DYNAMICS AND CONTROLLABILITY OF NONLINEAR DISCRETE-TIME CONTROL SYSTEMS

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Abstract: We discuss controllability properties of non-invertible nonlinear discrete-time systems under the assumption of forward accessibility. It is shown how regularity properties of control sequences can be used in the construction of regions of complete controllability, i.e. control sets. Some properties of control sets are discussed, and the existence of regular controls with periodic points in a prescribed region of a control set is shown. Furthermore, the parameter dependence of control sets and chain control sets is analyzed.

Keywords: Discrete-time nonlinear control, topological dynamics, control sets, chain control sets, parameter dependent control systems.

1. INTRODUCTION

At the basis of this paper lies the approach to control systems as dynamical systems that has been introduced for the continuous time case in (Colonius and Kliemann, 1993), see also (Colonius and Kliemann, 1996*a*) for a recent overview over the available results. In particular, these results may be used to obtain a spectral theory for control systems see (Colonius and Kliemann, 1996*b*), (Colonius and Kliemann, 1996*c*).
The dynamical systems approach to nonlinear discrete-time systems has been studied to (Albertini and Sontag, 1990), (Albertini and Sontag, 1993), (Bauer, 1996), and references in these papers. The common point in these references was to investigate the properties of so-called control sets and chain control sets. That is sets that describe particularly nice regions of the state space, in which some controllability properties hold. We aim to extend these results in some points, also generalizing some of the results presented in (Wirth, 1998) for a special class of systems.
In the discrete time case some results on topological dynamics of discrete-time control systems have been presented in (Kotsios, 1993). Unfortunately, not all the results stated in this paper state correctly the assumptions necessary for the conclusion to hold.
In this paper we make particular use of the results obtained in (Sontag and Wirth, 1998), which show conditions under which the generic existence of universally regular control sequence can be guaranteed.
In the following Section 2 we give a precise description of the class of systems we want to consider. We will not restrict ourselves to discrete time systems that come from sampling of continuous time-systems (which would imply that for fixed controls we have a diffeomorphism) as this is sometimes too restrictive in the discrete-time setup. We discuss regularity and accessibility properties of our system and point out some connections between them. In the ensuing Section 3 we show how regularity may be used to characterize the core of control sets, introduced in (Albertini and Sontag, 1990). Parameter dependence of control sets and chain control sets is analyzed in Section 4.

2. PRELIMINARIES

Let the *state space* M be a connected, Riemannian, C^ω-manifold of dimension n. On M we consider a metric d to be fixed. Let the *set of control values* $U \subset \mathbb{R}^m$ be a connected set, satisfying $U \subset \operatorname{cl}\operatorname{int} U$. Let $\tilde{U}$ be an open set with $\operatorname{cl} U \subset \tilde{U}$. Assume there exists an *exceptional set* X which describes those pairs of states and controls where the transition map is not defined.

We assume that X is analytic in $M \times \tilde{U}$ and satisfies furthermore that for all $x \in M$ it holds that $\{x\} \times \tilde{U} \not\subset X$ (so that from every point x we find an admissible control value u). Define $W := (M \times \tilde{U}) \setminus X$ and the set of *admissible control values* $U(x)$ by $\{x\} \times U(x) = \{x\} \times U \setminus X$. For an analytic map

$$f : W \to M \tag{1}$$

we consider the discrete-time system on M of the form

$$x(t+1) = f(x(t), u(t)), \quad t \in \mathbb{N}, \tag{2}$$

$$x(0) = x_0 \ \in M, \tag{3}$$

$$u(t) \in U(x(t)). \tag{4}$$

For all $t \in \mathbb{N}$ the exceptional sets X_t in $M \times U^{t+1}$ are in a natural way defined by $(x, u(0), \ldots, u(t)) \in X_t$ iff for some $0 \le s \le t$ it holds that $(x(s; x, (u(0), \ldots, u(s-1))), u(s)) \in X$. We assume that the sets X_t are given as the intersection of an analytic set $\tilde{X}_t$ in $M \times \tilde{U}^t$ with $M \times U^t$. Assume furthermore that the set $U_{sub} := \{u \in \cap_{x \in M} U(x) \ ; \ f(\cdot, u) \text{ is submersive}\}$ is the complement of a proper analytic subset in $\tilde{U}$ intersected with U.

