WIRTH, FABIAN

Stability of Linear Parameter Varying and Linear Switching Systems

We consider stability of families of linear time-varying systems, that are determined by a set of time-varying parameters which adhere to certain rules. The conditions are general enough to encompass on the one hand stability questions for systems that are frequently called linear parameter varying systems in the literature and on the other hand also linear switching systems, in which parameter variations are allowed to have discontinuities. Combinations of these two sets of assumptions are also possible within the framework studied here.
Under the assumption of irreducibility of the sets of system matrices, we show how to construct parameter dependent Lyapunov functions for the systems under consideration that exactly characterize the exponential growth rate. It is clear that such Lyapunov functions do not exist in general. But every system of our class can be reduced to a finite number of subsystems for which irreducibility holds.

1. Introduction and Preliminaries

In recent years there has been an increasing interest in time-varying linear systems with parameter variations that vary in a prescribed set and satisfy constraints on the derivative. Especially the class of linear time varying systems has received considerable interest, see [1],[2],[3],[6]. More recently, also linear switching systems have been investigated where the parameter variations can have discontinuities, see e.g. [4].

In the following we define a fairly general class of linear systems with parameter variations that encompasses a large number of the systems commonly treated in the literature. For this class we show how to construct Lyapunov functions that characterize the exponential growth rate of the system if an irreducibility condition is satisfied. Irreducibility is a generic condition in the space of systems.

Once the existence of such Lyapunov functions is established, they can be used to show that the largest exponential growth rate of the system (that is, the maximal Lyapunov exponent) can be approximated by exponential growth rates of periodic parameter variations, resp. periodic switching sequences. Furthermore, it can be shown that the growth rate depends in a continuous manner on the systems data and that this dependence is even locally Lipschitz continuous in neighborhoods of irreducible systems. Due to the size constraints of this paper these points are not explained here but will be treated in a forthcoming paper, see also [6].

The system is given by the data $\Sigma = (\Theta, \Theta_1, h, A)$, where $\Theta \subset \mathbb{R}^m$ denotes the space of possible parameter values, $\Theta_1 \subset \mathbb{R}^m$ describes a restriction for the variations of the parameters, $h \in [0, \infty]$ is a dwell-time, that is, the distance between discontinuities of the parameter variations and A is a map assigning a matrix to a parameter value. We will always assume that the following assumptions are satisfied.

(i) $h \in (0, \infty]$,

(ii) $\Theta \subset \mathbb{R}^m$ is a finite disjoint union of compact convex sets $\Omega_j \subset \mathbb{R}^m$, $j \in \{1, \dots, k\}$, if $h = \infty$ then $k = 1$, i.e. Θ is compact and convex,

(iii) Θ_1 a convex compact set with $0 \in \operatorname{int} \Theta_1$,

(iv) $A : \Theta \to \mathbb{R}^{n \times n}$ is a continuous map from the parameter space to the space of matrices.

Definition 1. *A function $\theta : \mathbb{R}_+ \to \Theta$ is called an admissible parameter variation if there are times $0 < t_0 < t_1 < \dots < t_k < \dots$ such that*

$$h \leq t_{k+1} - t_k \,, \ \textit{for } k = 0, 1, 2, \dots ,$$

and so that θ is absolutely continuous on the intervals $[t_k, t_{k+1})$, and satisfies

$$\dot{\theta}(t) \in \Theta_1 \,, \quad \textit{a.e. } t \in (t_k, t_{k+1}) \,.$$

The set of admissible parameter variations is denoted by $\mathcal{U}$.

We now consider the linear parameter varying system

$$\dot{x}(t) \quad = \quad A(\theta(t))x(t)\,, \quad t \geq 0\,, \quad \theta \in \mathcal{U}\,. \tag{1}$$

The evolution operator generated by (1) for a fixed parameter variation $\theta \in \mathcal{U}$ is denoted by $\Phi_\theta(t,s), t \geq s \geq 0$. In the following we want to characterize the exponential growth rate and stability of this family of systems. We have explicitly excluded the case $h = 0$ in our assumption. The natural interpretation of this condition would be that all measurable functions $\theta : \mathbb{R}_+ \to \Theta$ are admissible parameter variations. This case has been treated in [5] and the corresponding results (that are of strikingly similar nature) can be found in this reference.

