.4

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On Lipschitz continuity of the joint spectral radius

Exponential stability of a discrete linear inclusion is characterized by the value of the joint (or generalized) spectral radius. This quantity is locally Lipschitz continuous on the space of compact irreducible sets of matrices. We give a brief outline of the proof.

1. Preliminaries

Given a set of matrices $\mathcal{M} \subset \mathbb{K}^{n\times n}$, where $\mathbb{K} = \mathbb{R},\mathbb{C}$, we are interested in the asymptotic behavior of solutions of the discrete inclusion

$$x(t+1) \in \{Ax(t) \mid A \in \mathcal{M}\}, \quad t \in \mathbb{N}. \tag{1}$$

One of the main tools in the study of discrete inclusions consists of the generalized (or joint) spectral radius.We now define these two numbers. Associated to the set $\mathcal{M}$ we can consider the sets of products of length t

$$\mathcal{S}_t := \{A(t-1)\dots A(0) \mid A(s) \in \mathcal{M}\,, s = 0,\dots,t-1\}.$$

Let $\|\cdot\|$ be some operator norm on $\mathbb{K}^{n\times n}$ and define for $t \in \mathbb{N}$

$$\overline{\rho}_t(\mathcal{M}) := \sup\{r(S_t)^{1/t} \mid S_t \in \mathcal{S}_t\}\,, \qquad \hat{\rho}_t(\mathcal{M}) := \sup\{\|S_t\|^{1/t} \mid S_t \in \mathcal{S}_t\}\,. \tag{2}$$

The *joint spectral radius*, respectively the *generalized spectral radius*, are now defined as

$$\rho(\mathcal{M}) := \limsup_{t\to\infty} \overline{\rho}_t(\mathcal{M}) = \lim_{t\to\infty} \hat{\rho}_t(\mathcal{M})\,,$$

where the fact that equality holds is Theorem 4 in Berger and Wang [1]. The main result of this paper is the following. As usual we regard the set of compact subsets of $\mathbb{K}^{n\times n}$ as a complete metric space endowed with the Hausdorff metric. A set of matrices $\mathcal{M}$ is *irreducible* if only the trivial subspaces are invariant under $\mathcal{M}$.

Theorem 1. *The joint spectral radius is locally Lipschitz continuous on the set of irreducible compact subsets of* $\mathbb{K}^{n\times n}$.

2. Outline of Proof

Given an irreducible set $\mathcal{M} \subset \mathbb{K}^{n\times n}$ define the *limit semigroup* $\mathcal{S}_\infty$ by

$$\mathcal{S}_\infty := \{S \in \mathbb{K}^{n\times n} \mid \exists t_k \to \infty, S_{t_k} \in \mathcal{S}_{t_k} \text{ such that } \rho(\mathcal{M})^{-t_k} S_{t_k} \to S\}\,. \tag{3}$$

We note the following properties of $\mathcal{S}_\infty$, for details see [2].

Proposition 2. *Let* $\mathbb{K} = \mathbb{R},\mathbb{C}$ *and let* $\mathcal{M} \subset \mathbb{K}^{n\times n}$ *be compact and irreducible. The set* $\mathcal{S}_\infty$ *defined by* (3) *satisfies*

(i) $\mathcal{S}_\infty$ *is compact and nonempty,* $\mathcal{S}_\infty \neq \{0\}$,

(ii) $\mathcal{S}_\infty$ *is a semigroup,*

(iii) for $T \in \mathcal{S}_t, S \in \mathcal{S}_\infty$ *we have*

$$\rho(\mathcal{M})^{-t}TS\,, \quad \rho(\mathcal{M})^{-t}ST \in \mathcal{S}_\infty\,,$$

(iv) for all $t \in \mathbb{N}$, $S \in \mathcal{S}_\infty$ *there exist* $T \in \mathcal{S}_\infty, A \in \mathcal{S}_t$ *as well as* $R \in \mathcal{S}_\infty, B \in \mathcal{S}_t$ *such that*

$$S = \rho(\mathcal{M})^{-t}TA = \rho(\mathcal{M})^{-t}BR\,,$$

(v) for all $S \in \mathcal{S}_\infty$ there exist $R, T \in \mathcal{S}_\infty$ with

$$S = RT\,,$$

(vi) $\mathcal{S}_\infty$ is irreducible.

