On the calculation of time-varying stability radii

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Abstract

The problem of calculating the maximal Lyapunov exponent (generalized spectral radius) of a discrete inclusion is formulated as an average yield optimal control problem. It is shown that the maximal value of this problem can be approximated by the maximal value of discounted optimal control problems, where for irreducible inclusions the convergence is linear in the discount rate. This result is used to obtain convergence rates of an algorithm for the calculation of time-varying stability radii.

1 Introduction

The concept of stability radii was introduced in two papers by Hinrichsen and Pritchard [18], [19], who analyzed in particular robust stability of systems of the form

$$x(t+1) = (A + D\Delta E)x(t)\,, \quad t \in \mathbb{N}, \tag{1}$$

where A represents the unperturbed system, D and E are given *structure matrices* of appropriate sizes and Δ is an unknown perturbation matrix. The stability radius is then defined as the size of the smallest Δ (measured in some operator norm) for which system (1) becomes unstable. Both the case of real and complex perturbations were considered. The problem of calculating stability radii for different perturbation classes has attracted the interest of several researchers since then. For an overview of the theory the reader is referred to the survey article [20].

It was soon evident that in the complex time-invariant case the situation is far easier to analyze than if real perturbations are considered. A formula for the stability radius for a matrix subject to real time-invariant perturbations was obtained only recently by Qui et al. in [26] for the case that the perturbations Δ are measured in the spectral norm. In this case a feasible algorithm for the calculation of the real structured stability radius has been given by Sreedhar et al. [27]. However, it has been known from the work of Hinrichsen and Pritchard

Keywords: Real stability radius, discrete inclusions, generalized spectral radius, Lyapunov exponents, optimal control, discount rates, average yield
1991 Subject Classification: 34D08, 49N20, 90C30, 93C50, 93C55, 93D09

[21] that for the case of real perturbations the stability radius differs if time-invariant or time-varying perturbations are considered. In particular, it is known that for systems of type (1) the time-varying stability radius is bounded from above by the time-invariant real and from below by the complex stability radius. The ratio of these two stability radii may be arbitrarily large and so these bounds may contain little information. Thus a gap remained inasmuch as there was no algorithm available for the calculation of real time-varying stability radii.

In this paper discrete inclusions of the form

$$x(t+1) \in \{Ax(t)\ ;\ A \in \mathcal{M}\}, \quad t \in \mathbb{N}, \tag{2}$$

are studied, where $\mathcal{M}$ is a bounded set of real or complex matrices. Stability and dynamics of such systems has been studied extensively by Berger and Wang [5], Gurvits [16], Lagarias and Wang [23] and the author [30]. In particular, equality between the joint and generalized spectral radius has been shown, which will be our most useful tool. On the other hand there has been extensive research on the largest Lyapunov exponent of a discrete inclusion by Barabanov [1], [2], [3], [4], which is essentially the logarithm of the generalized spectral radius. The method proposed in [1] – [4] for the calculation of the largest Lyapunov exponent, however, does not lend itself easily to the calculation of stability radii as it becomes the more expensive the closer the exponential growth rate is to 0. This, however, is the interesting case when stability radii are considered. Algorithms that are based on the evaluation of matrix products have been proposed by Gripenberg [13] and Maesumi [24].

In our approach the maximal Lyapunov exponent is formulated as the value of an optimal control problem on the $n-1$ dimensional sphere. The main idea is to approximate the intrinsically hard problem of calculating maximal Lyapunov exponents by easier ones. These are the so called discounted optimal control problems with low discount rates. General convergence properties of value functions of discounted optimal control problems have been studied by the author [28], [29] and Grüne [14]. It has been shown in [28], that in general it is not possible to approximate average yield optimal control problems by discounted ones. Here we pursue a different approach which only yields convergence results for the maxima of the value functions, but which has the added advantage of supplying convergence rates in the discount rate, which have to our knowledge not been available previously. Also it is shown that the procedure we present for the calculation of stability radii is reliable in the sense that the estimates obtained for the stability radius are below the actual stability radius.

In the continuous-time case there has been substantial work on the Lyapunov spectrum of time-varying linear systems using an approach introduced by Colonius and Kliemann [9]. This has also led to an investigation of real time-varying stability radii in [7], [8] where they are examined via a Lyapunov exponent approach, under the assumption of further controllability properties. An alternative approach to robustness of time-varying systems via Bohl exponents has been undertaken in [17] and [31].

The paper is organized as follows. In Section 2 we present the class of systems that is studied, we introduce the time-varying stability radius and show a preliminary convergence result. Section 3 is devoted to the average yield optimal control problem that characterizes maximal Lyapunov exponents. The associated discounted optimal control problems are introduced and some known properties of the corresponding value functions are discussed. A further point is to analyze yields along periodic trajectories which may be used to approximate the optimal value. In Section 4 we discuss convergence of the maximum of the discounted value functions to the generalized spectral radius. It is then shown how these results apply to the problem of approximating time-varying stability radii. In Section 5 a numerical example is presented to illustrate the results.

2 Preliminaries

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and consider a stable time-invariant system

$$\begin{aligned} x(t+1) &= A_0 x(t)\,, \quad t \in \mathbb{N} \\ x(0) &= x_0 \quad \in \mathbb{K}^n\,, \end{aligned}$$

where $A_0 \in \mathbb{K}^{n\times n}$ and the spectral radius satisfies $r(A_0) < 1$. Time-varying uncertainty of this system may be modeled in the following manner: Let $A_0 \in \mathcal{M} \subset \mathbb{K}^{n\times n}$ be bounded and consider the discrete inclusion

$$\begin{aligned} x(t+1) &\in \{Ax(t)\ ;\ A \in \mathcal{M}\}\,, \quad t \in \mathbb{N} \\ x(0) &= x_0 \quad \in \mathbb{K}^n\,. \end{aligned} \tag{3}$$

A sequence $\{x(t)\}_{t\in\mathbb{N}}$ is called a solution of (3) with initial condition x_0 if $x(0) = x_0$ and for all $t \in \mathbb{N}$ there exists an $A(t) \in \mathcal{M}$ such that $x(t+1) = A(t)x(t)$. We denote the set of all finite products of length t by

$$\mathcal{S}_t := \{A(t-1)\dots A(0)\ ;\ A(s) \in \mathcal{M}\,, s = 0,\dots,t-1\}\,.$$

Exponential stability of the discrete inclusion (3) may now be defined as follows.

Definition 2.1 (Exponential stability) *The discrete inclusion (3) is called* exponentially stable, *if there exist constants $c \geq 1, \beta < 0$ such that*

$$\|S_t\| \leq c e^{\beta t}\,, \qquad \text{for all } t \geq 0\,, \quad S_t \in \mathcal{S}_t\,. \tag{4}$$

Two quantities which have been studied in [5] in order to analyze exponential stability of discrete inclusions are the joint and the generalized spectral radius. We depart slightly from the conventions in this area in that we take logarithms of all quantities, as is the custom if Lyapunov exponents are considered. Let $\mathcal{M} \subset \mathbb{K}^{n\times n}$ be fixed, let $\|\cdot\|$ be some operator norm on $\mathbb{K}^{n\times n}$ and define:

$$\overline{\rho}_t(\mathcal{M}) := \sup\{\frac{1}{t}\log r(S_t)\ ;\ S_t \in \mathcal{S}_t\}\,, \qquad \hat{\rho}_t(\mathcal{M}) := \sup\{\frac{1}{t}\log \|S_t\|\ ;\ S_t \in \mathcal{S}_t\}\,.$$