Before we turn to our object of investigation, let us briefly remark that for the sake of analysis of stability the class of systems we have just defined encompasses two system classes that have been widely studied in recent years, namely, that of linear parameter varying systems and that of linear switching systems. This also justifies the title of the present paper.

A standard setup that is encountered in the literature on linear parameter varying (LPV) systems considers systems of the form (1) where the parameter variations $\theta : \mathbb{R}_+ \to \Theta$ are assumed to be continuously differentiable and satisfy the requirement $\dot{\theta}(t) \in \Theta_1$ for all $t \geq 0$. It can be shown that the exponential growth rate of the overall system does not change if this condition is relaxed to the condition that the θ are Lipschitz continuous and satisfy the restriction on the derivative $\dot{\theta}(t) \in \Theta_1$ only almost everywhere, see [6]. The latter class is contained in our setup by setting $h = \infty$, so that from the point of view of analyzing stability the standard system class is contained in our setup. Linear switching systems on the other hand are often defined by a finite set of matrices $\Theta := \{A_1, \ldots, A_k\}$ and the admissible parameter variations are those piecewise constant functions with values in Θ that satisfy a constraint on the distance between discontinuities. These systems are contained in our setup by choosing the Ω_j to be singleton sets.

2. Exponential growth rates and stability

We now begin our analysis of stability. For system (1) with $\mathcal{U}$ as set of admissible parameter variations the maximal uniform exponential growth rate is defined by

$$\widehat{\rho} := \lim_{t\to\infty} \sup_{\theta\in\mathcal{U}} \frac{1}{t} \log \|\Phi_\theta(t,0)\|\,.$$

It may be shown that the limit exists by noting that the map $t \mapsto \sup_{\theta\in\mathcal{U}} \log \|\Phi_\theta(t,0)\|$ is subadditive and using standard results on subadditive functions. Furthermore, it is straightforward to see that

$$\widehat{\rho} = \inf\{\rho \mid \exists\, M \geq 1 : \|\Phi_\theta(t,0)\| \leq M e^{\rho t}, \text{ for all } \theta \in \mathcal{U}, t \geq 0\}\,,$$

so that the following definition is a natural definition of exponential stability.

Definition 2. *System (1) with parameter variations $\mathcal{U}$ is called* exponentially stable *if $\hat{\rho} < 0$.*

Now denote by $t_0(\theta)$ the smallest discontinuity point of $\theta \in \mathcal{U}$. In order to investigate the systems behavior starting from some initial value of the parameter variation we define for $\theta_0 \in \Theta, \tau \in (0,h)$ and $t \geq 0$ the set

$$\mathcal{S}_t(\theta_0,\tau) \quad := \quad \{\,\Phi_\theta(t,0) \mid \quad \theta \text{ admissible}\,, \theta(0) = \theta_0, \text{ and } t_0(\theta) + \tau \geq h\}\,,$$

and also

$$\mathcal{S}_t(\theta_0,h) \quad := \quad \{\,\Phi_\theta(t,0) \mid \; \theta \text{ admissible}\,, \theta(0) = \theta_0, \text{ or } t_0(\theta) \geq h\}.$$

Note that for $h = \infty$ the sets do not depend on τ. The corresponding growth rates are given by

$$\widehat{\rho}(\theta_0,\tau) := \limsup_{t\to\infty} \widehat{\rho}_t(\theta_0,\tau) := \limsup_{t\to\infty} \sup \left\{ \frac{1}{t} \log \|S\| \mid S \in \mathcal{S}_t(\theta_0,\tau) \right\}\,.$$

Note that $t \mapsto \sup\{\log \|S\| \mid S \in \mathcal{S}_t(\theta_0,\tau)\}$ is not subadditive. Still we have

Lemma 3. *For all $(\theta,\tau) \in \Theta \times (0,h]$ it holds that $\widehat{\rho} = \limsup_{t\to\infty} \widehat{\rho}_t(\theta,\tau)$.*

The previous lemma states that the exponential growth rate can be realized starting from any parameter value, which is an easy but crucial property in the following construction of Lyapunov functions.

