Controllability Properties of Numerical Eigenvalue Algorithms

Uwe Helmke[1] and Fabian Wirth[2*]

[1] Mathematisches Institut, Universität Würzburg, 97074 Würzburg, Germany, helmke@@mathematik.uni-wuerzburg.de

[2] Zentrum für Technomathematik, Universität Bremen, 28334 Bremen, Germany, fabian@@math.uni-bremen.de

Abstract. We analyze controllability properties of the inverse iteration and the QR-algorithm equipped with a shifting parameter as a control input. In the case of the inverse iteration with real shifts the theory of universally regular controls may be used to obtain necessary and sufficient conditions for complete controllability in terms of the solvability of a matrix equation. Partial results on conditions for the solvability of this matrix equation are given. We discuss an interpretation of the system in terms of control systems on rational functions. Finally, first results on the extension to inverse Rayleigh iteration on Grassmann manifolds using complex shifts is discussed.

For many numerical matrix eigenvalue methods such as the QR algorithm or inverse iterations shift strategies have been introduced in order to design algorithms that have faster (local) convergence. The shifted inverse iteration is studied in [3,4,15] and in [17,18], where the latter references concentrate on complex shifts. For an algorithm using multidimensional shifts for the QR-algorithm see the paper of Absil, Mahony, Sepulchre and van Dooren in this book.

In this paper we interpret the shifts as control inputs to the algorithm. With this point of view standard shift strategies as the well known Rayleigh iteration can be interpreted as feedbacks for the control system. It is known (for instance in the case of the inverse iteration or its multidimensional analogue, the QR-algorithm) that the behavior of the Rayleigh shifted algorithm can be very complicated, in particular if it is applied to non-Hermitian matrices A [4]. It is therefore of interest to obtain a better understanding of the underlying control system, which up to now has been hardly studied.

Here we focus on controllability properties of the corresponding systems on projective space for the case of inverse iteration, respectively the Grassmannian manifold for the QR-algorithm. As it turns out the results depend heavily on the question whether one uses *real* or *complex* shifts. The controllability of the inverse iteration with complex shifts has been studied in [13], while the real case is treated in [14].

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In Section 1 we introduce the shifted inverse power iteration with real shifts and the associated system on projective space and discuss its forward accessibility properties. In particular, there is an easy characterization of the set of universally regular control sequences, that is those sequences with the property, that they steer every point into the interior of its forward orbit. This will be used in Section 2 to give a characterization of complete controllability of the system on projective space in terms of solvability of a matrix equation. In Section 3 we investigate the obtained characterization and interpret it in terms of the characteristic polynomial of A. Some concrete cases in which it is possible to decide based on spectral information whether a matrix leads to complete controllable shifted inverse iteration are presented in Section 4. An interpretation of these results in terms of control systems on rational functions is given in Section 5. In Section 6 we turn to the analysis of the shifted QR algorithm. We show that the corresponding control system on the Grassmannian is never controllable except for few cases. The reachable sets are characterized in terms of Grassmann simplices. We conclude in Section 7.

1 The shifted inverse iteration on projective space

We begin by reviewing recent results on the shifted inverse iteration which will motivate the ideas employed in the case of the shifted QR algorithm. Let A denote a real $n \times n$-matrix with spectrum $\sigma(A) \subset \mathbb{C}$. The *shifted inverse iteration* in its controlled form is given by

$$x(t+1) = \frac{(A - u_t I)^{-1} x(t)}{\|(A - u_t I)^{-1} x(t)\|}, \quad t \in \mathbb{N}, \tag{1}$$

where $u_t \notin \sigma(A)$. This describes a nonlinear control system on the $(n-1)$-sphere. The trajectory corresponding to a normalized initial condition x_0 and a control sequence $u = (u_0, u_1, \dots)$ is denoted by $\phi(t; x_0, u)$. Via the choice $u_t = x^*(t)Ax(t)$ we obtain from (1) the Rayleigh quotient iteration studied in [3], [4].