Theorem 4 in [5] states that for bounded $\mathcal{M}$ the following equality holds

$$\rho(\mathcal{M}) := \lim_{t\to\infty} \hat{\rho}_t(\mathcal{M}) = \limsup_{t\to\infty} \overline{\rho}_t(\mathcal{M})\,. \tag{5}$$

It is easy to see that (3) is exponentially stable iff $\rho(\mathcal{M}) < 0$. Furthermore, we have for all $t \geq 1$

$$\overline{\rho}_t(\mathcal{M}) \leq \rho(\mathcal{M}) \leq \hat{\rho}_t(\mathcal{M})\,. \tag{6}$$

The concept of stability radius may now be formulated as follows, see also [7]. Assume we are given an increasing family $\mathcal{U} := \{\mathcal{M}_\gamma;\ \gamma \geq 0\}$ of bounded subsets of $\mathbb{K}^{n\times n}$, i.e. $\mathcal{M}_{\gamma_1} \subseteq \mathcal{M}_{\gamma_2}$

if $\gamma_1 \leq \gamma_2$. $\mathcal{M}_\gamma$ is the set of admissible perturbations at the *perturbation intensity* γ. We will assume that $\gamma = 0$ represents the case, when no perturbations are present i.e. $\mathcal{M}_0 = \{A_0\}$. The role model we have in mind when considering the family $\{\mathcal{M}_\gamma,\ \gamma \geq 0\}$ is given by an increasing family of convex sets where U_1 is a bounded convex subset of $\mathbb{K}^{n\times n}$ with $0 \in U_1$ and for all $\gamma \geq 0$ the uncertainty is modeled by $\mathcal{M}_\gamma := \gamma U_1 + A_0$. This set-up contains in particular affine linear perturbations, perturbation structures of feedback type considered in [18], [21], positive systems studied in [22], and periodic systems [31] as subclasses. Given the family $\mathcal{U}$ the problem is then to find the smallest γ such the discrete inclusion given by $\mathcal{M}_\gamma$ is not exponentially stable.

Definition 2.2 (Stability radius) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For an increasing family $\mathcal{U}$, such that $\{A_0\} = \mathcal{M}_0$ we define the* time-varying stability radius *of A_0 by*

$$r_{tv}(A_0, \mathcal{U}) := \inf\{\gamma;\ \rho(\mathcal{M}_\gamma) \geq 0\}\,. \tag{7}$$

For our later analysis we need the following result on the convergence of $\hat{\rho}_t(\mathcal{M})$ to $\rho(\mathcal{M})$. Recall that $\mathcal{M} \subset \mathbb{K}^{n\times n}$ is called *irreducible* if only the trivial subspaces $\{0\}$ and $\mathbb{K}^n$ are invariant under all matrices $A \in \mathcal{M}$. Otherwise $\mathcal{M}$ is called *reducible.*

Lemma 2.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that $\mathcal{M} \subset \mathbb{K}^{n\times n}$ is bounded.*

(i) If $\mathcal{M}$ is irreducible, then there exists a constant $M > 0$ such that for all $t \geq 1$

$$|\hat{\rho}_t(\mathcal{M}) - \rho(\mathcal{M})| < Mt^{-1}\,.$$

(ii) If $\mathcal{M}$ is reducible then there exists an $M > 0$ such that for all $t \geq 1$

$$|\hat{\rho}_t(\mathcal{M}) - \rho(\mathcal{M})| < M\frac{1+\log t}{t}\,.$$

Proof: (i) As $\mathcal{M}$ is irreducible, there exists a norm on $\mathbb{K}^n$ that induces an operator norm v on $\mathbb{K}^{n\times n}$ satisfying for all $t \geq 1$ $\sup\{v(S_t)\ ;\ S_t \in \mathcal{S}_t\} = e^{\rho(\mathcal{M})t}$, see [1] Theorem 2. As all norms on finite dimensional vector spaces are equivalent it follows with (6) that

$$0 \leq \frac{1}{t}\log \sup_{S_t\in\mathcal{S}_t} \|S_t\| - \rho(\mathcal{M}) \leq \frac{1}{t}\log \sup_{S_t\in\mathcal{S}_t} cv(S_t) - \rho(\mathcal{M}) = \frac{1}{t}\log c\,. \tag{8}$$

This proves the assertion.
(ii) Without loss of generality (i.e. after a suitable change of basis) all matrices $A \in \mathcal{M}$ are of the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & \dots & A_{1d} \\ 0 & A_{22} & A_{23} & \dots & A_{2d} \\ 0 & 0 & A_{33} & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & \\ 0 & & \dots & 0 & A_{dd} \end{bmatrix},$$

where each of the sets $\mathcal{M}_{ii} := \{A_{ii}; A \in \mathcal{M}\}, i = 1\dots d$ is irreducible. If $\rho(\mathcal{M}) = -\infty$ then $\hat{\rho}_d(\mathcal{M}) = \rho(\mathcal{M})$ and there is nothing to show. Otherwise, let $\mathbb{K}^n = V_1 \bigoplus \dots \bigoplus V_d$ be the corresponding decomposition. By Lemma 2 (c) in [5] it holds that $\rho(\mathcal{M}) = \max_{i=1,\dots,d} \rho(\mathcal{M}_{ii})$

and as in part (i) we may choose norms v_i on V_i, $i=1,\dots,d$ which induce matrix norms (also denoted by v_i) satisfying $\max_{S_t\in\mathcal{S}_t} v_i(S_{ii,t}) = e^{\rho(\mathcal{M}_{ii})t}$. Let $\alpha := \max\{w_{ij}(A_{ij}),\ 1\le i<j\le d,\ A\in\mathcal{M}\}$ where w_{ij} is the operator norm for the linear maps from V_i to V_j induced by the norms v_i, v_j. On $\mathbb{K}^n$ we may consider the norm $\tilde{v}(x) = \max_{i=1,\dots,d} v_i(x_i)$ if $x=(x_1,\dots,x_d)$, which induces the operator norm $\tilde{v}(A) = \max_{i=1,\dots,d}\sum_{j=1}^d w_{ij}(A_{ij})$. Again this is an equivalent norm and so we obtain for $t>d$

$$0 \le \frac{1}{t}\log\sup_{S_t\in\mathcal{S}_t}\|S_t\| - \rho(\mathcal{M}) \le \frac{1}{t}\log\sup_{S_t\in\mathcal{S}_t} c\tilde{v}(S_t) - \rho(\mathcal{M})$$

$$\le \frac{1}{t}\log ce^{(t-d)\rho(\mathcal{M})}\left\|\begin{bmatrix} e^{d\rho(\mathcal{M})} & p_{12}(e^{\rho(\mathcal{M})},t) & \dots & p_{1d}(e^{\rho(\mathcal{M})},t) \\ 0 & e^{d\rho(\mathcal{M})} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & e^{d\rho(\mathcal{M})} \end{bmatrix}\right\|_\infty - \rho(\mathcal{M})\,, \qquad (9)$$

where the $p_{ij}, 1\le i<j\le d$ are polynomials in $e^{\rho(\mathcal{M})}$ and t whose coefficients depend on α and d and whose degrees depend on d. Thus there exists a polynomial $p(t)$ satisfying

$$0 \le \frac{1}{t}\log\sup_{S_t\in\mathcal{S}_t}\|S_t\| - \rho(\mathcal{M}) \le$$

$$\frac{1}{t}\log c + \left|\frac{t-d}{t}\rho(\mathcal{M}) - \rho(\mathcal{M})\right| + \left|\frac{1}{t}\log p(t)\right| \le \frac{1}{t}(\log c + d|\rho(\mathcal{M})| + M(1+\log(t)))\,,$$

for some M large enough. This implies the assertion. □

By the results of Barabanov [1] and Gurvits [16] it is known that if the set of matrices $\mathcal{M}$ is exponentially stable and bounded then the same holds true for the closure of its convex hull $\operatorname{cl}\operatorname{conv}(\mathcal{M})$. Thus all considerations can be restricted to affine perturbations and compact $\mathcal{M}$. The following general assumption will be made throughout the remainder of this paper.

