Robustness of nonlinear systems subject to time-varying perturbations

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Keywords: Robust stability, nonlinear systems, genericity, spectral theory, time-varying perturbations.

1 Introduction

The robustness analysis of linear systems via a state space approach has been of major interest in the last decade. A significant step in this development was the introduction of the *stability radius* as a measure of robustness by Hinrichsen and Pritchard [8]. This methodology has subsequently been extended to several classes of linear systems and perturbations, see the survey [9]. There has also been a great deal of work done on extending these results to more general perturbation classes, see for example the survey paper [11].

Recently, in [13], stability radii for linear discrete-time systems with time-varying perturbations have been investigated based on results on the joint and the generalized spectral radius obtained in [2], [3], [4]. In this paper we study robustness of discrete-time nonlinear systems based on these results. The key idea is to define a stability radius for the perturbed nonlinear system, and then to examine the related stability radii for the linearized system.

Following the approach outlined in [6, 7] we assume there exists a fixed point x^* of the nonlinear system, and that it is singular with respect to the perturbations, *i.e.* not perturbed under the perturbation class considered. For this fixed point we define the exponential stability radius. We show that lower and upper bounds of this stability radius can be obtained by studying the linearization in x^*. Using results of the spectral theory of linear time-varying systems and semi-algebraic geometry we show that in a certain sense the upper and the lower bound of the nonlinear stability radius generically coincide. This can be used to show that the set of systems for which the stability radii for the nonlinear system and the linearization are equal contains a countable union of open and dense subsets in the C^1-topology.

A further question of interest concerns extended stability radii, that is stability radii with respect to a prescribed rate of decay c. For a fixed matrix along with a fixed perturbation structure we describe the behavior of the extended stability radii under variation of c. It turns out that for all but at most countably many c the upper and the lower bound of the nonlinear stability radius coincide.

In the ensuing Section 2 we introduce the stability radii under consideration and note some preliminary properties. In Section 3 we study extended stability radii obtaining the main tool for the proof of our main result. In Section 4 the main result concerning generic equality of the linear stability radius to the nonlinear is shown. In Section 5 we summarize the results and give a short outlook on remaining problems.

2 Preliminaries

In this paper we study nonlinear systems of the form

$$\begin{aligned} x(t+1) &= f_0(x(t))\,, \quad t \in \mathbb{N} \qquad (1)\\ x(0) &= x_0 \in \mathbb{R}^n\,, \end{aligned}$$

which have an exponentially stable fixed point x^*. By this we mean that there exists a neighborhood U of x^* and constants $c > 1, \beta < 0$ such that the solution of (1) satisfies $\|x(t,x)\| \leq ce^{\beta t}$ for all $x \in U$. Assume that (1) is subject to perturbations of the form

$$x(t+1) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t))\,, \quad t \in \mathbb{N}, \quad (2)$$

where the perturbation functions f_i leave the fixed point invariant, i.e. $f_i(x^*) = 0, i = 1,\ldots,m$. It is our aim to analyze the corresponding time-varying stability radius

$$r_{tv}(f_0,(f_i)) := \inf\{\|u\|_\infty \,|\, u : \mathbb{N} \to \mathbb{R}^m \text{ s. t.}$$
$$\text{(2) is not exponentially stable}\}\,.$$

For time-invariant perturbations this problem has been studied in [12]. Assume that the functions f_i are continuously differentiable. Associated with the nonlinear system (2) we may study the linearization in x^* given by

$$y(t+1) = A_0 y(t) + \sum_{i=1}^{m} u_i(t) A_i y(t)\,, \quad t \in \mathbb{N}, \quad (3)$$

where A_i denotes the Jacobian of f_i in x^*. We abbreviate $A(u) = A_0 + \sum u_i A_i$. Given $(A_0,\ldots,A_m)$ denote

$$\mathcal{M}_\alpha(A_0,\ldots,A_m) := \{A(u) \,|\, \|u\| \leq \alpha\}\,,$$

where we will suppress the dependence on the matrices $(A_0,\ldots,A_m)$ when it is clear which matrices are considered. For a bounded set $\mathcal{M} \subset \mathbb{R}^{n\times n}$ we consider the set of all finite products of length t denoted by