If the initial condition x_0 for system (1) lies in an invariant subspace of A then the same holds true for the entire trajectory $\phi(t; x_0, u)$, regardless of the control sequence u. In order to understand the controllability properties from x_0 it would then suffice to study the system in the corresponding invariant subspace. Therefore we may restrict our attention to those points not lying in a nontrivial invariant subspace of A, i.e. those $x \in \mathbb{R}^n$ such that $\{x, Ax, \dots, A^{n-1}x\}$ is a basis of $\mathbb{R}^n$. Vectors with this property are called *cyclic* and a matrix A is called cyclic if it has a cyclic vector, which we will always assume in the following. To keep notation short let us introduce the union of A-invariant subspaces

$$\mathcal{V}(A) := \bigcup_{AV \subset V, 0 < dim V < n} V \,.$$

Using the fact that the interesting dynamics of (1) are on the unit sphere and identifying opposite points (which give no further information) we then define our state space of interest to be

$$M := \mathbb{RP}^{n-1} \setminus \mathcal{V}(A)\,, \tag{2}$$

where $\mathbb{RP}^{n-1}$ denotes the real projective space of dimension $n-1$. The natural projection from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{RP}^{n-1}$ will be denoted by $\mathbb{P}$. Thus M consists of the 1-dimensional linear subspaces of $\mathbb{R}^n$, defined by the cyclic vectors of A. Since a cyclic matrix has only a finite number of invariant subspaces, $\mathcal{V}(A)$ is a closed algebraic subset of $\mathbb{R}^n$. Moreover, M is an open and dense subset of $\mathbb{RP}^{n-1}$. The system on M is now given by

$$\begin{aligned} \xi(t+1) &= (A - u_t I)^{-1}\xi(t)\,, \quad t \in \mathbb{N} \\ \xi(0) &= \xi_0 \in M\,, \end{aligned} \tag{3}$$

where $u_t \in U := \mathbb{R} \setminus \sigma(A)$ (the set of admissible control values). We denote the space of finite and infinite admissible control sequences by U^t and $U^{\mathbb{N}}$, respectively. The solution of (3) corresponding to the initial value ξ_0 and a control sequence $u \in U^{\mathbb{N}}$ is denoted by $\varphi(t;\xi_0,u)$. The forward orbit of a point $\xi \in M$ is then given by

$$\mathcal{O}^+(\xi) := \{\eta \in M \mid \exists t \in \mathbb{N}, u \in U^t \text{ such that } \eta = \varphi(t;\xi,u)\}\,.$$

Similarly, the set of points reachable exactly in time t is denoted by $\mathcal{O}_t^+(\xi)$. System (3) is called *forward accessible* [2], if the forward orbit $\mathcal{O}^+(\xi)$ of every point $\xi \in M$ has nonempty interior and *uniformly forward accessible (in time t)* if there is a $t \in \mathbb{N}$ such that $\operatorname{int}\mathcal{O}_t^+(\xi) \neq \emptyset$ for all $\xi \in M$. Note that $\operatorname{int}\mathcal{O}^+(\xi) \neq \emptyset$ holds iff there is a $t \in \mathbb{N}$ such that $\operatorname{int}\mathcal{O}_t^+(\xi) \neq \emptyset$. Sard's theorem implies then the existence of a control $u \in U^t$ such that

$$\operatorname{rk}\frac{\partial\varphi(t;\xi,u)}{\partial u} = n-1\,.$$

A pair $(\xi,u) \in M \times U^t$ is called *regular* if this rank condition holds. The control sequence $u \in U^t$ is called *universally regular* if (ξ,u) is a regular pair for every $\xi \in M$. By [16, Corollaries 3.2 & 3.3] forward accessibility is equivalent to the fact that the set of universally regular control sequences U^t_{reg} is open and dense in U^t for all t large enough. (For a precise statement we refer to [16].)

The following result shows forward accessibility for (3) and gives an easy characterization of universally regular controls.