Assumption 2.4 *Let $\mathbb{K}=\mathbb{R},\mathbb{C}$. We assume that $\mathcal{U}=\{\mathcal{M}_\gamma\ ;\ \gamma\ge 0\}$ satisfies:*

(i) $\mathcal{M}_0=\{A_0\}$.

(ii) The family $\mathcal{U}$ is increasing in γ.

(iii) For all $\gamma\ge 0$ the set $\mathcal{M}_\gamma$ is compact and convex.

(iv) For all $\gamma>0$ it holds that $V_\gamma := \{x\in\mathbb{K}^n\ ;\ Ax=0,\ \forall A\in\mathcal{M}_\gamma\}=\{0\}$.

Note that Assumption (iv) is without loss of generality for robustness analysis as V_γ is a linear subspace and we can study exponential stability on the quotient space $\mathbb{K}^n/V_\gamma$ if necessary.

3 Infinite horizon optimal control

In this section we aim to show how to formulate the stability radius problem as an infinite horizon optimal control problem. In order to do this we introduce exponential growth rates of trajectories, following the approaches taken in [7], [14], [30].

Definition 3.1 (Lyapunov exponent) *Given a sequence* $\mathbf{A} \in \mathcal{M}^{\mathbb{N}}$ *and an initial condition* $x_0 \in \mathbb{K}^n \setminus \{0\}$ *the* Lyapunov exponent *corresponding to* $(x_0, \mathbf{A})$ *is defined by*

$$\lambda(x_0, \mathbf{A}) = \limsup_{t\to\infty} \frac{1}{t} \log \|\mathbf{A}(t-1)\dots\mathbf{A}(0)x_0\| \,, \tag{10}$$

where we use the convention $\log 0 = -\infty$.

From (6) it is easy to see that

$$\rho(\mathcal{M}) = \sup\{\lambda(x_0, \mathbf{A}); x_0 \in \mathbb{K}^n \setminus \{0\}, \mathbf{A} \in \mathcal{M}^{\mathbb{N}}\} \,,$$

which is the quantity studied in [1] – [4]. Note that in order to characterize exponential stability of time-varying systems it is not sufficient to consider Lyapunov exponents, but rather Bohl exponents have to be introduced, see [10] or [25]. However, it follows from (5) that for discrete inclusions determined by a bounded set of matrices the maximal Lyapunov and Bohl exponents coincide.

One tool for the study of Lyapunov exponents has been the projection onto the projective space, known as Bogolyubov's projection. It is based on the fact that in continuous time the angular component of the system may be decoupled from the radial one and can be studied independently. For our purposes it will be sufficient to consider the projection onto the sphere $\mathbb{S}^{n-1}_{\mathbb{K}}$ and to neglect the identification of opposite points. Note that all possible Lyapunov exponents can be realized starting from points on the sphere.

In our discrete–time system we do not exclude the possibility that the origin may be reached from non–zero states. However, Assumption 2.4 (iv) prevents this from happening too often. Denote $\mathcal{M}(x) := \{A \in \mathcal{M} \; ; \; Ax \neq 0\}$ and $\mathcal{M}^{\mathbb{N}}(x) := \{\mathbf{A} \in \mathcal{M}^{\mathbb{N}} \; ; \; \mathbf{A}(t)\dots\mathbf{A}(0)x \neq 0 \,, \forall t \in \mathbb{N}\}$. By Assumption 2.4 (iv) it holds for all $x \neq 0$ that $\mathcal{M}(x) \neq \emptyset, \mathcal{M}^{\mathbb{N}}(x) \neq \emptyset$. With this notation the projected inclusion corresponding to our linear inclusion (3) is given by

$$\begin{aligned} \xi(t+1) &\in \left\{ \frac{A\xi(t)}{\|A\xi(t)\|} \; ; \; A \in \mathcal{M}(\xi(t)) \right\} , \quad t \in \mathbb{N} \\ \xi(0) &= \xi_0 \in \mathbb{S}^{n-1}_{\mathbb{K}} . \end{aligned} \tag{11}$$

We denote the solution of (11) corresponding to an initial value $\xi_0 \in \mathbb{S}^{n-1}_{\mathbb{K}}$ and a control sequence $\mathbf{A} \in \mathcal{M}^{\mathbb{N}}(\xi_0)$ by $\xi(\cdot; \xi_0, \mathbf{A})$. In order to obtain the Lyapunov exponent $\lambda(x_0, \mathbf{A})$ from a trajectory $\xi(\cdot; x_0/\|x_0\|, \mathbf{A})$ of (11) we define for $\xi \in \mathbb{S}^{n-1}_{\mathbb{K}}$, $A \in \mathcal{M}(\xi)$

$$q(\xi, A) := \log \|A\xi\| \,. \tag{12}$$

A straightforward calculation yields the following expression for Lyapunov exponents.

Lemma 3.2 *Let* $\mathbb{K} = \mathbb{R}, \mathbb{C}$. *For* $\xi_0 \in \mathbb{S}^{n-1}_{\mathbb{K}}$, $\mathbf{A} \in \mathcal{M}^{\mathbb{N}}$ *it holds that*

$$\lambda(\xi_0, \mathbf{A}) = \begin{cases} \limsup_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} q(\xi(s;\xi_0,\mathbf{A}), \mathbf{A}(s)) & , \quad \mathbf{A} \in \mathcal{M}^{\mathbb{N}}(\xi_0) \\ -\infty & , \quad \textit{else.} \end{cases} \tag{13}$$

Thus Lyapunov exponents may interpreted as average yields along trajectories on the sphere. We can now begin to explore the relationship between discounted and average yield optimal control problems. For $\delta > 0$ consider the δ-discounted yield

$$J_\delta(\xi, \mathbf{A}) := \begin{cases} \sum_{t=0}^{\infty} e^{-\delta t} q(\xi(t;\xi,\mathbf{A}),\mathbf{A}(t)) & , \quad \mathbf{A} \in \mathcal{M}^{\mathbb{N}}(\xi) \\ -\infty & , \quad \text{else} . \end{cases} \tag{14}$$

The associated value functions are given by

$$V_\delta(\xi) := \sup_{\mathbf{A}\in\mathcal{M}^{\mathbb{N}}} J_\delta(\xi,\mathbf{A}), \qquad V_0(\xi) := \sup_{\mathbf{A}\in\mathcal{M}^{\mathbb{N}}} \lambda(\xi,\mathbf{A}). \tag{15}$$

Remark 3.3 (i) The stability radius can now be formulated in terms of value functions in the following sense. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $A_0 \in \mathbb{K}^{n\times n}$. Furthermore, let $\mathcal{U} \subset \mathbb{K}^{n\times n}$ satisfy Assumption 2.4 then

$$r_{tv}(A_0, \mathcal{U}) = \inf\{\gamma \geq 0; \sup_{\xi\in\mathbb{S}^{n-1}_{\mathbb{K}}} V_{0,\gamma}(\xi) \geq 0\},$$

where $V_{0,\gamma}$ denotes the value function of the average yield problem corresponding to $\mathcal{M}_\gamma$.
(ii) Note that for every $\mathbf{A} \in \mathcal{M}^{\mathbb{N}}(\xi)$ the expression for $J_\delta(\xi,\mathbf{A})$ is well defined. In fact, it holds that the infinite sum is either absolutely convergent, or the partial sums tend to $-\infty$. This may be seen as follows. Define $f_+(t) := \max\{0, e^{-\delta t} q(\xi(t;x,\mathbf{A}),\mathbf{A}(t))\}$ and $f_-(t) := \min\{0, e^{-\delta t} q(\xi(t;x,\mathbf{A}),\mathbf{A}(t))\}$. Then

$$\sum_{s=0}^{t} |e^{-\delta s} q(\xi(s;x,\mathbf{A}),\mathbf{A}(s))| = \sum_{s=0}^{t} f_+(s) + |f_-(s)|.$$

