Robustness analysis of domains of attraction of nonlinear systems

Andrew Paice
ABB Corporate Research C1
5405 Baden-Dättwil
Switzerland
Andrew.Paice@chcrc.abb.ch

Fabian Wirth
Zentrum für Technomathematik
Universität Bremen
Postfach 330 440, D-28334 Bremen, Germany
fabian@math.uni-bremen.de

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Abstract

We consider a perturbed nonlinear system with a fixed point that is invariant under all perturbations. Under the assumption that this fixed point is locally exponentially stable for the unperturbed system we propose a method for the approximation of the robust domain of attraction, that is, the set of points that are attracted to the fixed point under all time-varying perturbations taking values in a specified set.

1 Introduction

For nonlinear systems one basic question is that of the determination of domains of attraction of asymptotically stable fixed points. This question has received considerable attention over the last decade, see *e.g.* [9], [1], [7]. In this paper we study nonlinear systems with an affine perturbation structure that have a singular fixed point. This setup is a special case of the one treated in [6], where it has been proved that for uniformly asymptotically stable sets of families of time-varying systems there exist smooth global Lyapunov functions.

We proceed as follows. In Section 2 we introduce the class of systems we wish to consider. The concept of the robust domain of attraction is introduced and a few properties are discussed. In the ensuing section we analyze the linearization of the nonlinear systems, finding a ball of initial conditions yeilding trajectories which robustly converge to the origin. The determination of the domain of attraction, however, is clearly a nonlinear problem, thus in Section 4 we characterize the robust domain of attraction in terms of an optimal control problem, and present approximations to this problems whose value functions are computable as viscosity solutions of Hamilton-Jacobi-Bellman equations. In order to improve these approximations, we suggest how to use the information provided by the linearization in Section 5. The algorithm suggested by our results is then presented in Section 6.

2 Problem formulation

Consider an autonomous system subject to affine perturbations as follows:

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} d_i(t) f_i(x) =: F(x, d(t)) \tag{2.1}$$

where $f_i(0) = 0,\ i = 0 \dots m$ and the $d(\cdot) \in \ell^\infty(\mathbb{R}, \mathbb{R}^m)$ are such that existence and uniqueness of solutions is guaranteed. We assume that the vector fields f_i are locally Lipschitz continuous and continuously differentiable in 0. Solutions to the initial value problem (2.1) with $x(0) = x_0$ for a particular d will be denoted $\varphi(t; x_0, d)$. The *unknown* perturbation function d is assumed to take values in $D \subset \mathbb{R}^m$, where D is compact, convex, with nonempty interior, and $0 \in D$. Denote $\mathcal{D} := \{d \in \ell^\infty(\mathbb{R}, \mathbb{R}^m)\ ;\ d(t) \in D\}$.
Given $d \in \mathcal{D}$, the domain of attraction of 0 at time $t_0 = 0$ for (2.1) is

$$\mathcal{A}_d(0) := \{x \mid \varphi(t; x, d) \to 0, t \to \infty\}$$

A robust domain of attraction may now be defined.

Definition 2.1 [D-robust Domain of Attraction] Let $D \subset \mathbb{R}^m$ as before. The *D-robust domain of attraction of the equilibrium* 0 *of (2.1)*, is

$$\begin{aligned} \mathcal{A}_D(0) &:= \{x \mid \forall d \in \mathcal{D},\ \varphi(t; x, d) \to 0, t \to \infty\} \\ &= \textstyle\bigcap_{d \in D} A_d(0) \end{aligned}$$

Note that the definition implies that if 0 is locally uniformly asymptotically stable for all $d \in \mathcal{D}$, then $\mathcal{A}_D(0)$ is an open, connected, invariant set. This definition inspires a number of problems regarding the robustness of the perturbed system (2.1). The two main problems are:

(i) Given $D \subset \mathbb{R}^m$, determine $\mathcal{A}_D(0)$.

(ii) Given $A \subset \mathcal{A}(0)$ and $D \subset \mathbb{R}^m$, determine the largest α such that $A \subset \mathcal{A}_{\alpha D}(0)$.

In this paper we concentrate on the first question. Note that if the allowable perturbations are increased there are three different scenarios with which the the property $A \subset \mathcal{A}_{\alpha D}(0)$ is lost at some minimal α_0.

(i) Loss of stability at 0. *i.e.* $A \subset \mathcal{A}_{\alpha D}(0)$ for $\alpha < \alpha_0$ and $\mathrm{dist}(A, \partial\mathcal{A}_{\alpha D}(0)) > \delta > 0$ for all $0 < \alpha < \alpha_0$. *e.g.* linear systems.