As in (Albertini and Sontag, 1993) regularity properties of the iterates of the transition map will be of importance to us. We define

$$W_t := \{(x, u) \in M \times \operatorname{int} U^t; \quad u \in U^t(x)\}$$

and consider the map

$$F_t : W_t \to M \quad F_t(x, u) := x(t; x, u).$$

For fixed $x \in M$ and $u_0 \in \operatorname{int} U^t(x)$ we consider the rank of the linearization of $F_t(x, \cdot) : U^t(x) \to M$ at $u_0 \in U^t \subset \mathbb{R}^{mt}$ with respect to all control variables and define

$$r(t; x, u_0) := \operatorname{rk} \frac{\partial F_t}{\partial u}(x, u_0). \tag{5}$$

Definition 1. (Regularity). A pair $(x, u) \in M \times \operatorname{int} U^t$ is called *regular*, if $u \in \operatorname{int} U^t(x)$ and $r(t; x, u) = n$. A control $u \in \operatorname{int} U^t$ is called (V)-*universally regular*, if (x, u) is a regular pair for all $x \in M$, $(x \in V \subset M)$. $u \in U^{\mathbb{N}}$ is called universally regular if there exists a $T > 0$ such that for all $t \ge T$ the finite control sequence $(u(0), \ldots, u(t))$ is universally regular.

In the following we assume that $F_t(x, \cdot)$ is nontrivial with respect to u for all x, t, i.e. if there exists a regular pair $(x, u) \in W_t$, then there exists one in every connected component of $x \times U^t(x)$. Note that all these assumptions are automatically satisfied, if we consider systems that are defined everywhere on $M \times U$. A basic question in control theory is that of accessibility. Recall that the forward orbit at time t of system (2) is defined by $\mathcal{O}_t^+(x) := \{y \in M; \exists u \in U^t(x) \text{ with } y = x(t; x, u)\}$. The forward orbit of x is then defined by $\mathcal{O}^+(x) := \bigcup_{t \in \mathbb{N}} \mathcal{O}_t^+(x)$. The backward orbit of x at time t is given by $\mathcal{O}_t^-(x) := \{y \in M; \exists u \in U^t(y) \text{ with } x = x(t; y, u)\}$ and so $\mathcal{O}^-(x) := \bigcup_{t \in \mathbb{N}} \mathcal{O}_t^-(x)$.

Definition 2. (Accessibility). The system (2) is called *forward accessible from* x if $\operatorname{int} \mathcal{O}^+(x) \neq \emptyset$, and *forward accessible* if it is forward accessible from all $x \in M$.

We note the following properties of the forward orbit, which follows from simple continuity arguments.

Lemma 3. Consider system (2). Let $x_1, x_2 \in M$. If $x_2 \in \operatorname{cl} \mathcal{O}^+(x_1)$ then $\operatorname{cl} \mathcal{O}^+(x_2) \subset \operatorname{cl} \mathcal{O}^+(x_1)$.

The following lemma summarizes some easy properties in connection with regularity.

Lemma 4. Let $u_0 \in \operatorname{int} U^t$, $v_0 \in \operatorname{int} U^s$. For $x_0 \in M$ let $F_{t+s}(x_0, (u_0, v_0))$ be defined, then

(i) $r(t+s; x_0, (u_0, v_0)) \ge r(s; x(t; x_0, u_0), v_0)$.
(ii) If $v_0 \in \operatorname{int} U^s_{sub}$ then $r(t+s; x_0, (u_0, v_0)) \ge r(t; x_0, u_0)$.

Let us state some geometric properties in connection with regularity.

Lemma 5. Consider system (2).