Given our irreducible set $\mathcal{M}$ and the associated limit semigroup $\mathcal{S}_\infty$ we now define the function

$$v_{\mathcal{M}}(x) := \max_{S \in \mathcal{S}_\infty} \|Sx\| \tag{4}$$

and note that this defines a norm with an interesting extremal property.

Lemma 3. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and let $\mathcal{M} \subset \mathbb{K}^{n \times n}$ be irreducible. Then $v_{\mathcal{M}}$ is a norm on $\mathbb{K}^n$ satisfying*

$$v_{\mathcal{M}}(Ax) \leq \rho(\mathcal{M}) v_{\mathcal{M}}(x)\,, \quad \forall A \in \mathcal{M}\,.$$

For irreducible $\mathcal{M}$ we will need to know how much the original norm is deformed under the definition (4). Therefore we introduce the quantities

$$c^-(\mathcal{M}) := \min\{v_{\mathcal{M}}(x) \mid \|x\| = 1\}\,, \tag{5}$$

$$c^+(\mathcal{M}) := \max\{v_{\mathcal{M}}(x) \mid \|x\| = 1\}\,. \tag{6}$$

Note that for any $A \in \mathbb{K}^{n \times n}$ we have for the induced operator norm that

$$\frac{c^-(\mathcal{M})}{c^+(\mathcal{M})}\|A\| \leq v_{\mathcal{M}}(A) \leq \frac{c^+(\mathcal{M})}{c^-(\mathcal{M})}\|A\|\,.$$

Theorem 4. *[2] Let P be a compact set of compact irreducible subsets of $\mathbb{K}^{n \times n}$. Then there is a constant $C > 0$ such that*

$$1 \leq \frac{c^+(\mathcal{M})}{c^-(\mathcal{M})} \leq C\,, \text{ for all } \mathcal{M} \in P\,.$$

Proof. **(of Theorem 1)** Let P be a compact set with respect to the Hausdorff metric and assume that P consists of irreducible sets. Fix $\mathcal{M}, \mathcal{N} \in P$ arbitrary and let $v_{\mathcal{M}}$ denote the Barabanov norm with respect to $\mathcal{M}$. In the Hausdorff metric induced by our original norm $\|\cdot\|$ we have $H(\mathcal{M}, \mathcal{N}) =: a$, which implies the in the Hausdorff metric $H_{\mathcal{M}}$ induced by $v_{\mathcal{M}}$ it holds that

$$H_{\mathcal{M}}(\mathcal{M}, \mathcal{N}) \leq \frac{c^+(\mathcal{M})}{c^-(\mathcal{M})} a \leq Ca\,,$$

where C is a constant only depending on P which exists by Theorem 4. Hence for all $x \in \mathbb{K}^n, A \in \mathcal{N}$ it holds that there exists a $B \in \mathcal{M}$ with $v_{\mathcal{M}}(A - B) \leq Ca$ and thus

$$v_{\mathcal{M}}(Ax) \leq v_{\mathcal{M}}(Bx) + v_{\mathcal{M}}((A - B)x) \leq (\rho(\mathcal{M}) + Ca)\, v_{\mathcal{M}}(x)\,.$$

Hence $\rho(\mathcal{N}) \leq \rho(\mathcal{M}) + Ca$ and by symmetry we obtain

$$|\rho(\mathcal{N}) - \rho(\mathcal{M})| \leq CH(\mathcal{M}, \mathcal{N})\,,$$

as desired.

3. References

1 Berger M. A., Wang Y.: Bounded semigroups of matrices; Lin. Alg. Appl. **166** (1992), 21–27.
2 Wirth, F.: The generalized spectral radius and extremal norms; Berichte aus der Technomathematik 00-16. Zentrum für Technomathematik, Universität Bremen, (2000), submitted to Linear Algebra and its Applications.

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