(ii) As $\alpha \to \alpha_0$ it holds that $\mathrm{dist}(A, \partial\mathcal{A}_{\alpha D}(0)) \to 0$.

(iii) Birth of an attractor in $\operatorname{int} A$, while $\mathrm{dist}(A, \partial\mathcal{A}_{\alpha D}(0)) > \delta > 0$ for all $0 < \alpha < \alpha_0$.

An example for the last scenario is given by the following system on $\mathbb{R}$:

$$\dot{x} = -x + d(t)\frac{2}{\pi}x\sin(x)$$

with $A = [-1,1]$ and $D = [-1,1]$. Then, $\mathcal{A}_{\frac{1}{2}D}(0) = [-\pi,\pi]$, while for $0 < \alpha < \frac{1}{2}$ we have $\mathcal{A}_{\alpha D}(0) = \mathbb{R}$.

3 The linearized system

The first question to consider is under which conditions $\mathcal{A}_{\mathcal{D}}(0)$ contains a neighborhood of 0. To examine this question we study the linearization of (2.1) at 0:

$$\dot{y}(t) = A_0 y(t) + \sum_{i=1}^{m} d_i(t) A_i y(t) := A(d(t))\ y(t) \tag{3.1}$$

where A_i denotes the Jacobian of f_i in 0, $i = 0 \dots m$. For $d \in \mathcal{D}$ let $\Phi_d(t,s)$ denote the evolution operator of $\dot{x} = A(d(t))x(t)$. The corresponding Bohl exponent is defined by

$$\beta(d) := \inf\{\beta\ ;\ \exists M : \|\Phi_d(t,s)\| \le Me^{\beta(t-s)}\ \forall t \ge s \ge 0\}$$

Exponential stability of a linear time-varying system given by $d \in \mathcal{D}$ is equivalent to $\beta(d) < 0$, see [4]. We denote the maximal Bohl exponent by

$$\beta(A_0,\dots,A_m,D) := \max_{d\in\mathcal{D}} \beta(d)\,.$$

In [2] it is shown that the maximum is indeed attained. If the maximal Bohl exponent is negative this is equivalent to existence of constants $M \ge 1, \beta < 0$ such that

$$\|\Phi_d(t,s)\| \le Me^{\beta t}\,, \quad \forall\, d \in \mathcal{D}\,. \tag{3.2}$$

By [4, Th. VII.1.3] the Bohl exponent is upper semi-continuous even under nonlinear perturbations. Thus:

Lemma 3.1 *Consider (2.1) with linearization (3.1).*
(i) If $\beta(A_0,\dots,A_m,D) < 0$ then $\mathcal{A}_D(0)$ contains an open neighborhood of 0.
(ii) If $\beta(A_0,\dots,A_m,D) > 0$ then $0 \in \partial\mathcal{A}_D(0)$.

The following example shows that for the case $\beta(A_0,\dots,A_m,D) = 0$ both situations are possible.

Example 3.2 Let $A_1,\dots,A_m \in \mathbb{R}^{n\times n}$ and $D \subset \mathbb{R}^m$ be such that $\beta(0,A_1,\dots,A_m,D) = 0$ and consider the systems

$$\dot{x} = -x(t) < x(t), x(t) > + \sum_{i=1}^{m} d_i(t) A_i x(t) \tag{3.3}$$

$$\dot{x} = x(t) < x(t), x(t) > + \sum_{i=1}^{m} d_i(t) A_i x(t) \tag{3.4}$$

Clearly, for system (3.3) $\mathcal{A}_D(0) = \mathbb{R}^n$ while for (3.4) $\mathcal{A}_D(0) = \{0\}$ and the Bohl exponent of the linearization is 0 for both systems.

As in [8] for a given D the following can be shown. For all C^1-perturbation structures $f_0,\dots,f_m$ with singular fixed point 0, outside of a residual set, there exists at most one $\alpha \ge 0$ such that $\beta(A_0,\dots,A_m,\alpha D) = 0$. Thus "generically" the point where the Bohl exponent does not indicate whether $0 \in int\mathcal{A}_D(0)$ is exactly the perturbation intensity at which the system becomes exponentially unstable.
We conclude this section with a more precise statement on the size of the ball contained in $\mathcal{A}_D(0)$, which is a consequence of [4, Th.VII.1.3]. To this end denote $L(D) := \max_{d\in D} \|A(d)\|$.