$$\mathcal{S}_t := \{A(t-1)\ldots A(0) \mid A(s) \in \mathcal{M}\,, s = 0,\ldots,t-1\}\,.$$

and call $\mathcal{M}$ exponentially stable if there are constants $c > 1, \beta < 0$ such that $\|S_t\| \leq ce^{\beta t}$ for all $S_t \in \mathcal{S}_t$. The following quantities will be useful in characterizing exponential stability. Let $r(A)$ denote the spectral radius of A and let $\|\cdot\|$ be some operator norm. Define

$$\overline{\rho}_t(\mathcal{M}) := \sup\{\frac{1}{t}\log r(S_t) \mid S_t \in \mathcal{S}_t\}\,,$$

$$\hat{\rho}_t(\mathcal{M}) := \sup\{\frac{1}{t}\log \|S_t\| \mid S_t \in \mathcal{S}_t\}\,.$$

Theorem 4 in [4] states that for bounded $\mathcal{M}$ equality holds between the joint and the generalized spectral radius. This implies

$$\rho(\mathcal{M}) := \lim_{t\to\infty} \hat{\rho}_t(\mathcal{M}) = \limsup_{t\to\infty} \overline{\rho}_t(\mathcal{M})\,. \quad (4)$$

It is easy to see that (3) is exponentially stable for all $u : \mathbb{N} \to \mathbb{R}^m$, $\|u\|_\infty < \alpha$ iff $\rho(\mathcal{M}_\alpha) < 0$. Furthermore we have for all $t \geq 1$

$$\overline{\rho}_t(\mathcal{M}) \leq \rho(\mathcal{M}) \leq \hat{\rho}_t(\mathcal{M})\,. \quad (5)$$

We denote the Lyapunov exponent corresponding to an initial condition $y_0 \neq 0$ and a sequence $u(\cdot) \in \ell(\mathbb{N},\mathbb{R}^m)$ by $\lambda(y_0, u(\cdot))$. By (5) it follows that $\rho(\mathcal{M}_\alpha)$ is equal to the maximal Lyapunov exponent of the family of time-varying systems given by (3) and the condition $\|u\|_\infty \leq \alpha$. This is the quantity studied in [2] and [6]. Define the linear stability radii

$$r_{Ly}(A_0,(A_i)) = \inf\{\alpha \mid \rho(\mathcal{M}_\alpha) \geq 0\}\,, \quad (6)$$
$$\bar{r}_{Ly}(A_0,(A_i)) = \inf\{\alpha \mid \rho(\mathcal{M}_\alpha) > 0\}\,. \quad (7)$$

For linear systems a method for the calculation of $r_{Ly}(A_0,(A_i))$ based on the relation (5) has been presented in [13]. The following lemma states a basic relationship between the nonlinear and the linear stability radii.

Lemma 2.1

$$r_{Ly}(A_0,(A_i)) \leq r_{tv}(f_0,(f_i)) \leq \bar{r}_{Ly}(A_0,(A_i))\,.$$

□

Proof: Suppose $\gamma < r_{Ly}(A_0,(A_i))$. Then by definition, $\forall u \in \ell^\infty(\mathbb{N},\mathbb{R}^m)$ such that $||u||_\infty \leq \gamma$, system (3) is exponentially stable. Thus it follows via Theorem 5.6.2 of [1] that $\forall u$ satisfying $||u||_\infty \leq \gamma$, (2) is exponentially stable. Thus $\gamma < r_{ex}(f_0;(f_i))$ and $r_{Ly}(A_0,(A_i)) \leq r_{tv}(f_0,(f_i))$.

In order to prove the second inequality note that $\gamma > \bar{r}_{Ly}(A_0,(A_i))$ implies that for a particular u there exists a solution with positive Lyapunov exponent. Again a standard linearization argument shows that the corresponding nonlinear system cannot be exponentially stable. ∎

The previous lemma gives rise to the question whether $r_{Ly}(A_0,(A_i))$ and $\bar{r}_{Ly}(A_0,(A_i))$ commonly coincide. A simple example shows that this need not always be the case.