Lemma 1. *System* (3) *is uniformly forward accessible in time $n-1$. A control sequence $u \in U^t$ is universally regular if and only if there are $n-1$ pairwise different values in the sequence $u_0,\ldots,u_{t-1}$.*

2 Controllability of the projected system

By the results of the previous section we know that every point in M has a forward orbit with interior points and it is reasonable to wonder about controllability properties of system (3). As usual, we will call a point $\xi \in M$ controllable to $\eta \in M$ if $\eta \in \mathcal{O}^+(\xi)$. System (3) is said to be completely controllable on a subset $N \subset M$ if for all $\xi \in N$ we have $N \subset \mathcal{O}^+(\xi)$.

In order to analyze the controllability properties of (3) we introduce the following definition of what can be thought of as regions of approximate controllability in $M \subset \mathbb{RP}^{n-1}$. A *control set* of system (3) is a set $D \subset M$ satisfying

(i) $D \subset \operatorname{cl} \mathcal{O}^+(\xi)$ for all $\xi \in D$.
(ii) For every $\xi \in D$ there exists a $u \in U$ such that $\varphi(1; x, u) \in D$.
(iii) D is a maximal set (with respect to inclusion) satisfying (i).

An important subset of a control set D is its *core* defined by

$$\operatorname{core}(D) := \{\xi \in D \mid \operatorname{int} \hat{\mathcal{O}}^-(\xi) \cap D \neq \emptyset \text{ and } \operatorname{int} \hat{\mathcal{O}}^+(\xi) \cap D \neq \emptyset\}\,.$$

Here $\hat{\mathcal{O}}^-(\xi)$ denotes the points $\eta \in \mathbb{RP}^{n-1}$ such that there exist $t \in \mathbb{N}$, $u_0 \in \operatorname{int} U^t$ such that $\varphi(t; \eta, u_0) = \xi$ and (η, u_0) is a regular pair. By this assumption it is evident that on the core of a control set the system is completely controllable.

We are now in a position to state a result characterizing controllability of (3), see [14].

Theorem 1. *Let $A \in \mathbb{R}^{n\times n}$ be cyclic. Consider the system* (3) *on M. The following statements are equivalent:*

(i) There exists a $\xi \in M$ such that $\mathcal{O}^+(\xi)$ is dense in M.
(ii) There exists a control set $D \subset M$ with $\operatorname{int} D \neq \emptyset$.
(iii) M is a control set of system (3).
(iv) System (3) *is completely controllable on M.*
(v) There exists a universally regular control sequence $u \in U^t$ such that

$$\prod_{s=0}^{t-1} (A - u_s I)^{-1} \in \mathbb{R}^* I\,. \tag{4}$$

The unusual fact about the system we are studying is thus that by the universally regular representation of one element of the system's semigroup we can immediately conclude that the system is completely controllable. Furthermore, already the fact that there is a control set of the system implies complete controllability on the whole state space M. On the other hand it is worth pointing out, that if the conditions of the above theorem are not met, then no forward orbit of (3) is dense in M.

For brevity we will call a cyclic matrix A *II-controllable* (for inverse iteration controllable), if A satisfies any of the equivalent conditions of Theorem 1.

We have another simple characterization of II-controllability in terms of the existence of a universally regular periodic orbit through a cyclic vector v of A. This may come as a surprise.

Corollary 1. *Let $A \in \mathbb{R}^{n\times n}$ be cyclic with cyclic vector v and characteristic polynomial q_A. Consider the system* (3) *on M. The following statements are equivalent:*

(i) the matrix A is II-controllable.
(ii) There exist $t \in \mathbb{N}$, $u \in U^t_{reg}$ such that $\mathbb{P}v$ is a periodic point for system (3) *under the control sequence u.*

3 Polynomial characterizations of II-controllability

As has already become evident in the last result of the previous section the question of II-controllability is closely linked to properties of real polynomials. We will now further investigate this relationship. Here we follow the ideas for the complex case in [13] and discuss comparable results for the real case, see [14].

In the following theorem we use the notation $p \wedge q = 1$ to denote the fact that the two polynomials $p, q \in \mathbb{R}[z]$ are coprime.