3. Construction of exact parameterized Lyapunov functions

In the construction of the parameterized Lyapunov functions a further condition becomes crucial. In the following we always assume that

the set $A(\Theta) \subset \mathbb{R}^{n\times n}$ is irreducible.

Recall that a set of matrices $\mathcal{M} \subset \mathbb{R}^{n\times n}$ is called irreducible if only the trivial subspaces $\{0\}$ and $\mathbb{R}^n$ are invariant under all matrices $A \in \mathcal{M}$. Note that in the analysis of exponential growth rates the assumption of irreducibility can be made without loss of generality. To see this note that if the set $A(\Theta)$ is not irreducible we may by a similarity transformation transform all matrices $A(\theta)$ into a common block-diagonal form such that the individual diagonal blocks are irreducible. A simple application of the variation of constants formula shows that the exponential growth rate is given as the maximum of the growth rates of the blocks on the diagonal. So that the analysis may be restricted to the diagonal blocks.

Under the assumption of irreducibility we define for $(\theta,\tau) \in \Theta \times (0,h]$ the limit sets

$$\mathcal{S}_\infty(\theta,\tau) := \{\, S \in \mathbb{R}^{n\times n} \mid \exists t_k \to \infty \,, S_k \in \mathcal{S}_{t_k}(\theta,\tau) \text{ such that } e^{-\hat{\rho} t_k} S_k \to S \,\}.$$

The previous set is vital in our construction of Lyapunov functions. We note the following properties.

Lemma 4. *(i) $\mathcal{S}_\infty(\theta,\tau)$ is compact, nonempty, not equal to $\{0\}$ and irreducible,*

(ii) the map $(\theta,\tau) \to \mathcal{S}_\infty(\theta,\tau)$ is Lipschitz continuous with respect to the Hausdorff topology.

Proof. For reasons of space we only prove (ii). To this end it is sufficient to prove Lipschitz continuity in each variable separately. So fix $\theta \in \Theta$ and let $\tau_1,\tau_2 \in (0,h]$. We may assume without loss of generality that $\tau_1 < \tau_2$. Note that in this case for all $t \geq 0$ we have $\mathcal{S}_t(\theta,\tau_1) \subset \mathcal{S}_t(\theta,\tau_2)$ whence also $\mathcal{S}_\infty(\theta,\tau_1) \subset \mathcal{S}_\infty(\theta,\tau_2)$, so that we only have to obtain an estimate for the converse direction. So let $S_k \in \mathcal{S}_{t_k}(\theta,\tau_2)$ be a sequence such that $e^{-\hat{\rho} t_k} S_k \to S \in \mathcal{S}_\infty(\theta,\tau_2)$. As $0 \in \Theta_1$ this implies that $S_k e^{A(\theta)(\tau_2-\tau_1)} \in \mathcal{S}_{t_k+\tau_2-\tau_1}(\theta,\tau_1)$ and $e^{-\hat{\rho}(t_k+\tau_2-\tau_1)} S_k e^{A(\theta)(\tau_2-\tau_1)} \to S e^{(A(\theta)-\hat{\rho} I)(\tau_2-\tau_1)} \in \mathcal{S}_\infty(\theta,\tau_1)$. Thus for an arbitrary $S \in \mathcal{S}_\infty(\theta,\tau_2)$ there is an $\tilde{S} \in \mathcal{S}_\infty(\theta,\tau_1)$ with

$$\|S - \tilde{S}\| \leq \|S\| \|I - e^{(A(\theta)-\hat{\rho} I)(\tau_2-\tau_1)}\| \leq L|\tau_2 - \tau_1| \,,$$

