,$$

then $\mathcal{A}_D(0) = v_0^{-1}(0)$. In other words, there clearly exists a $c > 0$ such that $x \notin \mathcal{A}_D(0)$ implies that $v_0(x) > c$.

The problem with this value function is obviously its discontinuity at the boundary of $\mathcal{A}_D(0)$. As v_0 is hard to calculate we use a the approximation scheme already introduced in Section 4. For $\delta > 0$ define

$$J_\delta(x, d) := \int_0^\infty \delta e^{-\delta t} \|\varphi(t; x, d)\| dt$$

with value function $v_\delta(x) = \sup_{d \in \mathcal{D}} J_\delta(x, d)$. Note that v_δ is continuous w.r.t. x.

Although it is not generally true that v_δ is strictly decreasing w.r.t. δ, it is possible to obtain a convergence result. For $M \geq 1$ and $0 > \beta > \beta(A_0, \dots, A_m, D)$ denote

$$X(M, \beta) := \{x \mid \forall t > 0 : \sup_{d \in \mathcal{D}} \|\varphi(t; x_0, d)\| \leq M e^{\beta t}\}$$

Note that $\mathcal{A}_D(0) \supset X(M, \beta)$. We note the following properties of $X(M, \beta)$.

Proposition 6.1 *Consider* (2) *with linearization* (3) *and assume that* $\beta(A_0, \dots, A_m, D) < \beta < 0$, *then*

(i) $\mathcal{A}_D(0) = \bigcup_{M \geq 1} \operatorname{int} X(M, \beta)$,

(ii) $x \in X(M, \beta) \Rightarrow v_\delta(x) \leq M \frac{\delta}{\delta - \beta}$.

Proof. (i) Note that by Lemma 5.6 for some M_0 large enough and some $\varepsilon > 0$ we have $B(0, \varepsilon) \subset \operatorname{int} X(M_0, \beta)$. Let $x \in \mathcal{A}_D(0)$. By Proposition 5.3 (i) there exists a $T > 0$ such that for all $d \in \mathcal{D}$ we have $\varphi(T; x, d) \in B(0, \varepsilon/(2M_0))$. It follows that for a relatively compact neighborhood U of x we have $\varphi(T; y, d) \in B(0, \varepsilon/M_0)$ for all $y \in U, d \in \mathcal{D}$. Defining $M := \max\{M_0, e^{-\beta T} \sup\{ \|\varphi(t; y, d)\| \mid t \in [0, T], d \in \mathcal{D}, y \in \operatorname{cl} U\}\}$ we have that $U \subset X(M, \beta)$ as desired.

(ii) This follows from

$$v_\delta(x) = \sup_{d \in \mathcal{D}} J_\delta(x, d) \leq \sup_{d \in \mathcal{D}} \int_0^\infty \delta e^{-\delta t} M e^{\beta t} = M \frac{\delta}{\delta - \beta} .$$

□

Corollary 6.2 *Consider* (2) *with linearization* (3).

$$v_\delta \to v_0 \text{ uniformly on compact subsets of } \mathcal{A}_D(0) \text{ as } \delta \to 0.$$

Proof.

Let $K \subset \mathcal{A}_D(0)$ be compact and fix $0 > \beta > \beta(A_0, \dots, A_m, D)$, then by compactness and Proposition 6.1 (i) there exists an M, such that $K \subset X(M, \beta)$. Now the assertion follows from Proposition 6.1 (ii). □

The previous statement implies that v_δ converges linearly on compact subsets of $\mathcal{A}_D(0)$ to 0. To obtain an estimate for $\mathcal{A}_D(0)$ define

$$\mathcal{A}(\delta, \varepsilon) := \{x \in \mathbb{R}^n \mid v_\delta(x) < \varepsilon\}.$$

Then we have

Proposition 6.3 *Consider* (2) *and assume that* $\beta(A_0, \dots, A_m, D) < 0$ *then for all* $0 < \varepsilon \leq c_0 :=$ $\mathrm{dist}(0, \partial\mathcal{A}_D(0))$

$$\mathcal{A}_D(0) = \bigcup_{\delta > 0} \mathcal{A}(\delta, \varepsilon) = \bigcup_{\delta^* > \delta > 0} \mathcal{A}(\delta, \varepsilon), \quad \forall \delta^* > 0$$

Proof. This is an immediate consequence of Proposition 6.1 as we have for any $M \geq 1$ and $\beta(A_0, \dots, A_m, D) < \beta < 0$ and $\delta > 0$ small enough that $X(M, \beta) \subset \mathcal{A}(\delta, \varepsilon)$. □

In general, information about c_0 amounts to the solution of the original problem itself, so we need a lower bound on c_0. Using the quantities introduced in Lemma 5.6, assume that $\|F(x, d) - A(d)x\| < q$ for all $x \in B(0, \varepsilon), d \in D$, then $B(0, \varepsilon/M) \subset \mathcal{A}_D(0)$ and so $\varepsilon/M \leq c_0$ is the lower bound we require.

