On the structure of the set of extremal norms of a linear inclusion

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***Abstract*— A systematic study of the set of extremal norms of an irreducible linear inclusion is undertaken. We recall basic methods for the construction of extremal norms, and consider the action of basic operations from convex analysis on these norms. It is shown that the set of extremal norms of an irreducible linear inclusion is a convex cone with a compact basis in an appropriate Banach space. Furthermore, the compact basis may be chosen to depend upper semi-continuously on the data. We explain that this is the reason for the local Lipschitz continuity of the joint spectral radius as a function of the data.**

***Index Terms*— Joint spectral radius, extremal norms, convex cones, compact basis, Lipschitz continuity.**

I. Introduction

The joint spectral radius describes the worst case exponential growth behavior of a linear semigroup defined via a generating set of compact matrices and a linear inclusion in discrete or continuous time. This concept is important in the study of systems under time-varying perturbations, but surprisingly the idea has given rise to important applications in diverse aras of mathematics, e.g. in wavelet theory, coding theory, stochastics, combinatorics, iterated function systems. The definition is due to Rota and Strang, [1]. Applications in many areas are discussed in this invited session, e.g. [9].

A recurrent theme in the analysis of the joint spectral radius has been the use of so-called extremal norms, which are norms characterizing the exponential growth rate instantaneously (in discrete time) or infinitesimally (in continuous time). Using the existence of such norms (at least for the generic case of irreducible inclusions) numerous results have been shown, as for instance continuity of the joint spectral radius [2], [3], invariance of the joint spectral radius under convexification of the generating set [2], equality of joint and generalized spectral radius [4], [5], local Lipschitz continuity of the joint spectral radius, [6], [7] and (in the discrete time case) a strict monotonicity property [8], [7].

Despite this obvious importance of the concept of extremal norms, there has been no systematic analysis of these norms as objects in their own right, at least to the best of the knowledge of the author. In this paper we begin this analysis and derive several important properties. In particular, we will see that the set of extremal norms is a convex cone with compact basis in an appropriate Banach space. This cone is invariant under several operations that are well-known in convex analysis. As an application of these results we show how the properties may be used to give a simplified proof of the local Lipschitz continuity of the joint spectral radius on the set of irreducible generators.

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The paper is organized as follows. In Section II we briefly define linear inclusions in discrete and continuous time and we recall how they generate semigroups of matrices and define the joint spectral radius. In Section III we introduce extremal norms and two special cases: Barabanov and Protasov norms. We recall a method for constructing these norms. A fixed point property associated with the unit ball of Barabanov norms is presented and we introduce basic operations that leave the set of extremal norms invariant. In the ensuing Section IV it is shown that the set of extremal norms is a convex cone in an appropriate Banach space: the space of positively homogeneous, continuous functions on $\mathbb{K}^n$. In Section V it is shown that the cones of extremal norms are "small" in that space, i.e. they have a compact basis. This basis may be chosen to depend upper semicontinuously on the generator. Finally in Section VI the previous results are used to outline a simple proof of local Lipschitz continuity. For reasons of space some proofs in this paper have been omitted. These will appear elsewhere.

II. Linear Inclusions

In the following we study linear inclusions in continuous and discrete time. Whenever necessary we specify the time set $\mathbb{T}$, which is thus either equal to $\mathbb{R}_+ := [0,\infty)$ in the continuous time case or equal to $\mathbb{N}$ in the discrete time case.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Given a compact set $\emptyset \neq \mathcal{M} \subset \mathbb{K}^{n\times n}$ and the time set $\mathbb{T} = \mathbb{N}$ we consider the discrete inclusion

$$\begin{aligned} x(t+1) &\in \{Ax(t) \mid A \in \mathcal{M}\}, \quad t \in \mathbb{N} \\ x(0) &= x_0 \quad \in \mathbb{K}^n . \end{aligned} \tag{1}$$

A sequence $\{x(t)\}_{t\in\mathbb{N}}$ is called a solution of (1) with initial condition x_0 if $x(0) = x_0$ and if for all $t \in \mathbb{N}$ there exists an $A(t) \in \mathcal{M}$ such that $x(t+1) = A(t)x(t)$. Associated to (1) we can consider the sets of products of length t

$$\mathcal{S}_t := \{A(t-1)\dots A(0) \mid A(s) \in \mathcal{M}, s = 0,\dots,t-1\},$$

and the semigroup given by $\mathcal{S} := \bigcup_{t=1}^{\infty} \mathcal{S}_t$.