Theorem 2. *Let $A \in \mathbb{R}^{n\times n}$ be cyclic with characteristic polynomial q. Consider the system* (3) *on M. The following statements are equivalent:*

(i) the matrix A is II-controllable.
(ii) For every $B \in \Gamma_A := \{p(A) \mid p \in \mathbb{R}[z], p \wedge q = 1\}$ there exist $t \in \mathbb{N}$, $u \in U^t_{reg}$, $\alpha \in \mathbb{R}^*$ such that

$$B = \alpha \prod_{s=0}^{t-1} (A - u_s I)\,,$$

i.e. $\Gamma_A = \Gamma_A^{\mathbb{R}} := \{p(A) \mid p(z) = \alpha \prod_{s=0}^{t-1} (z - u_s), u_s \in \mathbb{R}, p \wedge q = 1, \alpha \in \mathbb{R}^*\}$.

(iii) For every $p \in \mathbb{R}[z], p \wedge q = 1$ there exist $t \in \mathbb{N}$, $u \in U^t_{reg}$, $\alpha \in \mathbb{R}^*$ such that

$$p(z) = \alpha \prod_{s=0}^{t-1} (z - u_s) \mod q(z)\,.$$

(iv) There exists a monic polynomial f with only real roots and at least $n-1$ pairwise different roots, $\alpha \in \mathbb{R}^*$ and $r(z) \in \mathbb{R}[z]$ such that

$$f(z) = \alpha + r(z)q(z)\,. \qquad (5)$$

Remark 1. From (5) it is easy to deduce the following statement: If for a cyclic matrix A with characteristic polynomial q there exists a monic polynomial f with only real roots that are *all* pairwise distinct such that (5) is satisfied, then there is a neighborhood of A consisting of II-controllable matrices. The reason for this is that, keeping α and $r(z)$ fixed, small changes in the coefficients of q will only lead to small changes in the coefficients of f, and the assumption guarantees that all polynomials in a neighborhood of f have simple real roots.

As an immediate consequence of Theorem 1 we obtain a complete characterization of the reachable sets of the inverse power iteration given by

$$\xi(t+1) = (A - u_t I)^{-1}\xi(t)\,, \quad t \in \mathbb{N}, \qquad \xi(0) = \xi_0 \in \mathbb{RP}^{n-1}\,, \tag{6}$$

for II-controllable matrices $A \in \mathbb{R}^{n\times n}$. This extends a result in [13] for real matrices.

Corollary 2. *Let A be II-controllable with characteristic polynomial q_A, then (i) for each $\xi = \mathbb{P}x \in \mathbb{RP}^{n-1}$ we have*

$$\mathcal{O}^+(\xi) = \mathbb{P} \bigcap_{x\in V, AV\subset V} V \setminus \bigcup_{x\notin V, AV\subset V} V\,,$$

$$\operatorname{cl}\mathcal{O}^+(\xi) = \mathbb{P} \bigcap_{x\in V, AV\subset V} V = \mathbb{P}\operatorname{span}\{x, Ax, A^2x, \dots, A^{n-1}x\}\,.$$

(ii) There is a one-to-one correspondence between
a) The forward orbits of system (6).
b) The closures of the forward orbits of system (6).
c) The A-invariant subspaces of $\mathbb{R}^n$.
d) The factors of $q_A(z)$ over the polynomial ring $\mathbb{R}[z]$.

4 Conditions for II controllability

The result of the previous section raises the question which cyclic matrices A admit a representation of the form (4) or equivalently when (5) is possible. With respect to this question we have the following preliminary results.

Proposition 1. *Let $A \in \mathbb{R}^{n\times n}$ be cyclic with characteristic polynomial q_A.*

(i) A is not II-controllable, if it satisfies one of the following conditions
(a) A has a nonreal eigenvalue of multiplicity $\mu > 1$.
(b) A has a real eigenvalue of multiplicity $\mu > 2$.
(ii) A is II-controllable, if $\sigma(A) \subset \mathbb{R}$ and no eigenvalue has multiplicity $\mu > 2$.