Example 2.2 Let $n=2, m=1$ and

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly $r_{Ly}(A_0,A_1)=0$ and $\bar{r}_{Ly}(A_0,A_1)=1$. □

To investigate this problem further we use the theory of semi-algebraic sets. Recall that a set $X \subset \mathbb{R}^n$ is called semi-algebraic, if it is the finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f_1(x) = 0 = \ldots = f_l(x),$$
$$g_1(x) > 0, \ldots, g_k(x) > 0\},$$

where the f_i, g_j are all polynomials in $\mathbb{R}[X_1,\ldots,X_n]$. For properties of semi-algebraic sets we refer to [5]. It has been shown in [10] that already in the 2×2-case the set of matrix pairs $\mathcal{M}=(B_1,B_2)$ with $\rho(\mathcal{M})<0$ is not semi-algebraic. We can, however, describe the relevant sets as countable unions.

Lemma 2.3 *The sets*

$$T_- := \{(A_0,\ldots,A_m,\alpha) \in \mathbb{R}^{(n\times n)\times(m+1)} \times \mathbb{R} \mid$$
$$\rho(\mathcal{M}_\alpha(A_0,\ldots,A_m)) < 0\},$$
$$T_+ := \{(A_0,\ldots,A_m,\alpha) \in \mathbb{R}^{(n\times n)\times(m+1)} \times \mathbb{R} \mid$$
$$\rho(\mathcal{M}_\alpha(A_0,\ldots,A_m)) > 0\},$$

are each open, countable unions of semi-algebraic sets. Furthermore

$$T_0 := \{(A_0,\ldots,A_m,\alpha) \in \mathbb{R}^{(n\times n)\times(m+1)} \times \mathbb{R} \mid$$
$$\rho(\mathcal{M}_\alpha(A_0,\ldots,A_m)) = 0\}.$$

is a closed countable intersection of semi-algebraic sets. □

Proof: To abbreviate notation let $X := R^{(n\times n)\times(m+1)} \times \mathbb{R}$. In order to prove the assertion for T_-, T_+ define for $t \geq 1$

$$P_t := \{(A_0,\ldots,A_m,\alpha) \in X \mid \overline{\rho}_t(\mathcal{M}_\alpha) > 0\},$$

$$Q_t := \{(A_0,\ldots,A_m,\alpha) \in X \mid \hat{\rho}_t(\mathcal{M}_\alpha) < 0\}.$$

Then it is clear from (5) that

$$T_- = \bigcup_{t=1}^{\infty} Q_t\,, \quad T_+ = \bigcup_{t=1}^{\infty} P_t\,.$$

As P_t, Q_t are clearly open it follows that T_-, T_+ are open. Thus it remains to show that P_t and Q_t are semi-algebraic. In [12] it is shown that the spectral radius is a semi-algebraic function on $\mathbb{R}^{n\times n}$. Also the map $(A_1,\ldots,A_t) \mapsto A_1\ldots A_t$ is semi-algebraic. We denote by $\mathcal{B}(0,\alpha)$ the ball around 0 with radius α with respect to the norm on $\mathbb{R}^m$. With these remarks it may be seen that

$$P_t = \{(A_0,\ldots,A_m,\alpha) \in X \mid$$
$$\forall (u_1,\ldots,u_t) \in \mathcal{B}(0,\alpha)^t \,:\, r(A(u_1)\ldots A(u_t)) < 1\}.$$

From this representation it follows that P_t is semi-algebraic if $\mathcal{B}(0,1)$ is semi-algebraic. The Euclidean norm, amongst others, satisfies this property, which completes our proof. The same argumentation is valid for Q_t, if the norm used in the definition of $\hat{\rho}_t$ is semi-algebraic. This establishes the claim for T_- and T_+. As

$$T_0 = T_-^c \cap T_+^c = \bigcap_{t=1}^{\infty} Q_t^c \cap \bigcap_{s=1}^{\infty} P_s^c$$

and the complements of semi-algebraic sets are semi-algebraic the proof is complete. ∎

Recall the following definitions of semi-continuity. A function $f:\mathbb{R}^n \to \mathbb{R}$ is upper (lower) semi-continuous when, given $f(x_0) > c$ (resp. $f(x_0) < c$) for some $x_0 \in \mathbb{R}^n$ there exists a neighborhood $U \subset \mathbb{R}^n$ of x_0 such that $\forall x \in U,\ f(x) > c$ (resp. $f(x) < c$).