(i) For all $t \in \mathbb{N}$ the set of singular pairs

$$S_f(t) := \{(x, u) \in W_t; \ r(t; x, u) < n\}$$

is analytic in W_t.

(ii) For all $t \in \mathbb{N}$ and for every $u_0 \in \operatorname{int} U^t$ the set

$$Y(u_0, t) := \{x \in M; u_0 \in U^t(x) \text{ and } r(t; x, u_0) < n\}$$

is analytic in $M \setminus \{x \in M; \ u_0 \notin U^t(x)\}$.

(iii) For all $t \in \mathbb{N}$, $x \in M$ the set

$$Z(x,t) := \{u \in \operatorname{int} U^t(x);\; r(t;x,u) < n\}$$

is analytic in $\operatorname{int} U^t(x)$.

(iv) If for every $t \in \mathbb{N}$ $u \in \operatorname{int} U^t$ the set $\operatorname{cl} Y(u)$ is analytic in M then the set

$$Y := \{x \in M;\; \forall t \in \mathbb{N},\; \forall u \in \operatorname{int} U^t(x) \text{ it holds that } r(t;x,u) < n\}$$

is analytic in M.

(v) Under the assumption of (iv) Y is an invariant set. I.e. for every $t \in \mathbb{N}$, $u \in U^t$ it holds that

$$x \in Y,\; u \in U^t(x) \Rightarrow x(t;x,u) \in Y\,. \quad (6)$$

Remark 6. For complex analytic systems, it holds that if $Z(x,t)$ is a proper analytic subset of $\operatorname{int} U^t(x)$ then its complement is open and dense. This is true as $\operatorname{int} U^t$ is connected, and thus $\operatorname{int} U^t(x)$ is connected as proper analytic subsets are nowhere separating in complex spaces. For real systems this might not be true, as several disjoint connected components of $\operatorname{int} U^t(x)$ have to be considered. This is why we assumed nontriviality of the maps $F_t(x,\cdot)$. Only nontriviality of the maps F_t with respect to u does not suffice in some of the following statements.

Definition 7. (Regular orbit). Consider system (2). For $x \in M$ we define the *regular forward orbit at time t* by $\hat{\mathcal{O}}_t^+(x) := \{y;\; \exists u \in \operatorname{int} U^t(x)$ such that (x,u) is regular and $y = x(t;x,u)\}$. Similarly, $\hat{\mathcal{O}}_t^-(x) := \{y;\; \exists u \in \operatorname{int} U^t(y)$ such that (y,u) is regular and $x = x(t;y,u)$.

The definitions of $\hat{\mathcal{O}}^+(x)$ and $\hat{\mathcal{O}}^-(x)$ are then clear.

The following result is only a slight extension of similar statements in (Albertini and Sontag, 1993).

Lemma 8. Consider system (2) and let $x \in M$, then

(i) $\hat{\mathcal{O}}_t^+(x)$ is open in M.
(ii) $\hat{\mathcal{O}}_t^-(x)$ is open in M.
(iii) $\operatorname{int} \mathcal{O}_t^+(x) \neq \emptyset$ iff $\hat{\mathcal{O}}_t^+(x) \neq \emptyset$.
(iv) If, for $t \in \mathbb{N}$, $\hat{\mathcal{O}}_t^+(x) \neq \emptyset$, then $\hat{\mathcal{O}}_s^+(x) \neq \emptyset$ for all $s \geq t$.
(v) $\operatorname{int} \mathcal{O}_t^+(x) \neq \emptyset \Rightarrow \operatorname{cl} \mathcal{O}_t^+(x) = \operatorname{cl} \hat{\mathcal{O}}_t^+(x)$.

3. CONTROL SETS AND CHAIN CONTROL SETS

The case that x is a fixed point under a control u such that (x,u) is a regular pair, can immediately be characterized as follows.

Proposition 9. For $x \in M$ there exist $u_x \in \operatorname{int} U^t, t \in \mathbb{N}$ such that (x,u_x) is a regular pair and

$$x = x(t;x,u_x) \quad (7)$$

if and only if there exists an open neighborhood V of x such that $V \subset \hat{\mathcal{O}}_t^+(x) \cap \hat{\mathcal{O}}_t^-(x)$.