The infinite sum over $f_+(s)$ exists as $\mathcal{M}$ is bounded, so that $\sum_{s=0}^{t} f_+(s) + f_-(s)$ converges absolutely iff $\lim_{t\to\infty} \sum_{s=0}^{t} f_-(s)$ is a real number. If this is not the case then for every $c > 0$ there exists a $T \in \mathbb{N}$ such that for all $t \geq T$ it holds that $\sum_{s=0}^{t} e^{-\delta s} q(\xi(s;x,\mathbf{A}),\mathbf{A}(s)) < -c$. □

The discounted optimal control problem is far easier to analyze, which is why one tries to obtain a relation between it and the average yield problem. The following theorem summarizes some known properties of V_δ and V_0. For details we refer to [6] Chapter V.

Theorem 3.4 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Let $\mathcal{M} \subset \mathbb{K}^{n\times n}$ satisfy Assumption 2.4 (iii) and (iv) and consider the optimal control problems given by (11), (13)–(15). The following properties hold*

(i) (Bellman's principle of optimality)
For all $t \in \mathbb{N}$, $\xi_0 \in \mathbb{S}^{n-1}_{\mathbb{K}}$ it holds that

$$V_\delta(\xi_0) = \sup_{\mathbf{A}\in\mathcal{M}^t(\xi_0)} \left[\sum_{s=0}^{t-1} e^{-\delta s} q(\xi(s;x_0,\mathbf{A}),\mathbf{A}(s)) + e^{-\delta t} V_\delta(\xi(t;x_0,\mathbf{A})) \right]. \tag{16}$$

(ii) V_δ is bounded and continuous.

(iii) (Hamilton–Jacobi–Bellman equation)
V_δ is the unique bounded solution of the difference equation

$$\inf_{A\in\mathcal{M}}[V(\xi_0) - e^{-\delta}V(\xi(1;\xi_0,A)) - q(\xi_0,A)] = 0. \tag{17}$$

(iv) (Bellman's principle of optimality II)
For all $t\in\mathbb{N}$ it holds that

$$V_0(\xi) = \sup_{\mathbf{A}\in\mathcal{M}^t(\xi_0)} V_0(\xi(t;x,\mathbf{A}))\,. \tag{18}$$

4 Convergence analysis and numerical results

The important fact given by the previous Theorem 3.4 is that V_δ may be characterized via the difference equation (17). A considerable amount of effort has been spent in recent years on numerical methods for such equations. It remains to analyze the relation between V_δ and V_0, so that we can make use of the results of these efforts. To this end let us first examine properties of the different values along periodic trajectories. This may then be employed in the analysis of trajectories evolving in eigenspaces. In the following statement we use the symbol $\mathbf{A}(t+\cdot)$ to denote the sequence obtained by shifting $\mathbf{A}$.

Proposition 4.1 *Let $\xi_0\in\mathbb{S}^{n-1}_{\mathbb{K}}$, $\mathbf{A}\in\mathcal{M}^{\mathbb{N}}(\xi_0)$ be such that there exists a $p\geq 1$ satisfying*

(i) $\mathbf{A}(t+p)=\mathbf{A}(t)$ for all $t\in\mathbb{N}$,

(ii) $\xi(t+p;\xi_0,\mathbf{A}) = \xi(t;\xi_0,\mathbf{A})$ for all $t\in\mathbb{N}$.

Then the following statements hold:

(i) $\lambda(\xi_0,\mathbf{A}) = \frac{1}{p}\sum_{t=0}^{p-1} q(\xi(t;\xi_0,\mathbf{A}),\mathbf{A}(t))$.

(ii) $\max_{0\leq t\leq p-1}(1-e^{-\delta})J_\delta(\xi(t;\xi_0,\mathbf{A}),\mathbf{A}(t+\cdot)) \geq \lambda(\xi_0,\mathbf{A})$.

Proof: To abbreviate notation let $f(t) := q(\xi(t;\xi_0,\mathbf{A}),\mathbf{A}(t))$. It holds that $f(t+p)=f(t)$ for all $t\in\mathbb{N}$. Assertion (i) follows as the following limits exist

$$\lim_{t\to\infty}\frac{1}{t}\sum_{s=0}^{t-1} f(s) = \lim_{k\to\infty}\frac{1}{kp}\sum_{s=0}^{kp-1} f(s) = \frac{1}{p}\sum_{s=0}^{p-1} f(s).$$

To show (ii) let $\delta>0$ be arbitrary. Note that for $0\leq t\leq p-1$ it holds by periodicity that

$$\begin{aligned} &J_\delta(\xi(t;\xi_0,\mathbf{A}),\mathbf{A}(t+\cdot)) \\ &= \frac{1}{1-e^{-\delta p}}f(t) + \frac{e^{-\delta}}{1-e^{-\delta p}}f(t+1) + \ldots + \frac{e^{-\delta(p-1)}}{1-e^{-\delta p}}f(t+p-1). \end{aligned}$$

Summing up these equalities for $0\leq t\leq p-1$ we obtain

$$\sum_{t=0}^{p-1} J_\delta(\xi(t;\xi_0,\mathbf{A}),\mathbf{A}(t+\cdot)) = \sum_{t=0}^{p-1}\sum_{k=0}^{p-1}\frac{e^{-\delta k}}{1-e^{-\delta p}}f(t+k) = \frac{1}{1-e^{-\delta}}\sum_{t=0}^{p-1} f(t)\,, \tag{19}$$

Dividing by p we obtain that the average of the discounted values exceeds $(1-e^{-\delta})^{-1}\lambda(\xi_0,\mathbf{A})$, which implies (ii). □

The preceding proposition is particularly useful, when considering trajectories evolving in eigenspaces given by a periodic sequence $\mathbf{A}$. For an eigenvalue μ of $S_t \in \mathcal{S}_t$ let $E(\mu)$ denote the corresponding eigenspace, or if $\mathbb{K} = \mathbb{R}$ and $\mu \notin \mathbb{R}$, let $E(\mu)$ denote the real part of the sum of the eigenspaces corresponding to $\mu, \bar{\mu}$.