Using continuity of the maximal Lyapunov exponent of a linear discrete inclusion given by a bounded set $\mathcal{M}$ (see [2]), we may now prove the following semi-continuity properties for r_{Ly} and $\bar{r}_{Ly}$, generalizing the results for time-invariant perturbations.

Lemma 2.4

(i) $r_{Ly}(A_0;(A_i))$ *is an upper semi-continuous function of* $(A_0;(A_i))$.

(ii) $\bar{r}_{Ly}(A_0;(A_i))$ *is a lower semi-continuous function of* $(A_0;(A_i))$.

□

Proof: Suppose that $\alpha_0 := r_{Ly}(A_0;(A_i)) > c$. Then $\forall u$, such that $||u|| < c$, $\max \lambda(y, u(\cdot)) \leq \epsilon < 0$. By continuity of $\rho(\cdot)$, there exists a neighborhood U of $(A_0;(A_i))$ such that for all $(B_0;(B_i)) \in U$, $\rho(\mathcal{M}_c(B_0,\ldots,B_m)) \leq \epsilon/2 < 0$, so that $r_{Ly}(B_0;(B_i)) > c$. Thus r_{Ly} is upper semi-continuous.

A similar argument establishes lower semi-continuity of $\bar{r}_{Ly}$. ■

3 Extended Stability Radii

In some situations it may be interesting to consider an extended version of the stability radius for the linearized system.

$$r^c_{Ly}(A_0;(A_i)) := \inf\{\alpha > 0 \mid \rho(\mathcal{M}_\alpha) \geq c\} \qquad (8)$$

$$\overline{r}^c_{Ly}(A_0;(A_i)) := \inf\{\alpha > 0 \mid \rho(\mathcal{M}_\alpha) > c\} \qquad (9)$$

This allows measurement of the robustness of the system with respect to a guaranteed level of exponential convergence or divergence. These new stability radii may be linked to those of (6) as follows.

Lemma 3.5 *Let* $(A_0,\ldots,A_m) \in \mathbb{R}^{(n\times n)\times(m+1)}$, *then*

$$r^c_{Ly}(A_0;(A_i)) = r_{Ly}(e^{-c}A_0;(e^{-c}A_i)), \qquad (10)$$

$$\overline{r}^c_{Ly}(A_0;(A_i)) = \overline{r}_{Ly}(e^{-c}A_0;(e^{-c}A_i)). \qquad (11)$$

□

Proof: This can be shown via a straightforward calculation. ■

Using this proposition, the question whether the equality $r^c_{Ly}(A_0;(A_i)) = \overline{r}^c_{Ly}(A_0;(A_i))$ holds generically may be answered for every c. A further question of interest involves those matrices/structures $(A_0,\ldots,A_m)$ for which r^c_{Ly} and $\overline{r}^c_{Ly}$ coincide for all $c \in \mathbb{R}$. In the time-invariant case it has been shown in [12] that this cannot be expected. It has to be pointed out however that the argument employed in this reference does not transfer to the time-varying case. The reason for this is that in the time-invariant case it is possible that for two constant $a < b$ it is possible that

$$\max\{r(A(u)) \mid \|u\| = b\} < \max\{r(A(u)) \mid \|u\| = a\}.$$

On the other hand if we denote $\mathcal{M}(a) := \{A(u) \mid \|u\| = a\}$ and denote by $\mathcal{M}(b)$ the corresponding set for b it follows that

$$\rho(\mathcal{M}(b)) \geq \rho(\mathcal{M}(a)).$$

by the results of Barabanov [3, Theorem 1].