PROOF.

(i) "$\Rightarrow$" This follows as $x \in \hat{\mathcal{O}}_t^+(x) \cap \hat{\mathcal{O}}_t^-(x)$ and the fact that both $\hat{\mathcal{O}}_t^+(x)$ and $\hat{\mathcal{O}}_t^-(x)$ are open by Lemma 8.
(ii) "$\Leftarrow$" This is obvious as $x \in \hat{\mathcal{O}}_t^+(x)$. $\square$

Let us now extend this property to connected sets.

Proposition 10. If $\Gamma \subset M$ is a connected set such that for every $x \in \Gamma$ the assumption of Proposition 9 holds for some $t(x) \in \mathbb{N}$ then there exists a connected open set V such that

$$\Gamma \subset V \subset \bigcap_{x\in\Gamma} \hat{\mathcal{O}}^+(x) \cap \hat{\mathcal{O}}^-(x)\,. \quad (8)$$

PROOF. Let $x \in \Gamma$ and consider the set $\hat{\mathcal{O}}^+(x) \cap \Gamma$, which is open in Γ by Lemma 8 (i). Let $y \in \Gamma \cap \operatorname{cl} \hat{\mathcal{O}}^+(x)$. As $y \in \hat{\mathcal{O}}^-(y)$, which is open, it follows that $\hat{\mathcal{O}}^+(x) \cap \hat{\mathcal{O}}^-(y) \neq \emptyset$ and hence $y \in \hat{\mathcal{O}}^+(x)$. Thus $\hat{\mathcal{O}}^+(x) \cap \Gamma$ is open and closed in Γ and nonempty. As Γ is connected it follows that $\Gamma \subset \hat{\mathcal{O}}^+(x)$ and as $x \in \Gamma$ was arbitrary it holds for all $x_1, x_2 \in \Gamma$ that $x_1 \in \hat{\mathcal{O}}^+(x_2)$ and thus $\hat{\mathcal{O}}^+(x_1) \subset \hat{\mathcal{O}}^+(x_2)$ by Lemma 4 (i). By symmetry we obtain $\hat{\mathcal{O}}^+(x_1) = \hat{\mathcal{O}}^+(x_2)$. Furthermore it follows for every $y \in \Gamma$ that $\Gamma \subset \hat{\mathcal{O}}^-(y)$ and again for all $x_1, x_2 \in \Gamma$ it holds that $\hat{\mathcal{O}}^-(x_1) = \hat{\mathcal{O}}^-(x_2)$. As Γ is connected we can thus choose V to be the connected component of $\hat{\mathcal{O}}^+(x) \cap \hat{\mathcal{O}}^-(x)$ containing Γ for some $x \in \Gamma$. $\square$

To describe the sets that have the properties guaranteed by the previous lemmata we introduce the notion of control sets.

Definition 11. (Control set). Consider system (2). A set $\emptyset \neq D \subset M$ is called a *precontrol set*, if

(i) $D \subset \operatorname{cl} \mathcal{O}^+(x)$, $\forall x \in D$.
(ii) For every $x \in D$ there exists a $u \in U^{\mathbb{N}}(x)$ and an increasing sequence $(t_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $x(t_k;x,u) \in D$ for all $k \in \mathbb{N}$.

A precontrol set D is called *control set*, if furthermore

(iii) D is a maximal set with respect to inclusion satisfying (i).

A control set C is called *invariant control set*, if

$$\mathrm{cl}\, C = \mathrm{cl}\, \mathcal{O}^+(x), \forall x \in C\,. \tag{9}$$

One of the main difficulties with the discrete-time case is that complete controllability need not hold in the interior of control sets (a trivial consequence of the definition in continuous time). Following (Albertini and Sontag, 1993) we define the core of a control set, where we also require regularity property to account for problems due to the non-invertibility of the system.