Corollary 4.2 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $S_t \in \mathcal{S}_t$ and $\xi_0 \in E(\mu), \|\xi_0\| = 1$ for some $\mu \in \sigma(S_t)$. Let $\mathbf{A} \in \mathcal{M}^{\mathbb{N}}$ be a t-periodic sequence satisfying $S_t = \mathbf{A}(t-1)\dots\mathbf{A}(0)$. Then for all $\delta > 0$*

$$\max_{0 \le s \le t-1} (1 - e^{-\delta}) J_\delta(\xi(s;\xi_0,\mathbf{A}), \mathbf{A}(s+\cdot)) \ge \lambda(\xi_0, \mathbf{A}) = \frac{1}{t} \log|\mu|\,.$$

Proof: If $\mu = 0$ all quantities involved are $-\infty$ and there is nothing to show. If $0 \ne \mu \in \mathbb{K}$ then $\mu\xi_0 = S_t\xi_0$ so ξ_0 is a fixed point under the projected system and we can directly apply Proposition 4.1. The same argumentation is valid if $\mathbb{K} = \mathbb{R}$ and $\mu = re^{i\omega}$ with $\frac{\omega}{\pi} \in \mathbb{Q}$ as then $\mu^p\xi_0 = |\mu|^p\xi_0 = S_t^p\xi_0$ for some $p \ge 1$ and again Proposition 4.1 may be applied. To consider the remaining case that $\mu = re^{i\omega}$ with $\frac{\omega}{\pi} \notin \mathbb{Q}$ note that in any neighborhood of the matrix S_t there exist matrices B such that $E(\mu)$ is still an eigenspace to a complex pair of eigenvalues $\mu', \overline{\mu'}$ arbitrarily close to $\mu, \overline{\mu}$, but such that $\mu' = re^{i\omega'}$ where ω' is a rational multiple of π. (This may be done by transforming S_t into (complex) Jordan normal form and changing the diagonal of the Jordan blocks corresponding to μ to μ' and transforming back. Note that we do not claim that $B \in \mathcal{S}_t$, in general this will be false. Here we just make use of the fact that the definition of J_δ makes sense for such B.) Then B may be represented as $B = B(t-1)\dots B(0)$ where for each s $B(s)$ is arbitrarily close to $\mathbf{A}(s)$. The assertion now follows from the fact that for such matrices J_δ and λ depend continuously on the periodic continuation of S_t, resp. B and for B the assertion holds by an application of Proposition 4.1. □

We are now in a position to prove the first main result on the convergence of the value functions.

Theorem 4.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\mathcal{M} \subset \mathbb{K}^{n\times n}$ be bounded.*

(i) If $\mathcal{M}$ is irreducible then there exists a constant $M > 0$ such that for all $\delta > 0$

$$|\max_{\xi \in \mathbb{S}^{n-1}_{\mathbb{K}}} (1 - e^{-\delta}) V_\delta(\xi) - \rho(\mathcal{M})| \le M(1 - e^{-\delta}) < M\delta\,.$$

(ii) If $\mathcal{M}$ is reducible then there exists a constant $M > 0$ such that

$$|\max_{\xi \in \mathbb{S}^{n-1}_{\mathbb{K}}} (1 - e^{-\delta}) V_\delta(\xi) - \rho(\mathcal{M})| \le M(1 - e^{-\delta})(1 - \log(1 - e^{-\delta}))\,.$$

(iii)

$$\lim_{\delta \to 0} \max_{\xi \in \mathbb{S}^{n-1}_{\mathbb{K}}} (1 - e^{-\delta}) V_\delta(\xi) = \inf_{\delta > 0} \max_{\xi \in \mathbb{S}^{n-1}_{\mathbb{K}}} (1 - e^{-\delta}) V_\delta(\xi) = \rho(\mathcal{M})\,.$$

Proof: We need the following preliminary remarks. Replacing $\mathcal{M}$ by $e^{(-\rho(\mathcal{M}))}\mathcal{M}$ we obtain a new discrete inclusion with $\rho(e^{(-\rho(\mathcal{M}))}\mathcal{M}) = 0$. Note that the dynamics of the projected system (11) remain unchanged. Denote the discounted value function given by the new discrete inclusion by $\tilde{V}_\delta$. An easy calculation shows that

$$\left|(1-e^{-\delta})V_\delta(\xi) - \rho(\mathcal{M})\right| = \left|(1-e^{-\delta})\tilde{V}_\delta(\xi)\right|$$

and thus it is sufficient to prove (i) and (ii) for $\rho(\mathcal{M}) = 0$, which we assume from now on.

For all $t \in \mathbb{N}$ we have that $\overline{\rho}_t(\mathcal{M}) \leq \rho(\mathcal{M}) = 0$ by (6). Using Corollary 4.2 we obtain that for all $t \geq 1$

$$\max_{\xi \in \mathbb{S}^{n-1}_{\mathbb{K}}} (1-e^{-\delta})V_\delta(\xi) \geq \overline{\rho}_t(\mathcal{M}),$$

and thus $\max_{\xi \in \mathbb{S}^{n-1}_{\mathbb{K}}} (1-e^{-\delta})V_\delta(\xi) \geq \sup_{t \in \mathbb{N}} \overline{\rho}_t(\mathcal{M}) = 0$ for all $\delta > 0$. It remains to find upper bounds for the maximum of the discounted value functions.

As an intermediate step we claim that for all finite trajectories $\xi(s, \xi_0, \mathbf{A})$, $s = 0, \ldots, t$ it holds that

$$\sum_{s=0}^{t} e^{-\delta s} q(\xi(s, \xi_0, \mathbf{A}), \mathbf{A}(s)) \leq$$

$$\hat{\rho}_1(\mathcal{M}) + e^{-\delta}(2\hat{\rho}_2(\mathcal{M}) - \hat{\rho}_1(\mathcal{M})) + \ldots + e^{-\delta t}((t+1)\hat{\rho}_{t+1}(\mathcal{M}) - t\hat{\rho}_t(\mathcal{M})).$$

This may be seen by induction. For $t = 0$ the assertion is clear by definition of $\hat{\rho}_1(\mathcal{M})$. Assume the statement is shown for $t-1$ and consider a sequence $q(\xi(s, \xi_0, \mathbf{A}), \mathbf{A}(s))$, $s = 0, \ldots, t$, where $t \geq 1$. By definition of $\hat{\rho}_t(\mathcal{M})$ it holds that

$$\sum_{s=0}^{t-1} q(\xi(s, \xi_0, \mathbf{A}), \mathbf{A}(s)) \leq t\hat{\rho}_t(\mathcal{M}).$$

In order to maximize the expression

$$\sum_{s=0}^{t} e^{-\delta s} q(\xi(s, \xi_0, \mathbf{A}), \mathbf{A}(s)) \tag{20}$$

we may assume that

$$\sum_{s=0}^{t} q(\xi(s, \xi_0, \mathbf{A}), \mathbf{A}(s)) = (t+1)\hat{\rho}_{t+1}(\mathcal{M}).$$

As the factor of $q(\xi(t, \xi_0, \mathbf{A}), \mathbf{A}(t))$ in (20) is the smallest it is optimal to choose the first t elements of the sequence such that their sum is as large as possible i.e. equal to $t\hat{\rho}_t(\mathcal{M})$. This choice implies that

$$q(\xi(t, \xi_0, \mathbf{A}), \mathbf{A}(t)) \leq (t+1)\hat{\rho}_{t+1}(\mathcal{M}) - t\hat{\rho}_t(\mathcal{M}).$$

Together with the induction assumption this shows our claim. Setting $\hat{\rho}_0(\mathcal{M})$ equal to some constant, it follows that

$$\max_{\xi \in \mathbb{S}^{n-1}_{\mathbb{K}}} V_\delta(\xi) \leq \sum_{s=0}^{\infty} e^{-\delta s}((s+1)\hat{\rho}_{s+1}(\mathcal{M}) - s\hat{\rho}_s(\mathcal{M})) = (1-e^{-\delta}) \sum_{s=0}^{\infty} e^{-\delta s}(s+1)\hat{\rho}_{s+1}(\mathcal{M}). \tag{21}$$

(i) Multiplying by $(1-e^{-\delta})$ and using Lemma 2.3 (i) there exists an $M_1 > 0$ such that the right hand side of (21) is bounded as follows

$$\max_{\xi \in \mathbb{S}_{\mathbb{K}}^{n-1}} (1-e^{-\delta}) V_\delta(\xi) \leq M_1 (1-e^{-\delta})^2 \sum_{s=0}^{\infty} e^{-\delta s}(s+1)(s+1)^{-1} = M_1(1-e^{-\delta}) \,.$$