In order to use the information provided by Proposition 6.3 we have to obtain estimates for the quantities $\beta(A_0, \dots, A_m, D)$ and M_β as used in Lemma 5.6, from these the quantities q and ε are determinable and Proposition 6.3 is then applicable.

In order to estimate ε and M, we need some information about the local growth properties of the perturbed system. Via Lemma 5.6 these may be obtained by examining the linearization at 0, defined in Section 3. In the following we use the notation of Section 4, where one problem has already been discussed, namely that of approximating the maximal Bohl exponent. Thus if $\beta(A_0, \dots, A_m, D) < 0$, then choosing $\delta > 0$ small enough in the optimization problem defined by (14) we can obtain $0 > \kappa_\delta(D) > \beta(A_0, \dots, A_m, D)$. It remains to obtain a constant M, such that (4) is satisfied.

Let $0 > \kappa > \kappa_\delta(D)$. By (13) it is sufficient to find $T > 0$ such that

$$\sup_{||x||=1, d \in \mathcal{D}} \int_0^T q(\psi(s; x, d), d(s)) - \kappa ds < 0$$

Then it follows that $||\Phi_d(T, 0)|| < e^{\kappa T}$, $\forall d \in \mathcal{D}$, and so

$$||\Phi_d(t, 0)|| < e^{L(D)T} e^{\kappa t}, \ \forall d \in \mathcal{D}, \forall t > 0$$

Note that in order to find T, the value function v_δ that has already been calculated can be used, and it is sufficient to find T such that

$$\sup_{||x||=1, d \in \mathcal{D}} \int_0^T \delta e^{-\delta s}(q(\psi(s; x, d), d(s)) - \kappa) ds < 0 \tag{20}$$

Note that solvability of (20) depends on the fact that $\kappa > \kappa_\delta(D)$, as for $\kappa_\delta(D)$, the expression on the left is always nonnegative.

With the estimates obtained so far, we are now in a position to describe an algorithm for determining $\mathcal{A}_D(0)$, which is the main contribution of this section.

Algorithm 6.4 *Given $f_0, \dots, f_m$ and D such that $\beta(A_0, \dots, A_m, D) < 0$:*

1. *Calculate κ_δ for small δ, such that $\kappa_\delta < 0$.*
2. *With the data κ_δ, M satisfying (4), determine a ball $B(0,\varepsilon)$ contained in $\mathcal{A}_D(0)$ via Lemma 5.6.*
3. *Let $\varepsilon_0 = \varepsilon$, $\hat{\mathcal{A}}_0 = B(0,\varepsilon)$.*
4. *Determine the value function $v_{\delta,k}$ associated with the cost functional*

$$J_{\delta,k}(x,d) := \int_0^\infty \delta e^{-\delta t} g_k(\varphi(t;x,d))dt\,, \tag{21}$$

where $g_k(x) = \|x\|$ if $\|x\| \notin \hat{\mathcal{A}}_k$, $g_k(x) = 0$, otherwise.

5. *Determine ε_{k+1} such that $B(0,\varepsilon_{k+1}) \subset \hat{\mathcal{A}}_{k+1} := v_{\delta,k}^{-1}([0,\varepsilon_k]) \cup B(0,\varepsilon)$. Continue with step 4.*

Theorem 6.5 *Consider system (1) with perturbation structure (2). If $\beta(A_0, \dots, A_m, D) < 0$ then the sets $\hat{\mathcal{A}}_k$, $k = 1, 2, \dots$ generated by Algorithm 6.4 form a monotonically increasing sequence such that $\cup_{k=0}^\infty \hat{\mathcal{A}}_k = \mathcal{A}_D(0)$.*