Definition 12. (Regular core). Let $D \subset M$ be a control set with $\mathrm{int}\, D \neq \emptyset$. The *(regular) core* of D is defined to be the set of points in D for which regular forward and backward orbit intersect D. It is denoted by $\mathrm{core}(D)$.

Proposition 13. Consider system (2). It holds that $x \in \hat{\mathcal{O}}^+(x)$, iff there exists a control set $D \subset M$ such that $x \in \mathrm{core}(D)$.

PROOF. "$\Rightarrow$" This follows from Proposition 9.

"$\Leftarrow$" Let $x \in \mathrm{core}(D)$ and $y \in \hat{\mathcal{O}}^-(x) \cap D$. By Lemma 8 (ii) there exists an open neighborhood V of y with $V \subset \hat{\mathcal{O}}^-(x)$. As $y \in D$ it follows that $V \cap \mathcal{O}^+(x) \neq \emptyset$. Hence, for some $t \in \mathbb{N}$ we may choose a control $u \in \mathrm{int}\, U^t(x)$ to steer from x to some $z \in V$ by continuous dependence on u. Then we may steer from z back to x using a control u_z such that (z, u_z) is a regular pair. Thus $x \in \hat{\mathcal{O}}^+(x)$. □

Again we extend some results in (Albertini and Sontag, 1993) to the present situation.

Proposition 14. Consider system (2) and let $D \subset M$ be a control set with $\mathrm{int}\, D \neq \emptyset$. If system (2) is forward accessible from every $x \in D$, then

(i) $\mathrm{core}(D)$ is open in M.
(ii) $\mathrm{cl}\,\mathrm{core}(D) = \mathrm{cl}\,\mathrm{int}(D) = \mathrm{cl}\, D$.
(iii) If $x \in D$ then $\mathrm{core}(D) \subset \hat{\mathcal{O}}^+(x)$. If $x \in \mathrm{core}(D)$ then $D \subset \hat{\mathcal{O}}^-(x)$.
(iv) If $x \in \mathrm{core}(D)$, $t \in \mathbb{N}$, $u \in \mathrm{int}\, U^t_{sub}$ and $x(t; x, u) \in D$ then $x(s; x, u) \in \mathrm{core}(D)$ for $s = 0, \ldots, t$.

PROOF.

(i) If $x \in \mathrm{core}(D)$, then by Proposition 13 $x \in \hat{\mathcal{O}}^+(x)$. Thus the assertion follows from Proposition 9, as there exists an open neighborhood V of x satisfying $V \subset \hat{\mathcal{O}}^+(x) \cap \hat{\mathcal{O}}^-(x)$. V is an open precontrol set satisfying $V \subset \hat{\mathcal{O}}^+(y) \cap \hat{\mathcal{O}}^-(y)$ for all $y \in V$ and thus contained in $\mathrm{core}(D)$.

(ii) Clearly $\mathrm{cl}\,\mathrm{core}(D) \subset \mathrm{cl}\,\mathrm{int}\, D \subset \mathrm{cl}\, D$. Let $x \in \mathrm{cl}\, D$ and V any open neighborhood of x. In order to prove the assertion it suffices to construct a $z \in V \cap \mathrm{core}(D)$.
Let $y \in D \cap V$. By Lemma 8 (v) we may choose $t \in \mathbb{N}$, $u \in \mathrm{int}\, U^t(y)$ such that $x(t; y, u) \in \mathrm{int}\, D \cap \hat{\mathcal{O}}^+(y)$. By continuous dependence on the initial values there exists a neighborhood $V_2 \subset V$ of y such that $x(t; V_2, u) \subset \mathrm{int}\, D$, where V_2 may be chosen such that $V_2 \cap Y(u) = \emptyset$, i.e. u is an admissible control for all points in V_2, by Lemma 5 (ii). Let $y' \in x(t; V_2, u) \subset D$. Therefore we obtain that $y \in D \subset \mathrm{cl}\, \mathcal{O}^+(y')$, and thus we may again apply Lemma 8 (v) to see that there exists a $z \in V_2$ such that $z \in \hat{\mathcal{O}}^+(y')$.
We claim that $z \in \mathrm{core}(D)$. The fact that $z \in \hat{\mathcal{O}}^+(y')$ implies in particular that $y' \in \hat{\mathcal{O}}^-(z) \cap \mathrm{int}\, D \neq \emptyset$, and as $y' \in \mathrm{cl}\, \mathcal{O}^+(z')$ for all $z' \in D$ it follows that $z \in \mathrm{cl}\, \mathcal{O}^+(z')$ for all $z' \in D$, by Lemma 3. Furthermore, by construction $x(t; z, u) \in \hat{\mathcal{O}}^+(z) \cap \mathrm{int}\, D \neq \emptyset$ and thus $D \subset \mathrm{cl}\, \mathcal{O}^+(x(t; z, u)) \subset \mathcal{O}^+(z)$. Combining these statements we obtain that $z \in V \cap \mathrm{core}(D)$, which shows that $x \in \mathrm{cl}\,\mathrm{core}(D)$.