In a similar manner we obtain a semigroup in the continuous time case. Given a compact set $\emptyset \neq \mathcal{M} \subset \mathbb{K}^{n\times n}$ and the time set $\mathbb{T} = \mathbb{R}_+$ we consider the semigroup generated by a differential inclusion

$$\dot{x} \in \{Ax(t) \mid A \in \mathcal{M}\}. \tag{2}$$

A function $x : \mathbb{R}_+ \to \mathbb{K}^n$ is called solution of (2) if it is absolutely continuous and satisfies $\dot{x}(t) \in \{Ax(t) \mid A \in \mathcal{M}\}$ almost everywhere. Equivalently, $x(\cdot)$ is the solution of a linear time-varying differential equation

$$\dot{x} = A(t)x(t) \tag{3}$$

for an appropriately chosen measurable map $A : \mathbb{R}_+ \to \mathcal{M}$. We denote the evolution operators of (3) by $\Phi_A(t, s)$. The set of time t transition operators is then given by

$$\mathcal{S}_t := \{\Phi_A(t, 0) \mid A : \mathbb{R}_+ \to \mathcal{M} \text{ measurable}\} .$$

Again $\mathcal{S} = \cup_{t \in \mathbb{T}} \mathcal{S}_t$ defines a semigroup. In the sequel, we will always tacitly assume that $\mathcal{S}$ is generated by a discrete inclusion of the form (1) or a differential inclusion of the form (2), if we just speak of a semigroup $(\mathcal{S}, \mathbb{T})$

In the following we wish to introduce several quantities that characterize the growth behavior of a semigroup $\mathcal{S}$. These are the *joint spectral radius* (or generalized spectral radius, or maximal Lyapunov exponent, or Lyapunov indicator), that characterizes the long term exponential growth behavior.

Remark 2.1: Whenever one considers discrete time and continuous time systems simultaneously the dilemma appears, that in discrete time it is natural to denote exponential growth in the form r^t, while in continuous time it is natural to be interested in estimates of the type $e^{\log rt}$. To keep notation short we have opted for a unified notation using the discrete time approach.

We begin our definitions with the joint spectral radius. Let $r(A)$ denote the spectral radius of A and let $\| \cdot \|$ be some operator norm on $\mathbb{K}^{n \times n}$. Define for $t \in \mathbb{N}$

$$\begin{aligned} \overline{\rho}_t(\mathcal{M}) &:= \sup\{r(S_t)^{1/t} \mid S_t \in \mathcal{S}_t\} , \\ \hat{\rho}_t(\mathcal{M}) &:= \sup\{\|S_t\|^{1/t} \mid S_t \in \mathcal{S}_t\} . \end{aligned} \qquad (4)$$

The *joint spectral radius* is defined by

$$\rho(\mathcal{M}) := \limsup_{t \to \infty} \overline{\rho}_t(\mathcal{M}) = \lim_{t \to \infty} \hat{\rho}_t(\mathcal{M}) .$$

By the results in [4] the above quantity is well-defined. Note that it does not depend on the choice of the norm $\|\cdot\|$.

If we fear that there is a chance of confusion we will denote the joint spectral radius given by a set $\mathcal{M}$ via the discrete inclusion (1) by $\rho(\mathcal{M}, \mathbb{N})$ and in the alternative case we write $\rho(\mathcal{M}, \mathbb{R}_+)$. Finally, we note that in discrete time the sets $\mathcal{S}_t$ are clearly compact if $\mathcal{M}$ is compact. In continuous time, this is true if $\mathcal{M}$ is compact and convex. As the joint spectral radius is invariant under convexification, we will tacitly assume that $\mathcal{M}$ is convex if $\mathbb{T} = \mathbb{R}_+$. Otherwise, we would have to write $\operatorname{cl} \mathcal{S}_t$ in many statements and we wish to avoid this cumbersome notation.

III. Extremal Norms

A fundamental concept in the analysis of linear inclusions as defined in the previous section are extremal norms. We introduce these in the following definitions.

Definition 3.1: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and let $(\mathcal{S}, \mathbb{T})$ be a semigroup in $\mathbb{K}^{n \times n}$.

(i) A norm v on $\mathbb{K}^n$ is called *extremal* for $\mathcal{S}$ if

$$v(S) \le \rho(\mathcal{S})^t , \text{ for all } t \in \mathbb{T}, S \in \mathcal{S}_t . \qquad (5)$$

(ii) an extremal norm v on $\mathbb{K}^n$ is called *Barabanov norm* corresponding to $\mathcal{S}$ if for all $x \in \mathbb{K}^n$, $t \in \mathbb{T}$ there is an $S \in \operatorname{cl} \mathcal{S}_t$ such that

$$v(Sx) = \rho(\mathcal{S})^t v(x) . \qquad (6)$$

(iii) a norm v on $\mathbb{K}^n$ is called *Protasov norm* corresponding to $\mathcal{S}$ if the unit ball $\mathcal{B}_v$ of v satisfies

$$\rho(\mathcal{S})^t \mathcal{B}_v = \operatorname{conv} \operatorname{cl} \mathcal{S}_t \mathcal{B}_v , \quad \forall\, t \in \mathbb{T}. \qquad (7)$$

Extremal norms do not exist for arbitrary compact sets $\mathcal{M}$. A sufficient condition for the existence of extremal, Barabanov and Protasov norms is that $\mathcal{M}$ is irreducible, i.e., that only the trivial subspaces $\{0\}$ and $\mathbb{K}^n$ are left invariant under all $A \in \mathcal{M}$, [2], [10], [7]. We denote the space of compact irreducible subsets of $\mathbb{K}^{n \times n}$ by $I(\mathbb{K}^{n \times n})$. Note that $I(\mathbb{K}^{n \times n})$ is an open and dense subset of the metric space of all nonempty compact subsets of $\mathbb{K}^{n \times n}$. (Here we use the Hausdorff metric to define the topology).