The remaining inequality is clear.
(ii) Using the assumption and Lemma 2.3 (ii) we obtain from (21) that

$$\max_{\xi \in \mathbb{S}_{\mathbb{K}}^{n-1}} (1-e^{-\delta}) V_\delta(\xi) \leq M_1 (1-e^{-\delta})^2 \sum_{s=0}^{\infty} e^{-\delta s}(s+1)(s+1)^{-1}(1+\log(s+1))$$

and it remains to obtain a bound for

$$(1-e^{-\delta})^2 \sum_{s=0}^{\infty} e^{-\delta s} \log(s+1) = (1-e^{-\delta}) \sum_{s=1}^{\infty} e^{-\delta s} (\log(s+1) - \log(s)) \,.$$

Using the mean value theorem this can be bounded by

$$\leq (1-e^{-\delta}) \sum_{s=1}^{\infty} \frac{e^{-\delta s}}{s} = (1-e^{-\delta}) \log\left(\frac{1}{1-e^{-\delta}} \right) \,. \tag{22}$$

This completes the proof of (i) and (ii). (iii) is an immediate consequence of these statements and Corollary 4.2 (ii). □

Remark 4.4 (i) Note that in both cases of the preceding proof the speed of convergence is determined by the factor M_1 which is intrinsically given by the problem as the difference between the norm originally used and the norm given as the invariant norm of the discrete inclusion, or in the reducible case, by the number of irreducible blocks and their respective constants relating the original norm and the Barabanov norm v.

(ii) The convergence results of the previous theorem are formulated with respect to the factor $(1-e^{-\delta})$ as this simplifies some arguments. It is however easy to derive statements on the convergence of δV_δ (which is the case usually considered for continuous time systems) as it can be easily seen that the convergence $\delta/(1-e^{-\delta}) \to 1$ is linear. □

Let us also note the consequences of the previous theorem for the approximate calculation of time-varying stability radii. By definition the time-varying stability radius is the infimum of the set $\{\gamma \; ; \; \rho(\mathcal{M}_\gamma) \geq 0\}$. Let us assume that the sets $\mathcal{M}_\gamma$ are compact and the map $\gamma \mapsto \mathcal{M}_\gamma$ is continuous with respect to the Hausdorff topology. Then the map

$$g : \gamma \mapsto \rho(\mathcal{M}_\gamma)$$

is also continuous (see [1]), $g(r_{tv}(A_0, \mathcal{U})) = 0$ and g is clearly monotone by Assumption 2.4 (ii). Thus it follows that

$$c(\mathcal{U}) := \sup \left\{ c \in \mathbb{R} \; ; \; \limsup_{h \downarrow 0} \frac{g(r_{tv}(A_0, \mathcal{U}) - h)}{h} \leq -c \right\} \geq 0 \,.$$

The number $c(\mathcal{U})$ may be interpreted as the supremum of the gradients of those linear functions that have their zero in $r_{tv}(A_0, \mathcal{U})$ and are larger than g on some interval of the form $[a, r_{tv}(A_0, \mathcal{U})]$, where $a < r_{tv}(A_0, \mathcal{U})$. Let us note in passing that this constant has relations to the superdifferential of g in $r_{tv}(A_0, \mathcal{U})$.

Theorem 4.5 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $A_0 \in \mathbb{K}^{n\times n}$. Let $\mathcal{U}$ satisfy Assumption 2.4, then the following properties hold.*

(i) For all $\delta > 0$ it holds that

$$r_{tv}(A_0, \mathcal{U}) \geq r_\delta(A_0, \mathcal{U}) := \inf_{\gamma>0}\{\gamma;\ \max_{\xi\in\mathbb{S}_{\mathbb{K}}^{n-1}} (1-e^{-\delta})V_{\delta,\gamma}(\xi) \geq 0\}\,. \tag{23}$$

(ii) $$r_{tv}(A_0, \mathcal{U}) = \lim_{\delta\to 0} r_\delta(A_0, \mathcal{U}).$$

(iii) If $c(\mathcal{U}) > 0$ then there exist $\bar{\delta} > 0$ and a constant $M > 0$ such that for all $0 < \delta < \bar{\delta}$

$$r_{tv}(A_0, \mathcal{U}) - r_\delta(A_0, \mathcal{U}) \leq M(1-e^{-\delta})(1-\log(1-e^{-\delta}))\,.$$

If, furthermore, $\mathcal{M}_\gamma$ is irreducible for all $\gamma > 0$ then M may be chosen such that for all $0 < \delta < \bar{\delta}$

$$r_{tv}(A_0, \mathcal{U}) - r_\delta(A_0, \mathcal{U}) \leq M(1-e^{-\delta})\,.$$

Proof:

(i) If $\gamma > r_{tv}(A_0, \mathcal{U})$, then $0 \leq \rho(\mathcal{M}_\gamma) \leq \max_{\xi\in\mathbb{S}_{\mathbb{K}}^{n-1}}(1-e^{-\delta})V_{\delta,\gamma}(\xi)$ by Theorem 4.3 (iii). Thus $\gamma \geq r_\delta(A_0, \mathcal{U})$.

(ii) If $\gamma < r_{tv}(A_0, \mathcal{U})$, then $\rho(\mathcal{M}_\gamma) < 0$ and by Theorem 4.3 (iii) there exists a δ_γ such that for all $0 < \delta < \delta_\gamma$ it holds that $\max_{\xi\in\mathbb{S}_{\mathbb{K}}^{n-1}}(1-e^{-\delta})V_{\delta,\gamma}(\xi) < 0$, and therefore for $0 < \delta < \delta_\gamma$ it follows $\gamma \leq r_\delta(A_0, \mathcal{U}) \leq r_{tv}(A_0, \mathcal{U})$. Letting γ tend to $r_{tv}(A_0, \mathcal{U})$ from below shows the assertion.

(iii) To abbreviate notation let $a(\delta) := (1-e^{-\delta})(1-\log(1-e^{-\delta}))$. Choose $\varepsilon > 0$ small enough such that $c(\mathcal{U}) - \varepsilon > 0$. Then there exists an $\eta > 0$ such that

$$g(\gamma) < (c(\mathcal{U}) - \varepsilon)(\gamma - r_{tv}(A_0, \mathcal{U})) \text{ for all } \gamma \in [r_{tv}(A_0, \mathcal{U}) - \eta, r_{tv}(A_0, \mathcal{U})].$$

By Theorem 4.3 (ii) for every $\gamma \in [r_{tv}(A_0, \mathcal{U}) - \eta, r_{tv}(A_0, \mathcal{U})]$ there exists an $M_\gamma > 0$ such that

$$\max_{\xi\in\mathbb{S}_{\mathbb{K}}^{n-1}}(1-e^{-\delta})V_{\delta,\gamma}(\xi) \leq (c(\mathcal{U}) - \varepsilon)(\gamma - r_{tv}(A_0, \mathcal{U})) + M_\gamma a(\delta)\,.$$

As the constant M given by the proof of Lemma 2.3 (ii) depends on the number d (which is bounded by n) and the norms v_i (which also determine α) it follows that $M := \sup\{M_\gamma\ ;\ \gamma \in [r_{tv}(A_0, \mathcal{U}) - \eta, r_{tv}(A_0, \mathcal{U})]\}$ exists. Denote the zero of the right hand side in the above equation by

$$\tilde{r}_\delta := r_{tv}(A_0, \mathcal{U}) - \frac{M}{c(\mathcal{U}) - \varepsilon}a(\delta) \leq r_\delta(A_0, \mathcal{U})\,.$$

Then for all $0 < \delta < \delta'$ small enough so that $Ma(\delta)(c(\mathcal{U}) - \varepsilon)^{-1} < \eta$ we obtain

$$r_{tv}(A_0, \mathcal{U}) - r_\delta(A_0, \mathcal{U}) \leq r_{tv}(A_0, \mathcal{U}) - \tilde{r}_\delta = \frac{M}{c(\mathcal{U}) - \varepsilon}a(\delta)\,.$$

The claim for the irreducible case follows the same way by replacing $a(\delta)$ by δ and using Theorem 4.3 (i).