(iii) If $x \in \mathrm{core}(D)$ then $x \in \mathrm{cl}\, \mathcal{O}^+(y)$ for every $y \in D$. By Proposition 9, $x \in \hat{\mathcal{O}}^-(x)$ and so $\mathcal{O}^+(y) \cap \hat{\mathcal{O}}^-(x) \neq \emptyset$ and hence $y \in \hat{\mathcal{O}}^-(x)$ as we may choose a $t \in \mathbb{N}$ and a control in $\mathrm{int}\, U^t(y)$ to steer from y to $\hat{\mathcal{O}}^-(x)$. This shows that $D \subset \hat{\mathcal{O}}^-(x)$. As $x \in \mathrm{core}(D)$ was arbitrary this implies also that $\mathrm{core}(D) \subset \hat{\mathcal{O}}^+(y)$ for every $y \in D$.

(iv) This is clear as $D \subset \hat{\mathcal{O}}^-(x) \subset \hat{\mathcal{O}}^-(x(s; x, u))$ for $s = 0, \ldots, t$ by Lemma 4 (ii), and $\mathrm{core}(D) \subset \hat{\mathcal{O}}^+(x(t; x, u)) \subset \hat{\mathcal{O}}^+(x(s; x, u))$ by (iii). □

We have to point out that even with M compact and (2) forward accessible not every point in the core of a control set can be represented as a fixed point under a universally regular control, although these are generic for t large enough and the core can be characterized via fixed points satisfying a regularity condition. An example to this effect can be found in (Wirth, 1998). We can, however, prove the following statement.

Theorem 15. Assume that system (2) is forward accessible. For every control set $D \subset M$ with $\mathrm{core}(D) \neq \emptyset$ and every open, relatively compact set $\emptyset \neq V \subset \mathrm{core}(D)$ there exist $x \in V$, $t \in \mathbb{N}$, u^t V-universally regular such that $x(t; x, u) = x$.

PROOF. Let $x \in V$. By Proposition 13 $x \in \hat{\mathcal{O}}^+(x)$, and we can choose $t \in \mathbb{N}$, $u \in \mathrm{int}\, U^t$ such that (x, u) is a regular pair and

$$x = x(t; x, u) . \tag{10}$$

Without loss of generality let t be large enough such that the set of V-universally regular controls is open and dense in $\mathrm{int}\, U^t$, see (Sontag and Wirth, 1998). By Proposition 13 we can choose u_1 V-universally regular such that $y_1 := x(t; x, u_1) \in \hat{\mathcal{O}}_t^+(x) \cap \hat{\mathcal{O}}_t^-(x) \cap V$. Using the V-universal regularity of u_1 and applying the implicit function theorem it may be concluded that there exists an open neighborhood $V(x) \subset V$ such that for every $y \in V(x)$ there exists a V-universally regular $u(y)$ with $y_1 = x(t; y, u(y))$. Furthermore as $y_1 \in \hat{\mathcal{O}}_t^-(x)$ we may choose $u_2 \in \mathrm{int}\, U_{sub}^t$ such that $y_2 := x(t; y_1, u_2) \in V(x)$. Hence