We are at times interested in embedding the norms on $\mathbb{K}^n$ in a suitable Banach space for which we choose the space of continuous, positively homogeneous functions on $\mathbb{K}^n$ which we denote

$$\operatorname{Hom}(\mathbb{K}^n, \mathbb{R}) := \{f : \mathbb{K}^n \to \mathbb{R} \mid \forall \alpha \ge 0 : f(\alpha x) = \alpha f(x) \text{ and } f \text{ is continuous on } \mathbb{K}^n\} .$$

It is easy to see that this space becomes a Banach space when endowed with the norm

$$\|f\|_{\infty,\mathrm{hom}} := \max\{|f(x)| \mid \|x\|_2 = 1\}$$

by noting that any continuous, positively homogeneous function f defines a continuous function on the unit sphere and vice versa.

We note that there is a natural positive cone K in $\operatorname{Hom}(\mathbb{K}^n, \mathbb{R})$ which is given by the functions which are positive on $\mathbb{K}^n \setminus \{0\}$. In the following we will write $w \le v$, if $v - w \in K$, $v, w \in \operatorname{Hom}(\mathbb{K}^n, \mathbb{R})$.

Remark 3.1: It has become common to use the name *Barabanov norms* because they have been introduced in [2], [11]. It is clear from the definition, that Barabanov norms are extremal.

In a similar vein, we use the name Protasov norm, because these norms have been introduced in [10]. It follows from the results of [12] that Protasov norms are extremal.

In the following we use a particular method for the construction of Barabanov and Protasov norms, which has been introduced in [6], [7]. We briefly recall the construction.

Given an irreducible semigroup $(\mathcal{S}, \mathbb{T})$ we define the *limit semigroup* $\mathcal{S}_\infty$ by

$$\mathcal{S}_\infty := \{S \in \mathbb{K}^{n \times n} \mid \exists t_k \to \infty, S_{t_k} \in \mathcal{S}_{t_k} \text{ such that } \rho(\mathcal{S})^{-t_k} S_{t_k} \to S\} . \qquad (8)$$

We recall that $\mathcal{S}_\infty$ is a irreducible, compact semigroup with a factorization property, see [6]. Given an irreducible semigroup $(\mathcal{S}, \mathbb{T})$ and the associated limit semigroup $\mathcal{S}_\infty$ we now present a general method to construct Barabanov and Protasov norms, see [7].

Theorem 3.1: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and let $(\mathcal{S}, \mathbb{T})$ be an irreducible semigroup in $\mathbb{K}^{n \times n}$ and consider an arbitrary norm $\| \cdot \|$ on $\mathbb{K}^n$ with unit ball B. Then

(i) the function

$$v(x) := \max_{S \in \mathcal{S}_\infty} \|Sx\| \qquad (9)$$

is a Barabanov norm for $\mathcal{S}$.

(ii) the set
$$B_\infty := \operatorname{conv} \mathcal{S}_\infty B \tag{10}$$
is the unit ball of a Protasov norm corresponding to $\mathcal{S}$.

A further interesting property of Barabanov norms is the following fixed point property. The proof is omitted for reasons of space.

Lemma 3.1: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and let $(\mathcal{S}, \mathbb{T})$ be an irreducible semigroup in $\mathbb{K}^{n\times n}$ with limit semigroup $\mathcal{S}_\infty$ and Barabanov norm v corresponding to the norm $\|\cdot\|$. Then for each $x \in \mathbb{R}^n$ and $R \in \mathcal{S}_\infty$ such that $v(x) = \|Rx\|$ there exist $S, T, \tilde{T} \in \mathcal{S}_\infty$ such that
$$v(Sx) = v(x)\,, \quad TSx = Sx\,, \quad \text{and} \quad \tilde{T}S = R\,.$$

In the following we will always say that a norm v is a Barabanov (Protasov) norm *corresponding* to a norm w if v is obtained from w via (9), resp. (10). We note the following interesting order relations between Barabanov, extremal and Protasov norms.

Proposition 3.1: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and let $(\mathcal{S}, \mathbb{T})$ be an irreducible semigroup in $\mathbb{K}^{n\times n}$ with limit semigroup $\mathcal{S}_\infty$. Let w be an extremal norm for $\mathcal{S}$ and let v be the corresponding Barabanov norm and v_∞ the corresponding Protasov norm, then

(i) the norm v_∞ is the Protasov norm corresponding to v, that is, for the unit ball B_∞ of v_∞ it holds that
$$B_\infty := \text{conv}\mathcal{S}_\infty B_w(0,1) = \text{conv}\mathcal{S}_\infty B_v(0,1)\,,$$
where $B_w(0,1)$ and $B_v(0,1)$ denote the unit balls of w and v, respectively,

(ii) the norm v is the Barabanov norm corresponding to the norm v_∞,

(iii) $v \le w \le v_\infty$.

Proof: (i) As w is an extremal norm we have $v \le w$ and hence $B_w(0,1) \subset B_v(0,1)$ so that we only have to show that $\mathcal{S}_\infty B_v(0,1) \subset \mathcal{S}_\infty B_w(0,1)$. To this end fix x with $v(x) = 1$ and $S \in \mathcal{S}_\infty$. By [6] we can factorize $S = RT$ with $R, T \in \mathcal{S}_\infty$. Now $w(Tx) \le 1$ as otherwise we would have $v(x) \ge w(Tx) > 1$ by definition of v. This shows that $Sx = RTx \in RB_w(0,1) \subset \mathcal{S}_\infty B_w(0,1)$ as desired.