□

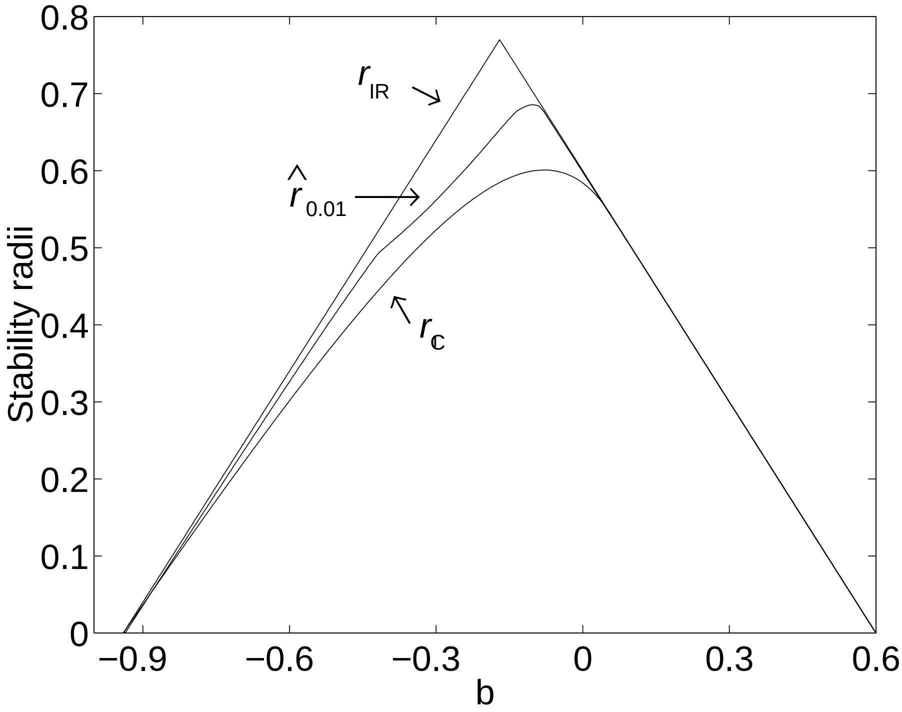

Figure 1: Comparison of the real structured, the complex structured and the time-varying stability radius

5 An Example

At the heart of the calculation of time-varying stability radii is the algorithm for the solution of the discrete Hamilton-Jacobi-Bellman equation, as described in [11], [12] and [14]. For this a discount rate δ, a discretization of $\mathbb{S}_{\mathbb{K}}^{n-1}$, and a discretization of the set of admissible control values U_ρ have to be chosen. A bound on the discretization error due to the discretization of the state space and the perturbation space has been obtained in Theorems 2.3 and 2.9 of [15] and we do not pursue this question here. Using these existing algorithms $\max_{\xi \in \mathbb{S}_{\mathbb{K}}^{n-1}}(1 - e^{-\delta})V_\delta(\xi)$ may be calculated and used as an approximation of the maximal Lyapunov exponent. A bisection algorithm may then be applied to obtain $r_\delta(A, \mathcal{U})$ as an approximation of the stability radius $r_{tv}(A, \mathcal{U})$. To make a clear distinction between the value $r_\delta(A, \mathcal{U})$, which is theoretically defined and the values that are the result of the numerical algorithm, we denote the latter by $\hat{r}_\delta(A, \mathcal{U})$.

Example 5.1 To consider a three-dimensional example let

$$A(u) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{10} & b & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[0\ 1\ 0] \tag{24}$$

$$=: A_0(b) + DuE\,. \tag{25}$$

and $U_1 := \{u \in \mathbb{R}, |u| \leq 1\}$. It is easy to see that for all $\gamma > 0$ and all $b \in \mathbb{R}$ the set $\mathcal{M}_\gamma(b) = A_0(b) + \gamma U_1$ is irreducible. With these data we obtain an uncertainty model of feedback type and the results presented in [20] may be compared with the results on the time-varying stability radius. The real stability radius may be obtained by the following simple calculation. The characteristic equation of $A_0(b)$ is given by

$$P_b(\lambda) = \lambda^3 + \frac{1}{2}\lambda^2 - b\lambda - \frac{1}{10} = 0\,, \tag{26}$$

and it is straightforward to see that $A_0(b)$ is stable iff $-0.94 < b < 0.6$ and for these values of b the real time-invariant stability radius satisfies

$$r_{\mathbb{R}}(A_0(b); D, E) = \min\{|-0.94 - b|, |0.6 - b|\}\,. \tag{27}$$

The complex stability radius $r_{\mathbb{C}}(A_0(b); D, E)$ can be calculated using the `MATLAB` routine Stabrad-Bruinsma, by L. Schwiedernoch. From [21] it follows that

$$r_{\mathbb{R}}(A_0(b); D, E) \geq r_{tv}(A_0(b); D, E) \geq r_{\mathbb{C}}(A_0(b); D, E)\,. \tag{28}$$

We calculated an approximation of the Lyapunov-stability radius with discount rate $\delta = 0.01$, a discretization of 100 sample points for the control value interval $U_\rho = [-\rho, \rho]$ and a grid on $\mathbb{S}^2_{\mathbb{R}}$ consisting of 900 vertices. The three different stability radii are shown in Figure 1.

The value $b = -0.165$ has been chosen to study the convergence of r_δ. To this end $\hat{r}_\delta(A_0(-0.165), \mathcal{U})$ was calculated for δ from 10^{-4} to $2 \cdot 10^{-2}$. The result is shown in Figure 2, where the discount rate is displayed on a logarithmic scale to the base 10. The rate of convergence is of order 1 in δ as predicted by the theory. In fact, using the MATLAB polyfit function we obtain that

$$r_\delta(A_0(-0.165); D, E) \approx -0.6212\,\delta + 0.6629\,,$$

where the least squares error is $6.7174 \cdot 10^{-04}$. The difference between the linear and the second order fit in $\delta = 0$ is $1.7686 \cdot 10^{-05}$ showing that a linear fit is adequate.

□

6 Conclusion

We have presented an approach to the calculation of time-varying stability radii that provides for linear convergence in the discount rate if the corresponding discrete inclusion is irreducible and a further condition on the growth of the maximal Lyapunov exponent with respect to the perturbation intensity γ at the stability radius is satisfied.

The results indicate that an algorithm of the following kind may be feasible in the irreducible case:
1. Calculate $r_{\delta_i}, i = 1, \dots, k$ for $\delta_1 > \dots > \delta_k > 0$.
2. Perform a linear fit through the data points.
3a. If the least squares error is below some precision constant then stop and use the zero of the linear fit as an estimation for r_{tv}.
3b. Otherwise increase k and calculate $r_{\delta_{k+1}}$ for $\delta_k > \delta_{k+1} > 0$. Continue with 2.

A topic of further research is to obtain error estimates for algorithms of this type.

Acknowledgment The author would like to thank Lars Grüne for his help in using his program for solving discrete Hamilton-Jacobi-Bellman equations and Andrew Paice for helpful discussions on several points of the paper.