Proof. Note that is is clear by definition that $\hat{\mathcal{A}}_0 \subset \hat{\mathcal{A}}_1$ and $g_1 \le g_0$. Thus we may proceed by induction assuming that $\hat{\mathcal{A}}_0 \subset \dots \hat{\mathcal{A}}_{k-1} \subset \hat{\mathcal{A}}_k$ and $g_0 \ge \dots g_{k-1} \ge g_k$. With this we obtain for $x \in \hat{\mathcal{A}}_k$,

$$\sup_{d\in\mathcal{D}} \int_0^\infty \delta e^{-\delta t} g_k(\varphi(t;x,d))dt \le \sup_{d\in\mathcal{D}} \int_0^\infty \delta e^{-\delta t} g_{k-1}(\varphi(t;x,d))dt \le \varepsilon_{k-1} \le \varepsilon_k\,.$$

It follows that $x \in \hat{\mathcal{A}}_{k+1}$. So $\hat{\mathcal{A}}_k \subset \hat{\mathcal{A}}_{k+1}$ and consequently, $g_k \ge g_{k+1}$.

Let $x \in \mathcal{A}_D(0)$ and assume $x \notin \hat{\mathcal{A}}_1$, as there is nothing to show otherwise. Define

$$T := \sup\{t \mid \exists d \in \mathcal{D} : \phi(t,x,d) \notin \hat{\mathcal{A}}_1\}$$

Note that T is finite by Proposition 5.3 (i) and the fact that $B(0,\varepsilon) \subset \hat{\mathcal{A}}_1$. Let

$$\mathcal{O}^+_{\le t}(x) := \{y \in \mathbb{R}^n \mid \exists d \in \mathcal{D}, 0 \le s \le t : y = \phi(s,x,d)\}\,.$$

Note that $\mathcal{O}^+_{\le T}(x)$ is compact and let R be such that $\mathcal{O}^+_{\le T}(x) \subset B(0,R)$. Let

$$h = -\frac{\log(1-\varepsilon/R)}{\delta}\,,$$

then it follows for $y \in \mathcal{O}^+_{\le T}(x) \setminus \mathcal{O}^+_{\le T-h}(x)$ that

$$h \ge \sup\{t \mid \exists d \in \mathcal{D} : \phi(t,y,d) \notin \hat{\mathcal{A}}_1\}\,,$$

as otherwise we have an immediate contradiction to the definition of T. It follows that for any $d \in \mathcal{D}$ we have

$$\int_0^\infty \delta e^{-\delta t} g_k(\phi(t,y,d))dt \leq \int_0^h \delta e^{-\delta t} R dt = R(1-e^{-\delta h}) = \varepsilon\,.$$

This implies $\mathcal{O}^+_{\leq T}(x) \setminus \mathcal{O}^+_{\leq T-h}(x) \subset \hat{\mathcal{A}}_2$. Continuing this argument we see that $x \in \hat{\mathcal{A}}_k$ where k is such that $kh > T$. □

Remark 6.6 *(i) A useful stopping criterion can be applied in step 5 as follows: If $\varepsilon_k - \varepsilon_{k+1}$ is bigger than some threshold go to 4. Otherwise, determine whether to decrease δ and go to 4 or stop, depending on the size of δ.*

(ii) In practice we would suggest to stop the algorithm in step 1 if $\kappa_\delta \geq 0$ for reasonably small δ. The reason for that is that although the nonlinear system may be exponentially stable, the Bohl exponent of the linearization is so small that the system is unlikely to be robustly stable in a meaningful sense.

(iii) The reason for choosing the particular form of g_k (21) is that once a trajectory enters $\hat{\mathcal{A}}_k$, it will robustly converge to 0, and thus there is no longer any need to penalize it in the cost.

(iv) Note that by construction $\hat{\mathcal{A}}_k \subset \mathcal{A}_D(0)$, thus the algorithm supplies an inner approximation of the robust domain of attraction.

7 Conclusion

In this paper we have studied the robustness of stability of a class of perturbed nonlinear systems, both from a local and a semi-global perspective.

First we introduced time-varying stability for nonlinear systems. Using linearization techniques and spectral theory for time-varying linear systems it was shown that the nonlinear stability radius equals the linear stability radii provided exponential stability of the unperturbed system holds. A scheme for the calculation of the time varying stability radius has been proposed, based on discounted optimal control.

Additionally we have discussed robust domains of attraction of singular fixed points. A scheme for the approximation of the robust domain of attraction has been presented. This involves the calculation of approximations of the maximal Bohl exponent of the linearized system and subsequently the solution of an optimal control problem given by the nonlinear system. In this way a sequence of interior approximations to the robust domain of attraction is produced, each approximation being contained in the next.