(ii) Again by extremality of w we have $B_\infty \subset B_w(0,1)$ and thus $v_\infty \ge w$. It follows for all $x \in \mathbb{K}^n$ that
$$\begin{aligned} v(x) &= \max\{w(Sx) \mid S \in \mathcal{S}_\infty\} \\ &\le \max\{v_\infty(Sx) \mid S \in \mathcal{S}_\infty\}\,, \end{aligned}$$
and we have to show the converse inequality. Now if $v_\infty(Sx) > 1$ for some $S \in \mathcal{S}_\infty$ we have that $Sx \notin B_\infty$. We claim that this implies that $w(S'x) > 1$ for some $S' \in \mathcal{S}_\infty$ and hence $v(x) > 1$. If $w(Sx) > 1$ this is clear, otherwise we factorize $S = RT$ again and obtain that $1 \ge w(RTx)$. Then either $w(Tx) > 1$ which was the claim or $w(Tx) \le 1$ which implies that $Sx = RTx \in RB_w(0,1) \subset B_\infty$, a contradiction. This shows that
$$v(x) \ge \max\{v_\infty(Sx) \mid S \in \mathcal{S}_\infty\}\,,$$
and concludes the proof. The remaining item is immediate from the definitions. ■

Given a finite number of norms $w_1, \dots, w_k$ with associated unit balls $B_1, \dots, B_k$ there are a number of basic operations well known in convex analysis in order to construct new norms. These are the following

(i) summation of norms, where we define for $x \in \mathbb{K}^n$ a norm
$$w_\Sigma(x) := \sum_{j=1}^k w_j(x), \tag{11}$$

(ii) summation of unit balls, where we consider the norm w_{Σ^*} the unit ball of which is given by
$$B_{\Sigma^*} := B_1 + \ldots + B_k.$$
For $x \in \mathbb{K}^n$ we thus define
$$w_{\Sigma^*}(x) := \inf\left\{\alpha > 0 \,|\, \alpha^{-1}x \in B_{\Sigma^*}\right\}, \tag{12}$$

(iii) maximization, where we obtain the norm by defining
$$w_{\max}(x) := \max\{w_j(x) \,|\, j = 1, \dots, k\}\,, \tag{13}$$

(iv) and infimal convolution, which is defined by
$$\begin{aligned} &\square_{j=1}^k w_k(x) := \\ &\inf\left\{\sum_{j=1}^k w_j(x_j) \,|\, x_1, \dots, x_k \in \mathbb{K}^n, \sum_{j=1}^k x_k = x\right\}. \end{aligned} \tag{14}$$

Recall that the dual norm corresponding to a norm w is defined by
$$w^*(x) := \max\{|\langle l, x\rangle| \mid w(l) \le 1\}\,.$$
It is well known [13, Corollary 16.4] that we have the following duality relations between summation and summation of unit balls, respectively maximization and infimal convolution
$$(w_{\Sigma^*})^* = \sum_{j=1}^k w_j^*\,, \tag{15}$$
$$\left(\max_{j=1,\dots,k} w_j\right)^* = w_1^* \square w_2^* \square \ldots \square w_k^*. \tag{16}$$

We will now show that these operations leave the set of extremal norm invariant.

Theorem 3.2: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and let $(\mathcal{S}, \mathbb{T})$ be an irreducible semigroup and let $w_1, \dots, w_k$ be extremal norms for $(\mathcal{S}, \mathbb{T})$. Then the following assertions hold.

(i) For any $\alpha > 0$ the norm αw_1 is an extremal norm.

(ii) the sum $\sum_{j=1}^k w_j$ is an extremal norm.

(iii) the norm w_{Σ^*} defined in (12) is extremal.

(iv) The norm $w_{\max}$ defined in (13) is extremal.

(v) The infimal convolution $\square_{j=1}^k w_k$ is extremal.

Proof: We may assume that $\rho(\mathcal{S}) = 1$.

(i) This is obvious from the definition.

(ii) Fix $x \in \mathbb{K}^n$, $t \in \mathbb{T}$ and $S \in \mathcal{S}_t$. Then it follows that
$$\sum_{j=1}^k w_j(Sx) \le \sum_{j=1}^k w_j(x)$$
by assumption and the assertion follows.

(iii) This follows by the duality relation (15) and using that the dual of extremal norms is extremal by [12, Lemma 6.1].

(iv) Fix $x \in \mathbb{K}^n$, $t \in \mathbb{T}$ and $S \in \mathcal{S}_t$. Then it follows that

$$\max\{w_1(Sx), \ldots, w_k(Sx)\} \leq \max\{w_1(x), \ldots, w_k(x)\}\,.$$

(v) This using duality by (16) and [12, Lemma 6.1].

∎

If we consider Barabanov and Protasov norms the situation is more delicate as we see now. The proof follows the ideas of the previous proof.

Proposition 3.2: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and let $(\mathcal{S}, \mathbb{T})$ be an irreducible semigroup. Then the following assertions hold.