For a fixed matrix and perturbation structure we now analyze the dependence of the extended stability radii on c.

Proposition 3.6 *Let* $m, n \in \mathbb{N}$ *and* $(A_0,\ldots,A_m) \in \mathbb{R}^{(n\times n)\times m}$ *be fixed. For the maps*

$$h : c \mapsto r^c_{Ly}(A_0;(A_i)), \quad \overline{h} : c \mapsto \overline{r}^c_{Ly}(A_0;(A_i)),$$

the following statements hold:

(i) h *is upper semi-continuous,* $\overline{h}$ *is lower semi-continuous.*

(ii) $h, \overline{h}$ *are discontinuous at* c_0 *iff* $h(c_0) < \overline{h}(c_0)$.

(iii) $h, \overline{h}$ *have at most countably many discontinuities.*

(iv) $h(c) = \overline{h}(c)$ *for all* $c \in \mathbb{R}$ *with the exception of at most countably many points.*

□

Proof:

(i) Semi-continuity is an immediate consequence of Lemma 3.5 and Lemma 2.4.

(ii) Let $h(c_0) < \overline{h}(c_0)$ then by definition and continuity of the maximal Lyapunov exponent $h(c_0+\epsilon) \geq \overline{h}(c_0)$ for any $\epsilon > 0$. Thus the assumption implies discontinuity of h at c_0. Discontinuity of $\overline{h}$ at c_0

follows from $h(c_0) \geq \overline{h}(c_0 - \epsilon)$. Conversely, let h be discontinuous at c_0. By semi-continuity this implies that $h(c_0) < h(c_0{+}0) \leq \overline{h}(c_0{+}0) = \overline{h}(c_0)$. The same argument works for $\overline{h}$.

(iii) This follows from the monotonicity of $h, \overline{h}$.

(iv) This follows from (ii) and (iii).

■

The following immediate corollary is needed in the proof of the main result of this paper.

Corollary 3.7 *Let $m, n \in \mathbb{N}$ and $(A_0, \ldots, A_m) \in \mathbb{R}^{(n\times n)\times m}$ be fixed. For the maps*

$$g : c \mapsto r_{Ly}(e^{-c}A_0; (e^{-c}A_i))\,,$$

$$\overline{g} : c \mapsto \overline{r}_{Ly}(e^{-c}A_0; (e^{-c}A_i))\,,$$

the following statements hold:

(i) g is upper semi-continuous, $\overline{g}$ is lower semi-continuous.

(ii) $g, \overline{g}$ are discontinuous at c_0 iff $g(c_0) < \overline{g}(c_0)$.

(iii) $g, \overline{g}$ have at most countably many discontinuities.

(iv) $g(c) = \overline{g}(c)$ for all $c \in \mathbb{R}$ with the exception of at most countably many points.

□

Proof: The statements follow from Proposition 3.6 and the fact that by Lemma 3.5 we have $g = h, \overline{g} = \overline{h}$.

■

4 Genericity

With the help of the previous results, it is possible to prove the following genericity result, which is the main result of our paper.

Theorem 4.8

(i) For fixed $m \geq 1$ the set $\mathcal{L}$ given by

$$\{(A_0, \ldots, A_m) \,|\, r_{Ly}(A_0, (A_i)) = \bar{r}_{Ly}(A_0, (A_i))\}$$

is a countable intersection of open and dense sets. Furthermore, the Lebesgue measure of the complement $\mathcal{L}^c$ is 0.

(ii) For fixed $m \geq 1$ the set $\mathcal{N}$ of maps $(f_0, \ldots, f_m)$ satisfying

$$r_{Ly}(A_0, (A_i)) = \bar{r}_{Ly}(A_0, (A_i)) = r_{tv}(f_0; (f_i))$$

contains a countable intersection of open and dense sets with respect to the C^1-topology on the space of C^1-maps $(f_0, \ldots, f_m)$ satisfying $f_0(x^*) = x^*$, $f_i(x^*) = 0$, $i = 1, \ldots, m$.