References

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References

[1] N. E. Barabanov. Absolute characteristic exponent of a class of linear nonstationary systems of differential equations. *Sib. Math. J.*, 29(4):521–530, 1988.

[2] H.-D. Chiang, M.W. Hirsch and F.F. Wu, Stability regions of nonlinear autonomous dynamical systems, *IEEE Trans. Auto. Control*, 33(1):16-27, 1988.

[3] F. Colonius and W. Kliemann. Stability radii and Lyapunov exponents. In D. Hinrichsen and B. Mårtensson, editors, *Control of Uncertain Systems*, number 6 in Progress in Systems and Control Theory, pages 19–55, Basel, 1990. Birkhäuser.

[4] F. Colonius and W. Kliemann. Maximal and minimal Lyapunov exponents of bilinear control systems. *J. Diff. Eqns.*, 101:232–275, 1993.

[5] F. Colonius and W. Kliemann. A stability radius for nonlinear differential equations subject to time varying perturbations. *3rd IFAC Symposium on Nonlinear Control Systems Design (NOLCOS'95)*, Lake Tahoe, USA, pages 44–46, June 1995.

[6] F. Colonius and W. Kliemann, *The Lyapunov spectrum of families of time varying matrices*, Trans. Amer. Math. Soc., 348 (1996), pp. 4389–4408

[7] F. Colonius and W. Kliemann, *The Dynamics of Control*, Birkhäuser, 1999. to appear.

[8] J. Daleckii and M. Krein. *Stability of Solutions of Differential Equations in Banach Spaces.* Number 43 in Translations of Mathematical Monographs. American Mathematical Society, Providence, Rhode Island, 1974.

[9] L. Grüne. Numerical stabilization of bilinear control systems. *SIAM J. Control Optim.*, 34(6):2024–2050, 1996.

[10] L. Grüne. An adaptive grid scheme for the discrete Hamilton-Jacobi-Bellman equation. *Numerische Mathematik*, 75:319–373, 1997.

[11] L. Grüne. *On the relation of discounted and average optimal value functions*, J. Differ. Equ., 148 (1998), pp. 65–99.

[12] L. Grüne. On the relation between discounted and average optimal control problems. *J. Diff. Eqns.*, 148:65-99, 1998.

[13] L. Grüne, F. Wirth , *On the rate of convergence of infinite horizon discounted optimal value functions.* Nonlinear Analysis. 1999. to appear.

[14] W. Hahn. *Stability of Motion.* Springer-Verlag, Berlin, 1967.

[15] D. Hinrichsen, A. Ilchmann, and A. Pritchard. Robustness of stability of time-varying linear systems. *J. Diff. Eqns.*, 82(2):219–250, 1989.

[16] Y. Lin, E.D. Sontag, Y. Wang, A smooth converse Lyapunov theorem for robust stability, *SIAM J. Control Optim.*, 34(1):124-160, 1996.

[17] M. Loccufier and E. Noldus, On the estimation of asymptotic stability regions for autonomous nonlinear systems, *IMA J. Math. Control & Information*, 12:91-109, 1995.

[18] J. Milnor. *Differential topology.* Number II in Lectures in Modern Mathematics. John Wiley & Sons, New York, 1964.

[19] A. Packard and J. Doyle. The complex structured singular value. *Automatica*, 29(1):71–109, 1993.

[20] A. D. Paice and F. R. Wirth. Robustness of nonlinear systems subject to time-varying perturbations. In *Proceedings of the 36th Conference on Decision and Control*, pages 4436–4441, San Diego, CA, December 1997.

[21] A. D. B. Paice and F. R. Wirth. Analysis of the local robustness of stability for flows. *Math. Contr., Sign., and Syst.*, 11(4):289–302, 1998.

[22] A.D.B. Paice and F. Wirth. *Robustness analysis of domains of attraction of nonlinear systems*, Proceedings of the Mathematical Theory of Networks and Systems MTNS98, pages 353 – 356, Padova, Italy, 1998.

[23] A. Vannelli and M. Vidyasagar. Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems. *Automatica*, 21(1):69–80, 1985.

[24] F.W. Wilson. The structure of the level surfaces of a Lyapunov function. *J. Differ. Equations* **3** (1967): 323–329.

[25] F. Wirth. On the calculation of real time-varying stability radii. *Int. J. Robust & Nonlinear Control*, 8:1043–1058, 1998.