In general, a complete characterization of the set of cyclic matrices that is not II-controllable is not known. Several examples, showing obstructions to this property in terms of the location of the eigenvalues are discussed in detail in [14]. These are obtained via the following result.

Proposition 2. *Let $A \in \mathbb{R}^{n\times n}$ be cyclic.*

(i) If for two eigenvalues $\lambda_1, \lambda_2 \in \sigma(A)$ we have $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2, |\lambda_1| \neq |\lambda_2|$ then A is not II-controllable.

(ii) If the spectrum $\sigma(A)$ is symmetric with respect to rotation by a root of unity, i.e. $\sigma(A) = \exp(2\pi i/m)\sigma(A)$ (taking into account multiplicities) and two eigenvalues of A^m satisfy the condition of (i) then A is not II-controllable.

If, furthermore, m is even, then it is sufficient that for two eigenvalues of A^m we have

$$|\lambda_1 - u| < |\lambda_2 - u|,$$

for all $u > 0$ in order that A is not II-controllable.

Using this corollary it is easy to construct examples of matrices that are not II-controllable. Such are e.g. the companion matrices of the polynomial $p(z) = z(z^2+1)$ and the 7-th degree polynomial whose roots are 0, the three cubic roots of i and their respective complex conjugates. Using the last statement, one sees that the matrix corresponding to $p(z) = (z^2-1)(z^2+1)$ is not II-controllable. Many more examples like this can be constructed, some more examples are discussed in [14].

For the case $n \leq 3$ the following complete result can be given.

Proposition 3. *Let $A \in \mathbb{R}^{n\times n}$ be cyclic.*

(i) If $n = 1, 2$ then A is II-controllable.

(ii) If $n = 3$ then A is II-controllable if and only if the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A do not have a common real part, i.e. do not satisfy $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = \operatorname{Re}\lambda_3$.

5 Control system on rational functions

There is an interesting reformulation of the inverse Rayleigh iteration as an equivalent control system on rational function spaces. This connects up with the work by Brockett and Krishnaprasad [9] on scaling actions on rational functions, as well as with divided difference schemes in interpolation theory.

Let $(c, A) \in \mathbb{R}^{1\times n} \times \mathbb{R}^{n\times n}$ be an observable pair and let $q(z) := \det(zI - A)$ denote the characteristic polynomial. Let

$$\operatorname{Rat}(q) := \left\{ \frac{p(z)}{q(z)} \in \mathbb{R}(z) \;\middle|\; \deg p < \deg q \right\}$$

denote the real vectorspace of all strictly proper real rational functions with fixed denomination polynomial $q(z)$. The map

$$\begin{aligned} \varphi : \mathbb{R}^n &\to \operatorname{Rat}(q) \\ x &\mapsto c(zI - A)^{-1}x \end{aligned}$$

defines a bijective isomorphism between $\mathbb{R}^n$ and $\mathrm{Rat}(q)$. We use it to transport the inverse Rayleigh iteration onto $\mathrm{Rat}(q)$. Let $\mathbb{P}(\mathrm{Rat}(q))$ denote the associated projective space; i.e. two rational functions $g_1(z), g_2(z) \in \mathrm{Rat}(q)$ define the same element $\mathbb{P}g_1 = \mathbb{P}g_2$ in $\mathbb{P}(\mathrm{Rat}(q))$ if and only if g_1 and g_2 differ by a nonzero constant factor. Then φ induces a homeomorphism

$$\begin{aligned} \phi : \mathbb{RP}^{n-1} &\to \mathbb{P}(\mathrm{Rat}(q)) \\ \phi(\mathbb{P}x) &:= \mathbb{P}c(zI - A)^{-1}x. \end{aligned}$$

Let $(\mathbb{P}x_t)_{t\in\mathbb{N}}$ denote the sequence in $\mathbb{RP}^{n-1}$ generated by the inverse power iteration (3). Then, for

$$g_t(z) := c(zI - A)^{-1}x_t$$

we obtain the divided difference scheme

$$\begin{aligned} \mathbb{P}g_{t+1}(z) &= \mathbb{P}c(zI - A)^{-1}(A - u_tI)^{-1}x_t \\ &= \mathbb{P}(\frac{g_t(z) - g_t(u_t)}{z - u_t}). \end{aligned}$$