$$y_1 = x\left(2t; y_1, (u_2, u(y_2))\right) , \tag{11}$$

where $(u_2, u(y_2))$ denotes the concatenation of $u_2, u(y_2))$. As $u_2 \in \mathrm{int}\, U_{sub}^t$ and $u(y_2)$ is V-universally regular it follows by Lemma 4 that $(u_2, u(y_2))$ is V-universally regular. □

Note that the formulation and proof of the previous proposition can be simplified if M is compact and then forward accessibility implies genericity of universally regular controls. If M is not compact this statement can only be shown for relatively compact subsets of M.

Another object that is of interest is given by the chain control sets of system (2). These are defined not via trajectories of the system but rather with parts of trajectories of (2) that can be connected with arbitrarily small jumps. For $x, y \in M$ an (ε, T)-chain from x to y is given by a sequence $x = x_0, x_1, \ldots, x_k = y$ with controls $u_1, \ldots, u_k$ such that $d(x(t_i, x_i, u_i), x_{i+1}) < \varepsilon$ for suitable $t_i \geq T$ and all $i = 0, \ldots, k-1$.

Definition 16. Consider system (2). A set $\emptyset \neq E \subset M$ is called a *chain control set*, if

(i) For every $x, y \in E$ and all $\varepsilon \geq 0, T \geq 0$ there exists an (ε, T)-chain from x to y.
(ii) For every $x \in E$ there exists a $u \in U^{\mathbb{N}}(x)$ such that $x(t; x, u) \in E$ for all $t \in \mathbb{N}$.
(iii) E is a maximal set w.r.t. to inclusion satisfying (i) and (ii).

It is easy to show that chain control sets are closed, and that to every control set D there exists a chain control set $E \supset \mathrm{cl}\, D$. However, it may happen that chain control sets contain several control sets. For examples we refer to the continuous time examples in (Colonius and Kliemann, 1996*a*), that exhibit all the necessary points.

4. PARAMETER DEPENDENCE

Assume now that our control system (2) depends analytically on a parameter $\alpha \in I \subset \mathbb{R}$, where I is an open interval with $0 \in I$. To be precise we consider an analytic map $f : W \times I \to M$ and assume that each of the systems

$$x(t+1) = f(x(t), u(t), \alpha) , \quad t \in \mathbb{N}$$

satisfies the conditions stated for system (2). (A more general description of the problem would also allow the dependence of the exceptional set on the parameter, but this is inessential here.) It is of interest to know how the regions of complete controllability can vary under small changes of α. Unfortunately, it is easy to see that neither control sets nor chain control sets depend continuously on parameters. An easy example is given by a system on $\mathbb{R}^n$ where the transmission maps may be any matrix with positive entries of norm ≤ 2. For such a system the positive orthant in $\mathbb{R}^n$ is an invariant control set, but for any open neighborhood of this set of matrices in $\mathbb{R}^{n \times n}$ the system is completely controllable on $\mathbb{R}^n$. However, we can state the following results, which we will show for compact M for ease of exposition. In the following the index α will denote objects defined via the system $(2)_\alpha$.

Proposition 17. Let M be compact and system $(2)_\alpha$ be forward accessible for every $\alpha \in I$.

Let D_0 be a control set of $(2)_0$ and K be a compact set with $K \subset \mathrm{core}(D)$. Then:
(i) There exists a neighborhood I_1 of 0 such that for every $\alpha \in I_1$ there is a control set D_α with $K \subset \mathrm{core}(D_\alpha)$.
(ii) The map $\alpha \mapsto \mathrm{cl}\, D_\alpha, \alpha \in I_1$ is lower semi-continuous w.r.t. the Hausdorff norm.