for a suitable constant L (which exists by the compactness of $\mathcal{S}_\infty(\theta,\tau_2)$ and as the matrix exponential function is analytic). Thus we have obtained the desired Lipschitz estimate in τ. To treat the first variable, note that we only have to prove Lipschitz continuity in the convex components Ω_j of Θ. One of these we now consider to be fixed. As $0 \in \operatorname{int}\Theta_1$ there is a constant $c > 0$ such that for all $\theta_1 \neq \theta_2 \in \Omega_j$ the vector $c(\theta_2-\theta_1)/\|\theta_2-\theta_1\| \in \Theta_1$, which implies that the map $t \mapsto \theta_1 + tc(\theta_2-\theta_1)/\|\theta_2-\theta_1\|$, $t \in [0, \|\theta_2-\theta_1\|/c]$ is the initial part of an admissible parameter variation connecting θ_1 and θ_2 and defining an operator $R \in \mathcal{S}_{\|\theta_2-\theta_1\|/c}(\theta_1,\tau)$. Hence if $S \in \mathcal{S}_s(\theta_2,\tau)$, then it follows that $SR \in \mathcal{S}_{s+\|\theta_2-\theta_1\|/c}(\theta_1,\tau)$, so that in particular for every $S \in \mathcal{S}_\infty(\theta_2,\tau)$ we have $e^{-\hat{\rho}\|\theta_2-\theta_1\|/c} SR \in \mathcal{S}_\infty(\theta_1,\tau)$. Now

$$\|S - e^{-\hat{\rho}\|\theta_2-\theta_1\|/c} SR\| \leq \|S\| \|I - e^{-\hat{\rho}\|\theta_2-\theta_1\|/c} R\| \leq \|S\| (\exp(\|\theta_2-\theta_1\| m/c) - 1) \,,$$

where $m := \max\{\|A - \hat{\rho} I\| \mid A \in A(\Theta)\}$. The last inequality on the right follows from an application of Gronwall's lemma. Also the expression on the right clearly allows for a Lipschitz estimate, so that the proof is complete.

We note the following immediate consequence.

Corollary 5. *(i) The function $v_{\theta,\tau}$ defined in the following way is a norm on $\mathbb{R}^n$*

$$v_{\theta,\tau}(x) := \max_{S \in \mathcal{S}_\infty(\theta,\tau)} \|Sx\|$$

(ii) the map $(\theta,\tau) \to v_{\theta,\tau}$ is Lipschitz continuous (with respect to the metric defined on the space of norms by taking the uniform norm on the standard sphere).

We denote the continuous extension of θ from the left in t by $\theta(t^-) := \lim_{s\uparrow t} \theta(s)$. Also for an admissible parameter variation θ with discontinuities at the points $0 < t_0 < \ldots < t_k < \ldots$ and an arbitrary time $t \geq 0$ we define

$$t^-(\theta,t) := \min\{t - \max\{t_k \mid t_k < t\}, h\} \,,$$

which can be interpreted as the time that has elapsed since the last discontinuity or h in case the last discontinuity has occurred before $t - h$. With this notation we can formulate the main result of this paper.

Theorem 6. *(i) For every parameter variation θ and $x \in \mathbb{R}^n$ it holds that*

$$v_{\theta(t^-),t^-(\theta,t)}(\Phi_\theta(t,0)x) \le e^{\hat\rho t}\, v_{\theta(0),h}(x)\,, \tag{2}$$

(ii) for every $x \in \mathbb{R}^n$, $t > 0$, $\tau \in (0,h]$ and every $\theta_0 \in \Theta$ there exists a $\theta \in \mathcal{U}$ such that $\Phi_\theta(t,0) \in \mathcal{S}_t(\theta_0,\tau)$ and

$$v_{\theta(t^-),t^-(\theta,t)}(\Phi_\theta(t,0)x) = e^{\hat\rho t}\, v_{\theta_0,\tau}(x)\,.$$

In particular, if $\tau \in (0,h)$ this implies $\theta(0) = \theta_0$.