(i) If w is a Barabanov norm for $\mathcal{S}$ then αw is a Barabanov norm for any $\alpha > 0$.

(ii) A sum of extremal norms w_Σ is a Barabanov norm if and only if for every $x \in \mathbb{K}^n, t \in \mathbb{T}$ there exists an $S \in \mathcal{S}_t$ such that $\rho(\mathcal{S})^t w_j(x) = w_j(Sx)$, $j = 1, \ldots, k$. In particular, a sum of extremal norms is a Barabanov norm only if all the summands are Barabanov norms.

(iii) The norm $w_{\max}$ defined in (13) is a Barabanov norm, if the norms w_j, $j = 1, \ldots, k$ are Barabanov norms.

The previous result has obvious consequences for Protasov norms.

Proposition 3.3: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and let $(\mathcal{S}, \mathbb{T})$ be an irreducible semigroups. Let $w_1, \ldots, w_k$ be Protasov norms for $(\mathcal{S}, \mathbb{T})$. Then the following assertions hold.

(i) For any $\alpha > 0$ the norm αw_1 is a Protasov norm.

(ii) The norm w_{Σ^*} is a Protasov norm if and only if for every $x \in \mathbb{K}^n, t \in \mathbb{T}$ there exists an $S \in \mathcal{S}_t$ such that $w_j^*(x) = w_j^*(S^* x), j = 1, \ldots, k$. In particular, summation of unit balls of extremal norms defines a Protasov norm only if all the summands are unit balls Protasov norms.

(iii) The infimal convolution $\Box_{j=1}^k w_k$ defined in (14) is a Protasov norm.

Proof: All assertions are immediate consequences of Equations (15) and (16) and the duality results in [12]. ∎

IV. The Cone of Extremal Norms

We will now investigate the structure of the set of extremal, Barabanov and Protasov norms given an irreducible set $\mathcal{M} \subset \mathbb{K}^{n \times n}$.

Lemma 4.1: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and let $\|\cdot\|$ be an arbitrary norm on $\mathbb{K}^n$. Let $\{\mathcal{M}_k\}_{k \in \mathbb{N}} \subset I(\mathbb{K}^{n \times n})$ be a converging sequence with $\mathcal{M}_k \to \mathcal{M} \in I(\mathbb{K}^{n \times n})$. Assume that for each k the norm v_k is an extremal (Barabanov, Protasov) norm for $(\mathcal{M}_k, \mathbb{T})$ with $\max\{v_k(x) \mid \|x\| = 1\} = 1$. Then for every uniformly convergent subsequence of $\{v_k\}_{k \in \mathbb{N}}$ the limit function v is an extremal (Barabanov, Protasov) norm for $(\mathcal{M}, \mathbb{T})$.

Remark 4.1: Note that any sequence of norms $\{w_k\}_{k \in \mathbb{N}}$ with $\max\{w_k(x) \mid \|x\| = 1\} = 1$ is uniformly bounded and equicontinuous on B, the unit sphere of $\|\cdot\|$. Thus the Arzela-Ascoli theorem guarantees the existence of a subsequence uniformly convergent on B. Thus under the assumption of Lemma 4.1 a uniformly convergent subsequence of $\{v_k\}_{k \in \mathbb{N}}$ always exists and is easily seen to be convex and Lipschitz on $\mathbb{K}^n$.

Proof: We consider the time scale $\mathbb{T}$ to be fixed and drop the dependence on $\mathbb{T}$ in the remainder of the proof as far as the notation is concerned. Now ρ is a continuous function of $\mathcal{M}$, [2], [3], [7]. Thus we see that $\rho(\mathcal{M}_k) \to \rho(\mathcal{M})$ and thus we may normalize all generalized spectral radii without destroying the convergence. In the following we will therefore assume that for all k we have $\rho(\mathcal{M}_k) = \rho(\mathcal{M}) = 1$.

Assume that v is the uniform limit of some sequence v_k of extremal norms for $\mathcal{M}_k$. As for every k there is an $x_k, \|x_k\| = 1$ such that $v_k(x_k) = 1$ a standard compactness argument shows that $v \not\equiv 0$. It is easy to see that v is convex and positively homogeneous and thus v vanishes on a linear subspace X of $\mathbb{K}^n$.

We now show that v satisfies an extremality property. To this end let $S \in \mathcal{S}_t(\mathcal{M})$ and $x \in \mathbb{K}^n$. By assumption there exists a sequence $\{S_k\}$ with $S_k \to S$ and such that $S_k \in \mathcal{S}_t(\mathcal{M}_k)$ for every $k \in \mathbb{N}$. Then for a given $\varepsilon > 0$ and all k large enough it follows that

$$\begin{aligned} v(Sx) &\leq v_k(Sx) + \varepsilon \leq v_k(S_k x) + 2\varepsilon \\ &\leq v_k(x) + 2\varepsilon \leq v(x) + 3\varepsilon\,. \end{aligned} \tag{17}$$

Letting ε tend to 0 shows extremality of v. To complete the proof for the case of extremal norms it remains to show that v is a norm which amounts to showing that $X = \{0\}$. If this is not the case then by irreducibility for every $0 \neq x \in X$ we find an $S \in \mathcal{S}(\mathcal{M})$ such that $Sx \notin X$. This, however, implies $v(Sx) > 0$, which would contradict (17), hence $X = \{0\}$, as desired.