Lemma 3.3 *Let $\beta(A_0,\dots,A_m,D) < \beta < 0$ and $M_\beta \ge 1$ such that (3.2) is satisfied and fix $\beta < \beta' < 0$ and $M > M_\beta$. Let $h > 0,\ q > 0$ be such that*

$$1 - M_\beta e^{-(\beta-\beta')h} > 0 \quad \textit{and}$$

$$qhe^{(2L(D)+q+\beta')h} = \min\{M - M_\beta, 1 - M_\beta e^{-(\beta-\beta')h}\}.$$

If $\|F(x,d) - A(d)\| < q$ for all $x \in B(0,\varepsilon), d \in D$ then

$$\|x(t,x,d) - A(d)x\| \le Me^{\beta' t}\|x\|\,, \quad \forall x \in B(0,\varepsilon/M), d \in \mathcal{D}\,.$$

In particular, it follows that $B(0,\varepsilon/M) \subset \mathcal{A}_D(0)$.

4 An optimal control characterization of the robust domain of attraction

By definition we have

$$x_0 \in \mathcal{A}_D(0) \Leftrightarrow \forall d \in \mathcal{D}\ \lim_{t\to\infty} \|\varphi(t;x_0,d)\| = 0\,,$$

Assuming that $\beta(A_0,\dots,A_m,D) < 0$ and applying Lemma 3.1 we obtain immediately that

$$x_0 \in \mathcal{A}_D(0) \quad \Leftrightarrow \quad \forall d \in \mathcal{D}, \quad \limsup_{t\to\infty} \|\varphi(t;x_0,d)\| = 0 \tag{4.1}$$

It is of interest to note that negativity of the Bohl exponent allows for a uniform bound.

Lemma 4.1 *Consider (2.1) and assume that $\beta((A_0,\dots,A_m),D) < 0$ then*

$$x_0 \in \mathcal{A}_D(0) \Leftrightarrow \lim_{t\to\infty} \sup_{d\in\mathcal{D}} \|\varphi(t;x_0,d)\| = 0\,.$$

Proof: Clearly we need only show "$\Rightarrow$". Assume that $x \in \mathcal{A}_D(0)$ and there exist sequences $d_k \in \mathcal{D}$, $T_k \to \infty$ such that $\|x(T_k,x,d_k)\| > \delta > 0$ for all $k \in \mathbb{N}$. Without loss of generality $d_k \to d \in \mathcal{D}$ in the weak-$*$ topology on $\mathcal{D}$. By assumption for every ε there exists a t_0 such that $x(t_0, x, d) \in B(0, \varepsilon)$ choosing $\varepsilon > 0$ small enough we can guarantee by Lemma 3.1 that all solutions starting in $B(0, 2\varepsilon)$ are bounded by $2\varepsilon M_\beta e^{\beta t}$ for some $\beta < 0$. As $x(t_0, x, d_k) \to x(t_0, x, d)$ this means for all k large enough $\|x(t, x, d_k)\| < 2\varepsilon M_\beta e^{\beta(t-t_0)}$ for $t \geq t_0$, a contradiction. □

Motivated by (4.1) and this lemma, the domain of attraction can thus be characterized via the following optimization problem. Define

$$J_0(x, d) := \limsup_{t\to\infty} \|\varphi(t; x, d)\|$$

and the corresponding value function

$$v_0(x) = \sup_{d\in D} J_0(x, d)\,,$$

then $\mathcal{A}_D(0) = v_0^{-1}(0)$. In other words, there clearly exists a $c > 0$ such that $x \notin \mathcal{A}_D(0)$ implies that $v_0(x) > c$.

The problem with this value function is obviously its discontinuity at the boundary of $\mathcal{A}_D(0)$. As v_0 is hard to calculate we use a standard approximation scheme, *e.g.* [5], for its calculation. For $\delta > 0$ define

$$J_\delta(x, d) := \int_0^\infty \delta e^{-\delta t} \|\varphi(t; x, d)\| dt$$

with value function $v_\delta(x) = \sup_{d\in\mathcal{D}} J_\delta(x, d)$. Note that v_δ is continuous w.r.t. x.