Conversely, if $g_0(z) = c(zI - A)^{-1}x_0 \in \mathrm{Rat}(q)$ and $(\mathbb{P}g_t)_{t\in\mathbb{N}}$ is recursively defined by

$$g_{t+1}(z) = \frac{g_t(t) - g_t(u_t)}{z - u_t}, t \in \mathbb{N}_0,$$

then $g_t \in \mathrm{Rat}(q)$ for all $t \in \mathbb{N}$ and

$$g_t(z) = c(zI - A)^{-1}x_t$$

for a sequence $(\mathbb{P}x_t)_{t\in\mathbb{N}_0}$ generated by the inverse Rayleigh iteration. Thus the inverse Rayleigh iteration (3) on $\mathbb{RP}^{n-1}$ is equivalent to the divided difference control system

$$\mathbb{P}g_{t+1}(z) = \mathbb{P}\left(\frac{g_t(z) - g_t(u_t)}{z - u_t}\right) \tag{7}$$

on $\mathbb{P}(\mathrm{Rat}(q))$.

Equivalently, we can reformulate this algorithm as a control system on polynomials of degree $< n$. To this end let

$$\mathbb{R}_n[z] := \{p \in \mathbb{R}[z] \mid \deg p < n\}$$

denote the vectorspace of polynomials of degree $< n$ and let $\mathbb{P}(\mathbb{R}_n[z])$ denote the associated projective space. Note that for any polynomial $p \in \mathbb{R}_n[z]$ and $u \in \mathbb{R}$

$$\hat{P}(z) := \frac{p(z)q(u) - p(u)q(z)}{z - u}$$

is again a polynomial of degree $< n$. Thus

$$\mathbb{P}(p_{t+1}(z)) = \mathbb{P}\left(\frac{p_t(z)q(u_t) - p_t(u_t)q(z)}{z - u_t}\right) \tag{8}$$

defines a control system on $\mathbb{P}(\mathbb{R}_n[z])$.

Since

$$\mathbb{P}\left(\frac{p_{t+1}}{q}(z)\right) = \mathbb{P}\left(\frac{\frac{p_t(z)}{q(z)} - \frac{p_t(u_t)}{q(u_t)}}{z - u_t}\right)$$

we see (for $q(u_t) \neq 0$) that the control systems (7) and (8) are equivalent.

6 Inverse iteration on flag manifolds

A well known extension of the inverse Rayleigh iteration (3) is the QR–algorithm. To include such algorithms in our approach we have to extend the analysis to inverse iterations on partial flag manifolds. A full analysis is beyond the scope of this paper and will be presented elsewhere.

For simplicity we focus on the complex case. Recall, that a partial *flag* in $\mathbb{C}^n$ is an increasing sequence $\{0\} \neq V_1 \subsetneqq \ldots \subsetneqq V_k \subset \mathbb{C}^n$ of $\mathbb{C}$–linear subspaces of $\mathbb{C}^n$. The *type* of the flag $(V_1, \ldots, V_k)$ is specified by the k–tuple $a = (a_1, \ldots, a_k)$ of dimensions

$$a_i = \dim_{\mathbb{C}} V_i, \quad i = 1, \ldots, k.$$

Thus

$$1 \leq a_1 < \ldots < a_k \leq n.$$

For any such sequence of integers $a = (a_1, \ldots, a_k)$, $1 \leq a_1 < \ldots < a_k \leq n$, let $\mathrm{Flag}(a, \mathbb{C}^n)$ denote the set of all flags $(V_1, \ldots, V_k)$ of type a. The set $\mathrm{Flag}(a, \mathbb{C}^n)$ is called a (partial) *flag manifold*. It is indeed a compact complex manifold. For $k = 1$, $a := a_1$, we obtain the *Grassmann manifold* $G_a(\mathbb{C}^n)$ as a special case while for $k = n$ and $a = (1, 2, \ldots, n)$ we obtain the (full) flag manifold

$$\mathrm{Flag}(\mathbb{C}^n) := \mathrm{Flag}((1, \ldots, n), \mathbb{C}^n).$$

For any linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ and any sequence (u_t), $u_t \notin \sigma(A)$, of complex numbers we obtain the *inverse iteration on flag manifolds*

$$(A - u_t I)^{-1} : \mathrm{Flag}(a, \mathbb{C}^n) \to \mathrm{Flag}(a, \mathbb{C}^n).$$