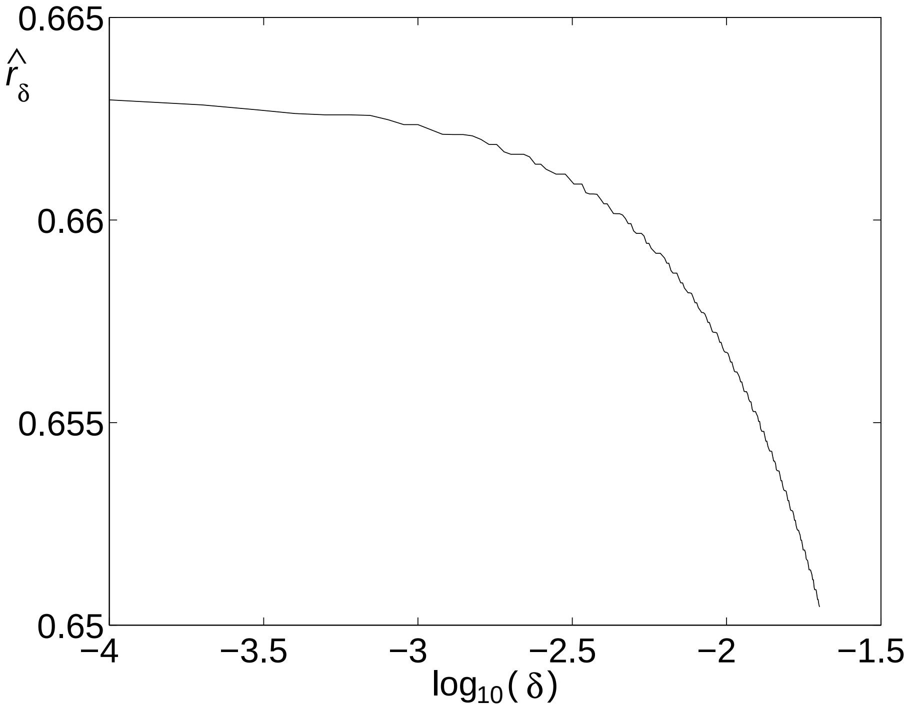

Figure 2: $\hat{r}_\delta$ for small discount rates, $b = -0.165$.

References

[1] N. E. Barabanov. Lyapunov indicator of discrete inclusions. I. *Autom. Remote Control*, 49(2):152–157, 1988.

[2] N. E. Barabanov. Lyapunov indicator of discrete inclusions. II. *Autom. Remote Control*, 49(3):283–287, 1988.

[3] N. E. Barabanov. Lyapunov indicator of discrete inclusions. III. *Autom. Remote Control*, 49(5):558–565, 1988.

[4] N. E. Barabanov. Method for the computation of the Lyapunov exponent of a differential inclusion. *Autom. Remote Control*, 50(4):475–479, 1989.

[5] M. A. Berger and Y. Wang. Bounded semigroups of matrices. *Lin. Alg. Appl.*, 166:21–27, 1992.

[6] D. Bertsekas. *Dynamic Programming Deterministic and Stochastic Models.* Prentice Hall, Englewood Cliffs, New Jersey, 1987.

[7] F. Colonius and W. Kliemann. Stability radii and Lyapunov exponents. In D. Hinrichsen and B. Mårtensson, editors, *Proc. Workshop Control of Uncertain Systems, Bremen 1989*, volume 6 of *Progress in System and Control Theory*, pages 19–55, Basel, 1990. Birkhäuser.

[8] F. Colonius and W. Kliemann. Maximal and minimal Lyapunov exponents of bilinear control systems. *J. Differ. Equations*, 101:232–275, 1993.

[9] F. Colonius and W. Kliemann. The Lyapunov spectrum of families of time varying matrices. *Trans. Am. Math. Soc.*, 348:4389–4408, 1996.

[10] J. L. Daleckii and M. G. Krein. *Stability of Solutions of Differential Equations in Banach Spaces*, volume 43 of *Translations of Mathematical Monographs*. A.M.S, Providence R.I., 1974.

[11] I. C. Dolcetta and H. Ishii. Approximate solutions of the Bellman equation of deterministic control theory. *Applied Mathematics and Optimization*, 11:161–181, 1984.

[12] M. Falcone. A numerical approach to the infinite horizon problem of deterministic control theory. *Applied Mathematics and Optimization*, 15:1–13, 1987. Corrigenda, ibid., 23:213-214, 1991.

[13] G. Gripenberg. Computing the joint spectral radius. *Linear Algebra and its Applications*, 234:43–60, 1996.

[14] L. Grüne. Numerical stabilization of bilinear control systems. *SIAM J. Contr. & Opt.*, 34(6):2024–2050, 1996.

[15] L. Grüne. An adaptive grid scheme for the discrete Hamilton-Jacobi-Bellman equation. *Numer. Math.*, 75:319–337, 1997.

[16] L. Gurvits. Stability of discrete linear inclusions. *Lin. Alg. Appl.*, 231:47–85, 1995.

[17] D. Hinrichsen, A. Ilchmann, and A. J. Pritchard. Robustness of stability of time-varying linear systems. *J. Differ. Equations*, 82(2):219–250, 1989.

[18] D. Hinrichsen and A. J. Pritchard. Stability radii of linear systems. *Syst. Contr. Let.*, 7:1–10, 1986.

[19] D. Hinrichsen and A. J. Pritchard. Stability radius for structured perturbations and the algebraic Riccati equation. *Syst. Contr. Let.*, 8:105–113, 1986.

[20] D. Hinrichsen and A. J. Pritchard. Real and complex stability radii: a survey. In D. Hinrichsen and B. Mårtensson, editors, *Control of Uncertain Systems*, volume 6 of *Progress in System and Control Theory*, pages 119–162, Basel, 1990. Birkhäuser.

[21] D. Hinrichsen and A. J. Pritchard. Destabilization by output feedback. *Differ. & Integr. Equations*, 5:357–386, 1992.

[22] D. Hinrichsen and N. K. Son. Stability radii of positive discrete-time systems. Report 329, Institut für Dynamische Systeme, Universität Bremen, 1994.

[23] J. C. Lagarias and Y. Wang. The finiteness conjecture for the generalized spectral radius of a set of matrices. *Lin. Alg. Appl.*, 214:17–42, 1995.

[24] M. Maesumi. An efficent lower bound for the generalized spectral radius of a set of matrices. *Linear Algebra and its Applications*, (240):1–7, 1996.

[25] K. M. Przyłuski. Remarks on the stability of linear infinite-dimensional discrete-time systems. *J. Differ. Equations*, 72(2):189–200, 1988.

[26] L. Qiu, B. Bernhardsson, A. Rantzer, E. J. Davison, P. M. Young, and J. C. Doyle. A formula for computation of the real stability radius. *Automatica*, 31:879–890, 1995.

[27] J. Sreedhar, P. Van Dooren, and A. L. Tits. A level-set idea to compute the real Hurwitz-stability radius. In *Proceedings of the 34th Conference on Decision & Control New Orleans, LA*, pages 126–127, December 1995.

[28] F. Wirth. Convergence of the value functions of discounted infinite horizon optimal control problems with low discount rates. *Mathematics of Operations Research*, 18(4):1006–1019, 1993.

[29] F. Wirth. Discrete time discounted optimal control with low discount rates. 1997. submitted.

[30] F. Wirth. Dynamics of time-varying discrete-time linear systems: Spectral theory and the projected system. *SIAM J. Contr. & Opt.*, 36(2), 1998. to appear.

[31] F. Wirth and D. Hinrichsen. On stability radii of infinite dimensional time-varying discrete-time systems. *IMA J. Math. Control & Information*, 11:253–276, 1994.