PROOF. Note that it is sufficient to prove (i) as (ii) follows immediately from this statement. To see (i) let K be as described. For each $x, y \in K$ there exists a $t(x, y) \in \mathbb{N}$ such that $y \in \hat{\mathcal{O}}^+_{t(x,y)}(x)$. Using a standard compactness argument it follows that $\max_{x,y \in K} t(x, y) < \infty$, see also (Wirth, 1998). Let $u \in U^t(x)$ be such that (x, u) is regular and $x(t; x, u) = y$. By regularity of the map $F_{t)}$ in $(x, u, 0)$ with respect to u we can apply the implicit function theorem to obtain that for every (z, α) a sufficiently small neighborhood of $(x, 0) \in M \times I$ there exists a control v that steers z to y under the system $(2)_\alpha$. In particular, this holds for the points (x, α) so that $y \in \hat{\mathcal{O}}^+_{t(x,y),\alpha}(x)$ for $|\alpha|$ small enough. A further compactness argument completes the proof. □

Proposition 18. Let M be compact and system $(2)_\alpha$ be forward accessible for every $\alpha \in I$.

Let E_0 be a chain control set of $(2)_0$ and V be an open neighborhood of E_0 separating E_0 from an open neighborhood of the other chain control sets of $(2)_0$. Assume that for some open interval $0 \in I_0 \subset I$ the map f is uniformly continuous on $V \times U \times I_0 \cap W \times I_0$, then the following hold:
(i) There exists a neighborhood I_2 of 0 such that for every $\alpha \in I_2$ and every chain control set E_α of $(2)_\alpha$ it holds that

$$E_\alpha \cap V \neq \emptyset \text{ implies } E_\alpha \subset V\,.$$

(ii) For $\alpha \in I_2$ let $E_{\alpha,j}, j \in J(\alpha)$ be the family of chain control sets for system $(2)_\alpha$ having nonempty intersection with V. Define

$$E_\alpha := \bigcup_{j \in J(\alpha)} E_{\alpha,j}$$

then, the map $\alpha \mapsto E_\alpha, \alpha \in I_2$ is upper semicontinuous w.r.t. the Hausdorff norm.

PROOF. Again (ii) is a consequence of (i). Here we merely supply an outline of the proof of (i).

Assume that there exists a sequence $\alpha_k \to 0$ such that for every $k \in \mathbb{N}$ there exists a chain control set E_k of $(2)_{\alpha_k}$ with points $x_k \in E_k \cap V$ and $y_k \in E_k \setminus V$. Without loss of generality there are $x \in \operatorname{cl} V, y \in M \setminus V$ such that $x_k \to x$ and $y_k \to y$ as $k \to \infty$. Fix $\varepsilon > 0$. Then there exists a $k_\varepsilon \in \mathbb{N}$ such that for all $k \geq k_\varepsilon$ it holds that $\max\{d(x_k, x), d(y_k, y)\} < \varepsilon$. Thus for all $k \geq k_\varepsilon$ there exist controlled $(2\varepsilon, T)_{\alpha_k}$-chains from x_k to y and from y_k to x, which can e.g. be constructed by going along an $(\varepsilon, T)_{\alpha_k}$-chain in E_k from x_k to y_k and then jumping to y. Using uniform continuity we may then construct a $(4\varepsilon, T)_0$-chain from x to y and back to x.

As $\varepsilon > 0$ was arbitrary this proves that x, y are contained in a chain control set E for $(2)_0$. This contradicts the assumptions on V. □

An interesting subproblem of the previous proposition is obtained when we assume that U is star shaped with respect to 0 and consider the parameter dependence in the form $f(x, u, \alpha) := f(x, \alpha u)$ $\alpha > 0$. For $\alpha \to 0$ these control systems approximate the free system

$$x(t+1) = f(x(t), 0)\,, \quad t \in \mathbb{N}\,.$$

The relation between the chain transitive sets of the free system and the control sets of the control sets with "small" control values has been studied in (Bauer, 1996).

5. CONCLUSION

We have presented some results make the connection between dynamical systems and discrete-time control systems motivated by similar results for the continuous time case.

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