This defines a nonlinear control system on the flag manifold. The reachable sets are again easily seen to be equal to

$$\begin{aligned}\mathcal{R}_A(\mathcal{V}) &:= \{\prod_{i=0}^{t-1}(A-u_iI)^{-1}\mathcal{V}\,|\,t\in\mathbb{N}, u_i\notin\sigma(A)\}\\ &= \Gamma_A^{\mathbb{C}}\cdot\mathcal{V} \qquad\qquad , \mathcal{V}\in \mathrm{Flag}(a,\mathbb{C}^n)\end{aligned}$$

where the semigroup

$$\Gamma_A^{\mathbb{C}} = \{p(A)\,|\,p(z)=\alpha\prod_{i=0}^{t-1}(z-u_i), \alpha\in\mathbb{C}^*, q(u_i)\neq 0\}$$

is defined as in Section 3. Since dim $\Gamma_A^{\mathbb{C}} \leq n$ and since $\Gamma_A^{\mathbb{C}}$ acts with a stabilizer of dimension ≥ 1 on $\mathrm{Flag}(a,\mathbb{C})$ we obtain

$$\dim \mathcal{R}_A(\mathcal{V}) \leq n-1\,, \quad \forall\, \mathcal{V}\in \mathrm{Flag}(a,\mathbb{C}^n).$$

Now dim $\mathrm{Flag}(a,\mathbb{C}^n) > n-1$, except for the cases $k=1$, $k=n-1$ or $n=2$.
Thus we conclude

Proposition 4. *Except for $k=1$, $k=n-1$ or $n=2$, the reachable sets of the inverse iteration on the flag manifold Flag$(a,\mathbb{C}^n)$ have empty interior.*

Moreover, since the QR–algorithm with shifts is equivalent to the inverse iteration

$$(A-u_tI)^{-1}: \mathrm{Flag}(\mathbb{C}^n)\to \mathrm{Flag}(\mathbb{C}^n)$$

we obtain

Corollary 3. *The QR–algorithm with origin shifts is not locally accessible nor controllable, if $n\geq 3$.*

We now describe in more detail the structure of the reachable sets. For simplicity we assume that $A\in\mathbb{C}^{n\times n}$ is a diagonal matrix with has distinct eigenvalues and we focus on the inverse iteration on Grassmann manifolds

$$(A-u_tI)^{-1}: G_k(\mathbb{C}^n)\to G_k(\mathbb{C}^n).$$

For any full rank matrices $X\in\mathbb{C}^{n\times k}$ let

$$[X] := \mathrm{Im}X \in G_k(\mathbb{C}^n)$$

denote the k–dimensional subspace spanned by the columns of X.

For any increasing sequence α of integers $1\leq\alpha_1<\ldots<\alpha_r\leq n$ let X_α denote the $r\times k$ submatrix formed by the rows $\alpha_1,\ldots,\alpha_r$ of X.

Definition 1. (a) Two complex linear subspaces $[X], [Y]$ in $\mathbb{C}^n$ of dimension k are called *rank equivalent* if

$$rkX_\alpha = rkY_\alpha$$

for all $1 \le \alpha_1 < \ldots < \alpha_r \le n$ and $r = 1, \ldots, n$.

(b) Rank equivalence defines an equivalence relation on $G_k(\mathbb{C}^n)$. The equivalence classes are called *Grassmann simplices* of $G_k(\mathbb{C}^n)$.