Proof. (i) Take an arbitrary parameter variation $\theta \in \mathcal{U}$, and arbitrary $x \in \mathbb{R}^n$ and $t \ge 0$. Note that by definition $\Phi_\theta(s,0) \in \mathcal{S}_s(\theta(0),h)$ for all $s \ge 0$ and $\Phi_\theta(t+s,t) \in \mathcal{S}_s(\theta(t^-),t^-(\theta,t))$, as for all discontinuities t_k of θ with $t_k > t$ we have by definition $t_k + t^-(\theta,t) \ge h$. Now choose $S \in \mathcal{S}_\infty(\theta(t^-),t^-(\theta,t))$ such that $v_{\theta(t^-),t^-(\theta,t)}(\Phi_\theta(t,0)x) = \|S\Phi_\theta(t,0)x\|$. It is easy to see that under the concatenation conditions given above, we have $e^{-\hat\rho t} S\Phi_\theta(t,0) \in \mathcal{S}_\infty(\theta_0,h)$ from which we obtain by definition of $v_{\theta_0,h}$ that

$$v_{\theta_0,h}(x) \ge \|e^{-\hat\rho t} S\Phi_\theta(t,0)x\| = e^{-\hat\rho t} v_{\theta(t^-),t^-(\theta,t)}(\Phi_\theta(t,0)x)\,,$$

as desired.

(ii) Pick $x \in \mathbb{R}^n$, $t \ge 0$, $\tau \in (0,h]$ and $\theta_0 \in \Theta$ and choose $S \in \mathcal{S}_\infty(\theta_0,\tau)$ such that $v_{\theta_0,\tau}(x) = \|Sx\|$. By construction there is a sequence $S_k \in \mathcal{S}_{t_k}(\theta_0,\tau)$ with $e^{-\hat\rho t_k} S_k \to S$ and $t_k \to \infty$. For all k large enough we can evaluate the evolution operator corresponding to $\mathcal{S}_k$ at $(t,0)$ and we denote the corresponding operator by $\Phi_k(t,0)$. It is easy to see that by going over to a subsequence we may assume that $\Phi_k(t,0) \to \Phi(t,0) \in \mathcal{S}_t(\theta_0,\tau)$. We denote by θ_k the parameter variations generating the $\Phi_k(t,0)$ and the parameter variation generating $\Phi(t,0)$ is by denoted by θ. Now it follows from (i) that $v_{\theta(t^-),t^-(\theta,t)}(\Phi_\theta(t,0)x) \le e^{\hat\rho t}\, v_{\theta(0),\tau}(x)$ (by a slight extension of the argument used in (i)). On the other hand if we decompose

$$S_k = R_k \Phi_k(t,0)$$

then we may assume that $e^{-\hat\rho(t_k - t)} R_k \to R$ (by going over to a subsequence if necessary) and it follows that $S = e^{-\hat\rho t} R\Phi(t,0)$. The construction implies that $R \in \mathcal{S}_\infty(\theta(t^-),t^-(\theta,t))$, so that

$$v_{\theta(t^-),t^-(\theta,t)}(\Phi(t,0)x) \ge \|R\Phi(t,0)x\| = e^{\hat\rho t}\|Sx\| = e^{\hat\rho t} v_{\theta_0,\tau}\,.$$

This shows the second assertion as the last statement is obvious.

4. Conclusions

We have shown how to construct parameter dependent Lyapunov functions that are in fact norms for a wide class of linear parameter varying systems in such a way that the norms $v_{\theta,\tau}$ characterize the growth rate of the system.

For reasons of space two interesting applications of this construction have not been presented. The existence of the Lyapunov norms may be used to show that the exponential growth rate may be approximated considering only the growth rates of periodic admissible parameter variations. Furthermore, they can be used to show that the exponential growth rate depends in a continuous manner on the data and that this dependence is even locally Lipschitz around systems that satisfy an irreducibility condition. For these statements we refer to [6] and a forthcoming paper.

5. References

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Fabian Wirth, Universität Bremen, Zentrum für Technomathematik, Postfach 330 440, 28334 Bremen, Germany