Assume now that the norms v_k are Barabanov norms. Then for fixed $t \in \mathbb{T}$ and every $k \in \mathbb{N}$ there exists some $S_k \in \mathcal{S}_t(\mathcal{M}_k)$ such that $v_k(S_k x) = v_k(x)$. Let $S \in \mathcal{S}_t(\mathcal{M})$ be any accumulation point of the sequence $\{S_k\}$. Then for a given $\varepsilon > 0$ and all k large enough it follows that

$$\begin{aligned} v(Sx) &\geq v_k(Sx) - \varepsilon \geq v_k(S_k x) - 2\varepsilon \\ &= v_k(x) - 2\varepsilon \geq v(x) - 3\varepsilon \end{aligned}$$

and again the assertion follows by letting ε tend to 0.

Finally, if the v_k are Protasov norms then $v_k \to v$ uniformly implies that $v_k^* \to v^*$ uniformly and the assertion follows from the previous considerations and by the duality between Barabanov and Protasov norms [12].

∎

The following theorem is the main result of this section.

Theorem 4.1: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$. The set valued maps

$$\mathcal{E}, \mathcal{B}, \mathcal{P} \quad : \quad I(\mathbb{K}^{n \times n}) \multimap \mathrm{Hom}(\mathbb{K}^n, \mathbb{R})\,,$$

given by

$$\mathcal{E} : \mathcal{M} \multimap \{v \mid v \text{ is an extremal norm for } (\mathcal{M}, \mathbb{T})\} \cup \{0\}\,,$$

$$\mathcal{B} : \mathcal{M} \multimap \{v \mid v \text{ is a Barabanov norm for } (\mathcal{M}, \mathbb{T})\} \cup \{0\}\,,$$

and

$$\mathcal{P} : \mathcal{M} \multimap \{v \,|\, v \quad \text{is a Protasov norm for} \quad (\mathcal{M}, \mathbb{T}) \;\} \cup \{0\}$$

have the following properties:

(i) The values of $\mathcal{E}$ are closed, convex cones.

(ii) The values of $\mathcal{B}$ and $\mathcal{P}$ are closed cones.

(iii) For every $\mathcal{M} \in I(\mathbb{K}^{n\times n})$ and $w \in \mathcal{B}(\mathcal{M})$ the minimal face of $\mathcal{E}(\mathcal{M})$ containing w is contained in $\mathcal{B}(\mathcal{M})$.

(iv) For every $r > 0$ the maps

$$\begin{aligned}
\mathcal{E}_r \quad &: \quad \mathcal{M} \quad \multimap \left\{ v \,|\, v \in \mathcal{E}\left(\mathcal{M}\right), \|v\|_{\infty,\text{hom}} \leq r \right\}, \\
\mathcal{B}_r \quad &: \quad \mathcal{M} \quad \multimap \left\{ v \,|\, v \in \mathcal{B}\left(\mathcal{M}\right), \|v\|_{\infty,\text{hom}} \leq r \right\}, \\
\mathcal{P}_r \quad &: \quad \mathcal{M} \quad \multimap \left\{ v \,|\, v \in \mathcal{P}\left(\mathcal{M}\right), \|v\|_{\infty,\text{hom}} \leq r \right\},
\end{aligned}$$

are upper semicontinuous with compact values.

Proof:

(i) Convexity and conicity of $\mathcal{E}(\mathcal{M})$ follow from Theorem 3.2 (i) and (ii). If $\{v_k\}_{k\in\mathbb{N}} \subset \mathcal{E}(\mathcal{M})$ is a convergent sequence, then either $v_k \to 0 \in \mathcal{E}(\mathcal{M})$ or

$$c_k := \max\left\{v_k\left(x\right) \mid \|x\|_2 = 1\right\} \to c > 0\,.$$

Then $v := \lim_{k\to\infty} v_k = \lim_{k\to\infty} (c/c_k)\, v_k$ and Lemma 4.1 shows that $v \in \mathcal{E}(\mathcal{M})$. Thus $\mathcal{E}\left(\mathcal{M}\right)$ is closed.

(ii) The values of $\mathcal{B}$ and $\mathcal{P}$ are cones by Theorem 3.2 (i). Closedness follows from an application of Lemma 4.1 as in (i).

(iii) This is immediate from Proposition 3.2 (ii).

(iv) To show upper semicontinuity of $\mathcal{E}_r, \mathcal{B}_r, \mathcal{P}_r$ let $\{\mathcal{M}_k\}_{k\in\mathbb{N}} \subset I(\mathbb{K}^{n\times n})$ be a converging sequence with $\mathcal{M}_k \to \mathcal{M} \in I\left(\mathbb{K}^{n\times n}\right)$. Assume that the norms $v_k \in \mathcal{E}_r\left(\mathcal{M}_k\right)$ are uniformly convergent to v. Then either $v \equiv 0 \in \mathcal{E}_r\left(\mathcal{M}\right)$ or $v \in \mathcal{E}_r\left(\mathcal{M}\right)$ by Lemma 4.1. The argument for $\mathcal{B}_r, \mathcal{P}_r$ is of course exactly the same. Compactness of the values also follows from an application of Lemma 4.1 by considering the sequence $\mathcal{M}_k \equiv \mathcal{M}$.