Although it is not generally true that v_δ is strictly decreasing w.r.t. δ, it is possible to obtain a convergence result.For $M \geq 1$ and $0 > \beta > \beta(A_0, \ldots, A_m, D)$ denote

$$X(M, \beta) := \{x\ ;\ \forall t > 0\ :\ \sup_{d\in\mathcal{D}} \|x(t, x_0, d)\| \leq M e^{\beta t}\}$$

Note that $\mathcal{A}_D(0) \supset X(M, \beta)$, and via Lemmas 3.3 and 4.1, $\mathcal{A}_D(0) = \bigcup_{M\geq 1, 0>\beta>\beta(A_0,\ldots,A_m,D)} X(M, \beta)$. The following result may then be shown.

Proposition 4.2 *Consider (2.1) and assume that* $\beta((A_0, \ldots, A_m), D) < 0$ *then*

$$v_\delta \to v_0 \text{ uniformly on compact subsets of } \mathcal{A}_D(0)\,.$$

Furthermore for $0 > \beta > \beta(A_0, \ldots, A_m, D)$ *it holds that*

$$x \in X(M, \beta) \Rightarrow v_\delta(x) \leq M \frac{\delta}{\delta - \beta}$$

The previous statement implies that v_δ converges linearly on compact subsets of $\mathcal{A}_D(0)$ to 0. To obtain an estimate for $\mathcal{A}_D(0)$ define

$$\mathcal{A}(\delta, \varepsilon) := \{x \in \mathbb{R}^n\ ;\ v_\delta(x) < \varepsilon\}\,.$$

Then we have

Proposition 4.3 *Consider (2.1) and assume that* $\beta(A_0, \ldots, A_m, D) < 0$ *then for all* $0 < \varepsilon \leq c_0 :=$ $\mathrm{dist}(0, \partial\mathcal{A}_D(0))$

$$\mathcal{A}_D(0) = \bigcup_{\delta>0} \mathcal{A}(\delta, \varepsilon) = \bigcup_{\delta^*>\delta>0} \mathcal{A}(\delta, \varepsilon)\,, \quad \forall \delta^* > 0$$

In general, information about c_0 amounts to the solution of the original problem itself, so we need a lower bound on c_0. Using the quantities introduced in Lemma 3.3, assume that $\|F(x, d) - A(d)x\| < q$ for all $x \in B(0, \varepsilon), d \in D$, then $B(0, \varepsilon/M) \subset A_D(0)$ and so $\varepsilon/M \leq c_0$ is the lower bound we require.

In order to use the information provided by Proposition 4.3 we have to obtain estimates for the quantities $\beta(A_0, \ldots, A_m, D)$ and M_β as used in Lemma 3.3, from these the quantities q and ε are determinable and Proposition 4.3 is then applicable.

5 Estimation of linear growth bounds

In order to estimate ε and M, we need some information about the local growth properties of the perturbed system. Via Lemma 3.3 these may be obtained by examining the linearization at 0 (3.1). Thus, in this section, we briefly review the theory of Lyapunov exponents of families of time-varying matrices, see [3] for further details. Recall that the Lyapunov exponent given by an initial condition x_0 and $d \in \mathcal{D}$ is defined by

$$\lambda(x_0, d) = \limsup_{t\to\infty} \frac{1}{t} \log \|x(t; x_0, d)\|$$

It is known [2] that

$$\beta(A_0, \ldots, A_m, D) = \max\{\lambda(x_0, d)\ ;\ x_0 \neq 0, d \in \mathcal{D}\}$$

The following scheme for an approximate calculation of $\beta(A_0, \ldots, A_m, D)$ based on the theory of Lyapunov exponents has been proposed, see [5], [10] and references therein. Via projection onto the sphere $\mathbb{S}^{n-1}$ we obtain from system (3.1) the system

$$\dot{s}(t) = (A(d(t)) - s(t)^T A(d(t)) s(t) \cdot \mathrm{Id}) s(t) \tag{5.1}$$

It is an easy calculation (see [2]) that for $x_0 \in \mathbb{S}^{n-1}$ there exists a q, determined by the A_i, such that

$$\|\Phi_d(t, 0)x_0\| = exp\left(\int_0^t q(\psi(s; x_0, d), d(s)) ds\right) \|x_0\| \tag{5.2}$$