For example, the following two matrices span rank equivalent subspaces.

$$X = \text{span}\begin{pmatrix} 1 & 0 \\ 3 & 4 \\ 0 & 1 \\ 0 & 2 \end{pmatrix}, \qquad Y = \text{span}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

The stabilization of Grassmann manifolds into Grassmann simplices has been introduced by Gelfand et.al. [7], [8]. For us they are of interest because of the following fact. (Remember that A is diagonal!)

Lemma 2. *Every reachable set $\mathcal{R}_A([X])$ in $G_k(\mathbb{C}^n)$ is contained in a Grassmann simplex.*

To obtain a more precise description of reachable sets and Grassmann simplices we consider a projection of the Grassmannian on a polytope.

For any subset $\alpha = \{\alpha_1, \ldots, \alpha_r\} \subset \overline{n} := \{1, \ldots, n\}$, $1 \le \alpha_1 < \ldots < \alpha_r \le n$, let

$$e_\alpha := e_{\alpha_1} + \cdots + e_{\alpha_r}$$

where e_i, $1 \le i \le n$, denotes the i–th standard basis vector of $\mathbb{C}^n$. For any full rank matrix $X \in \mathbb{C}^{n \times k}$ define

$$\mu(X) := \frac{\sum\limits_{1 \le \alpha_1 < \ldots < \alpha_k \le n} |\det X_\alpha|^2 e_\alpha}{\sum\limits_{\substack{1 \le \beta_1 \le \ldots \le \beta_r \le n \\ 1 \le r \le n}} |\det(X_\beta X_\beta^T)|}.$$

Then $\mu(X) = \mu(XS^{-1})$ for any invertible matrix $S \in \mathbb{C}^{k \times k}$ and thus $\mu(X)$ defines a smooth map

$$\mu : G_k(\mathbb{C}^n) \to \mathbb{R}^n, \; \mu([X]) := \mu(X)$$

on the Grassmann manifold. We refer to it as the *moment map* on $G_k(\mathbb{C}^n)$. It is easily seen that the image of μ in $\mathbb{R}^n$ is a convex polytope. More precisely we have

$$\mu(G_k(\mathbb{C}^n)) = \Delta_{k,n}$$

where $\Delta_{k,n}$ denotes the hypersimplex

$$\Delta_{k,k} := \{(t_1, \dots, t_n) \in \mathbb{R}_+^n \mid t_1 + \dots + t_n = k\}.$$

The following result by Gelfand et.al. [7] describes the geometry of Grassmann simplices in terms of the moment map.

Theorem 3. *(a) Every reachable set $\mathcal{R}_A([X])$, $[X] \in G_k(\mathbb{C}^n)$, is contained in a Grassmann simplex. More precisely, two subspaces $[X], [Y] \in G_k(\mathbb{C}^n)$ are rank equivalent if and only if*

$$\mu(\mathcal{R}_A([X])) = \mu(\mathcal{R}_A([Y])).$$

(b) $\mu(\overline{\mathcal{R}_A([X])})$ is a compact polytope in $\mathbb{R}^n$ with vertices $\{e_\alpha \mid \det X_\alpha \neq 0\}$. It is a closed subface of $\Delta_{k,n}$.
(c) There is a bijective correspondence between
(i) p–dimensional reachable sets in $\overline{\mathcal{R}_A([X])}$.
(ii) Open p–dimensional faces of $\mu(\overline{\mathcal{R}_A([X])})$.

7 Conclusions

Controllability properties of inverse iteration schemes provide fundamental limitation for any numerical algorithm defined by them in terms of suitable feedback strategies. In the complex case, reachable sets for the inverse iteration on projective space $\mathbb{CP}^{n-1}$ correspond bijectively to invariant subspaces of A. Moreover, complete controllability holds if and only if A is cyclic, see [13]. The real case is considerably harder and only partial results for complete controllability in terms of necessary or sufficient conditions are given.

Differences also occur for inverse iteration on Grassmannians or flag manifolds. The algorithms are never controllable, in particular the QR–algorithm is seen to be not controllable. Reachable sets are contained in Grassmann simplices and their adherence relation is described by the combinations of faces of a hypersimplex.

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