■

V. COMPACTNESS

Now for $\mathcal{M}$ irreducible there is an easy way to construct a compact basis for $\mathcal{E}(\mathcal{M})$.

Lemma 5.1: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and assume that $\mathcal{M}$ is irreducible. For any $0 \neq x \in \mathbb{K}^{n\times n}$ the set

$$\mathcal{A}(x) = \{v \in \mathcal{E}(\mathcal{M}) \mid v(x) = 1\}$$

is a compact basis of $\mathcal{E}(\mathcal{M})$.

Proof: Convexity of $\mathcal{A}(x)$ follows from convexity of $\mathcal{E}(\mathcal{M})$ and the convexity of the condition $v(x) = 1$. It is clear that for any $v \in \mathcal{E}(\mathcal{M})$ there is a unique $\alpha > 0$ such that $\alpha v(x) = 1$, so that $\alpha v \in \mathcal{A}(x)$. Hence, every ray in $\mathcal{E}(\mathcal{M})$ intersects $\mathcal{A}(x)$ exactly once, so that $\mathcal{A}(x)$ is a basis of $\mathcal{E}(\mathcal{M})$.

Finally to prove compactness, let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathcal{A}(x)$. Then $v_k(x) = 1$ for all k and by an application of Lemma 4.1 the sequence has a convergent subsequence, which has to converge to an extremal norm of $\mathcal{M}$. ■

Under variation of $\mathcal{M}$ we obtain an upper semicontinuous choice of the basis, i.e.

Proposition 5.1: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and let $q : I(\mathbb{K}^{n\times n}) \to \mathbb{K}^n \setminus \{0\}$ be a continuous map. Then the map

$$\mathcal{M} \mapsto \mathcal{A}(q(\mathcal{M}), \mathcal{M})$$

is an upper semicontinuous map from $\mathcal{M}$ to a basis of the cone of extremal norms corresponding to $\mathcal{M}$.

Proof: Let $\mathcal{M}_k \to \mathcal{M}$ be a convergent sequence in $I(\mathbb{K}^{n\times n})$. Denote $x_k = q(\mathcal{M}_k)$ and let

$$v_k \in \mathcal{A}(x_k, \mathcal{M}_k)$$

be a sequence converging to a function v. By continuity of q it follows that $x_k \to x = q(\mathcal{M}) \neq 0$ and by $v_k(x_k) = 1$ this implies that $v(x) = 1$, so that $v \not\equiv 0$ and it follows as in the proof of Lemma 4.1 that v is an extremal norm corresponding to $\mathcal{M}$. By construction

$$v \in \mathcal{A}(x, \mathcal{M}) = \mathcal{A}(q(\mathcal{M}), \mathcal{M})\,,$$

as desired. ■

A priori it is not clear, that if a Barabanov norm is contained in the compact basis $\mathcal{A}(x)$ then also its corresponding Protasov norm is contained in $\mathcal{A}(x)$. For certain choices of x this can be guaranteed recalling the fixed point property of Lemma 3.1.

Proposition 5.2: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and assume that $\mathcal{M}$ is irreducible. If $T \in \mathcal{S}_\infty$ and $0 \neq x \in \mathbb{K}^n$ satisfy $Tx = x$, then the compact basis $\mathcal{A}(x)$ of $\mathcal{E}(\mathcal{M})$ has the property that if v, v_∞ are a Barabanov norm and its corresponding Protasov norm then

$$v \in \mathcal{A}(x) \text{ if and only if } v_\infty \in \mathcal{A}(x)\,. \tag{18}$$

In particular, if $v \in \mathcal{A}(x)$ all extremal norms w for which v is the corresponding Barabanov norm satisfy $w \in \mathcal{A}(x)$.

Proof: If $v \in \mathcal{A}(x)$ then we have for the unit ball B_∞ of v_∞ that $B_\infty = \operatorname{conv} \mathcal{S}_\infty B_v \subset B_v$. As $Tx = x$ it follows that $x \in \partial B_\infty$, or in other words that $v_\infty(x) = 1$, so that $v_\infty \in \mathcal{A}(x)$. Conversely, if $v_\infty(x) = 1$, then $v(x) \geq v_\infty(Tx) = 1$ but also $v(x) \leq v_\infty(x) = 1$, as v_∞ is extremal. This shows the first assertion. The final statement is an immediate consequence of Proposition 3.1 (iii). ■

Under the assumption of the previous Proposition we can partition $\mathcal{A}(x)$ into sets of the form

$$[v, v_\infty] := \{w \in \mathcal{A}(x) \mid v \leq w \leq v_\infty\}\,,$$

where v and v_∞ are a Barabanov norm and the corresponding Protasov norm contained in $\mathcal{A}(x)$. We note the following properties.

Lemma 5.2: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and assume that $\mathcal{M}$ is irreducible. Let $x \in \mathbb{K}^n \setminus \{0\}$ satisfy $Tx = x$ for some $T \in \mathcal{S}_\infty$. Then

(i) $\mathcal{A}(x) = \bigcup [v, v_\infty]$, where the union is taken over all Barabanov norms $v \in \mathcal{A}(x)$,

(ii) for every Barabanov norm $v \in \mathcal{A}(x)$, the set $[v, v_\infty]$ is convex, invariant under maximization, convex combination of unit balls and infimal convolution.