Thus the Lyapunov exponent is of the form

$$\lambda(x_0, d) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t q(\psi(s; x_0, d), d(s)) ds$$

where $\psi(s; x_0, d)$ denotes the trajectory of (5.1). Interpreting this expression as an average yield optimal control problem on $\mathbb{S}^{n-1}$, we introduce the following approximating functional for $\delta > 0$

$$J_\delta(x_0, d) := \int_0^\infty \delta e^{-\delta s} q(\psi(s; x_0, d), d(s)) ds$$

with associated value function $V_\delta(x) := \sup_{d\in\mathcal{D}} J_\delta(x_0, d)$. For these optimal control problems it is known that

$$\kappa_\delta := \max_{x\in\mathbb{S}^{n-1}} V_\delta(x) \geq \beta(A_0, \ldots, A_m, D)\,,$$

and $\kappa_\delta \to \beta(A_0,\dots,A_m,D)$ with a linear convergence rate in δ, see [5, 10]. Thus if $\beta(A_0,\dots,A_m,D)<0$, then choosing $\delta>0$ small enough we can obtain $0>\kappa_\delta>\beta(A_0,\dots,A_m,D)$. In order to obtain our two required estimates it remains to obtain a constant M, such that M,κ_δ satisfy (3.2).
Let $0>\kappa>\kappa_\delta$. By (5.2) it is sufficient to find $T>0$ s.t.

$$\sup_{||x||=1,d\in\mathcal{D}} \int_0^T q(\psi(s,x,d),d(s) - \kappa ds < 0$$

Then it follows that $||\Phi_d(T,0)|| < e^{\kappa T},\ \forall d\in\mathcal{D}$, and so

$$||\Phi_d(t,0)|| < e^{L(D)T}e^{\kappa t},\ \forall d\in\mathcal{D}, \forall t>0$$

Note that in order to find T, the value function v_δ that has already been calculated can be used, and it is sufficient to find T such that

$$\sup_{||x||=1,d\in\mathcal{D}} \int_0^T \delta e^{-\delta s}(q(\psi(s,x,d),d(s) - \kappa)ds < 0 \quad (5.3)$$

Note that solvability of (5.3) depends on the fact that $\kappa>\kappa_\delta$, as for κ_δ, the expression on the left is always negative.

6 A description of the algorithm

With the estimates provided in the previous sections, we are in a position to describe an algorithm for determining $\mathcal{A}_\mathcal{D}(0)$, which is the main contribution of the paper. Given $f_0,\dots,f_m$ and D:

(i) Calculate κ_δ for small δ. If $\kappa_\delta \ge 0$ for all δ larger than some threshold stop.

(ii) With the data κ_δ, M satisfying (3.2), determine a ball $B(0,\varepsilon)$ contained in $\mathcal{A}_D(0)$ via Lemma 3.3.

(iii) Let $\varepsilon_0=\varepsilon$, $\hat{\mathcal{A}}_0 = B(0,\varepsilon)$.

(iv) Determine the value function v_δ associated with the cost functional

$$J_\delta(x,d) := \int_0^\infty \delta e^{-\delta t} g_k(\varphi(t;x,d))dt\,, \qquad (6.1)$$

where $g_k(x)=\|x\|$ if $\|x\|\notin \hat{\mathcal{A}}_k$, $g_k(x)=0$, otherwise.

(v) Determine ε_{k+1} such that $B(0,\varepsilon_{k+1})\subset \hat{\mathcal{A}}_{k+1} := v_\delta^{-1}([0,\varepsilon_k])$.

(vi) If $\varepsilon_k-\varepsilon_{k+1}$ is bigger than some threshold go to (iv). Otherwise, determine whether to decrease δ and go to (iv) or stop, depending on the size of δ.

Remark 6.1 (i) The reason to stop after the first step if $\kappa_\delta \ge 0$ for reasonably small δ is that although the nonlinear system may be exponentially stable, the Bohl exponent of the linearization is so small that the system is unlikely to be robustly stable.
(ii) The reason for choosing the particular form of g_k (6.1) is that once a trajectory enters $\hat{\mathcal{A}}_k$, it will robustly converge to 0, and thus no longer need to be penalized in the cost.
(iii) Note that by construction $\hat{\mathcal{A}}_k\subset\mathcal{A}_D(0)$, thus the algorithm supplies an inner approximation of the robust domain of attraction.

Theorem 6.2 *The $\hat{\mathcal{A}}_k$ generated by the above algorithm form a monotonically increasing sequence such that $\cup_{k=0}^\infty \hat{\mathcal{A}}_k = \mathcal{A}_\mathcal{D}(0)$.*

7 Conclusions

In this paper we have discussed robust domains of attraction of singular fixed points. A scheme for the approximation of the robust domain of attraction has been presented. This involves the calculation of approximations of the maximal Bohl exponent of the linearized system and subsequently the solution of an optimal control problem given by the nonlinear system.

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