□

Proof: **(i):** First note that $(A_0, \ldots, A_m, r_{Ly}) \in \partial T_- = T_0$, and $(A_0, \ldots, A_m, \bar{r}_{Ly}) \in \partial T_+ = T_0$ by continuity of the maximal Lyapunov exponent. Thus $(A_0, \ldots, A_m) \in \mathcal{L}^c$ iff $\exists\ a, b \geq 0,\ a \neq b$ such that $(A_0, \ldots, A_m, a), (A_0, \ldots, A_m, b) \in T_0$. Under this condition if follows for all $a < c < b$ that $(A_0, \ldots, A_m, c) \in T_0$. For $k \geq 1$ we denote

$$T_{0,k} := \Big\{(A_0, \ldots, A_m, \alpha) \in \mathbb{R}^{(n\times n)\times(m+1)} \times \mathbb{R} \mid (A_0, \ldots, A_m, \alpha + \frac{1}{k}) \in T_0\Big\}\,.$$

Thus $\mathcal{L}^c = \bigcup_{k=1}^{\infty} Q_k$ where

$$Q_k := \{(A_0, \ldots, A_m) \mid \exists a \geq 0 \text{ such that } (A_0, \ldots, A_m, a) \in T_0 \cap T_{0,k}\}\,.$$

The Q_k are projections of $T_0 \cap T_{0,k}$ onto $\mathbb{R}^{(n\times n)\times(m+1)}$. By Lemma 2.3 $T_0 \cap T_{0,k}$ is closed and thus Q_k is closed for every $k \geq 1$. Therefore we now need to show that all of the sets Q_k that compose $\mathcal{L}^c$ are nowhere dense in $\mathbb{R}^{n\times n\times(m+1)}$. For this it is sufficient that in every neighborhood of any point in $\mathcal{L}^c$ there exists a point that does not belong to $\mathcal{L}^c$, as any closed set either has interior points or is nowhere dense. As any ray from zero of the form $\{a(A_1, \ldots, A_m) \mid a \geq 0\}$ intersects $\mathcal{L}^c$ in at most countably many points by Corollary 3.7 the assertion is proved.

To prove that the measure of $\mathcal{L}^c$ is zero, let χ denote the indicator function of $\mathcal{L}^c$. We obtain with Fubini's theorem that

$$\int_{\mathbb{R}^{n\times n\times(m+1)}} \chi(x)dx =$$

$$\int_{\mathbb{S}^{(n\times n\times(m+1))-1}} \int_{\mathbb{R}+} \chi(as)\, da\, ds = 0\,,$$

where we have again used that any ray from zero of the form $\{as \mid a \geq 0\}$, $s = (A_1, \ldots, A_m)$, $\|s\| = 1$ intersects $\mathcal{L}^c$ in at most countably many points.

(ii): Note that for $(f_0, \ldots, f_m) \in \mathcal{N}$ it is sufficient that for the linearized system $(A_0, \ldots, A_m) \in \mathcal{L}$. It is thus sufficient to show that the preimage of an open and dense set under the continuous, linear map

$$\{f_0, f_1, \ldots, f_m\} \mapsto \{\frac{\partial f_0}{\partial x}(x^*), \ldots, \frac{\partial f_m}{\partial x}(x^*)\}$$

is open and dense. This, however, is clear by definition of the C^1-topology. ∎

A consequence of the previous theorem is that other stability radii which might be defined for the nonlinear system, e.g. with respect to Lyapunov, or asymptotic stability generically coincide with the exponential stability radius.

5 Conclusion

In this paper we have introduced time-varying stability for nonlinear systems. Using linearization techniques and spectral theory for time-varying linear systems is has been shown that the nonlinear stability radius is bounded by two linear stability radii. The set of matrices for which these two stability radii differ has measure zero. At the moment it is not known, whether this set is open. For extended stability radii with respect to different guaranteed exponential growth rates similar results have been obtained. Further research should be directed towards the question of robust domains of attraction of nonlinear systems.

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