(iii) The set

$$B_\infty := \operatorname{conv}\{\alpha S x \mid S \in \cup_{t>0} e^{-\rho(\mathcal{M})t}\mathcal{S}_t(\mathcal{M}) \cup \mathcal{S}_\infty\,,\ \alpha \in \mathbb{K}, |\alpha| = 1\}$$

is the unit ball of a Protasov norm contained in $\mathcal{A}(x)$ which is the maximal element contained in $\mathcal{A}(x)$.

VI. Lipschitz Continuity

As an application of the previous results we give a new proof for the property, that the joint spectral radius is locally Lipschitz continuous on $I(\mathbb{K}^{n\times n})$. To this end we need the following definition of eccentricity.

For irreducible $\mathcal{M}$ we are interested in the eccentricity that an extremal norm can have and we need a measure for this. Given a reference norm $\|\cdot\|$ we introduce for every norm v the quantities

$$c^-(v) := \quad \min\{v(x) \mid \|x\| = 1\}\,, \tag{19}$$

$$c^+(v) := \quad \max\{v(x) \mid \|x\| = 1\}\,. \tag{20}$$

Note that for any $A \in \mathbb{K}^{n\times n}$ we have for the induced operator norm that

$$\frac{c^-(v)}{c^+(v)}\|A\| \le v(A) \le \frac{c^+(v)}{c^-(v)}\|A\|\,. \tag{21}$$

Definition 6.1: Given two norms $\|\cdot\|, v$ on $\mathbb{K}^n$ the *eccentricity* of v with respect to $\|\cdot\|$ is defined by

$$\operatorname{ecc}(v) := \frac{c^+(v)}{c^-(v)}\,.$$

The result we wish to obtain relies on the following simple observation.

Lemma 6.1: Let $\|\cdot\|$ be a fixed norm on $\mathbb{K}^n$. If $\mathcal{V}$ is a set of norms that is compact in $\operatorname{Hom}(\mathbb{K}^n, \mathbb{R})$, then

$$1 \le \max\{\operatorname{ecc}(v) \mid v \in \mathcal{V}\} < \infty\,.$$

Proof: This follows from a standard compactness argument, after we note that by definition the eccentricity is bounded on open neighborhoods of norms. ■

We now define for a $\mathcal{M} \in I(\mathbb{K}^{n\times n})$

$$C(\mathcal{M}, \mathbb{T}) := \max\{\operatorname{ecc}(v) \mid v \quad \text{is} \quad \text{extremal} \quad \text{for} \quad (\mathcal{M}, \mathbb{T})\}$$

Corollary 6.1: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and let $\mathcal{M} \in I(\mathbb{K}^{n\times n})$. For every norm $\|\cdot\|$ on $\mathbb{K}^n$ it holds that

$$1 \le C(\mathcal{M}, \mathbb{T}) < \infty\,. \tag{22}$$

Proof: By Lemma 5.1 the cone of extremal norms $\mathcal{E}(\mathcal{M})$ has a compact basis $\mathcal{A}$. Clearly the eccentricity is constant on rays, i.e. $\operatorname{ecc}(v) = \operatorname{ecc}(\alpha v)$, for all $\alpha > 0$. Thus $C(\mathcal{M}, \mathbb{T})$ only depends on $\mathcal{A}$ and the result follows from Lemma 6.1. ■

Corollary 6.2: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$. For $\mathcal{M} \in I(\mathbb{K}^{n\times n})$ consider the constant $C(\mathcal{M}, \mathbb{T})$ defined in (22). The map

$$(\mathcal{M}, \mathbb{T}) \mapsto C(\mathcal{M}, \mathbb{T})$$

is upper semicontinuous on $I(\mathbb{K}^{n\times n})$. In particular, if $P \subset I(\mathbb{K}^{n\times n})$ is compact then there is a constant $C > 0$ such that

$$1 \le C(\mathcal{M}, \mathbb{T}) \le C\,, \text{ for all } \mathcal{M} \in P\,.$$

Proof: As we have noted $C(\mathcal{M}, \mathbb{T})$ only depends on a basis of $\mathcal{E}(\mathcal{M})$. By Proposition 5.1 we may choose the basis to depend upper semicontinuously on $\mathcal{M}$. Thus the assertion follows from Lemma 6.1 and Corollary 6.1. The second assertion is a standard property of upper semicontinuous functions. ■

We now have obtained easily all the ingredients necessary to obtain the following result on local Lipschitz continuity.

Theorem 6.1: Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathbb{T} = \mathbb{N}$. The joint spectral radius is locally Lipschitz continuous on $I(\mathbb{K}^{n\times n})$.

A similar argument applies for $\mathbb{T} = \mathbb{R}_+$, but we skip this part for reasons of space.

VII. Conclusions

For irreducible linear inclusions we have studied the set of extremal norms. It is shown that this set has a wealth of structural properties and we hope that these properties will give new insight into the joint spectral radius. As an application we have shown a proof of the Lipschitz continuity of the joint spectral radius. It remains to be investigated, what further results may be obtained using the methods developed in this paper.

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