Calculating the domain of attraction: Zubov's method and extensions

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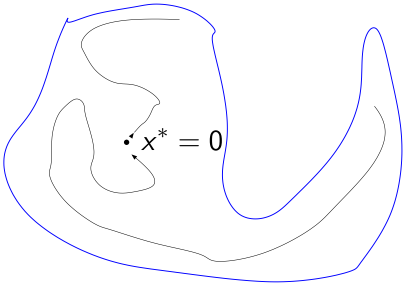
Overview

The domain of attraction

Consider a nonlinear system

$$\begin{aligned} \dot{x}(t) &= f(x(t)), \quad t \in \mathbb{R} \quad (1) \\ x(0) &= x_0 \in \mathbb{R}^n, \end{aligned}$$



f Lipschitz continuous, $f(0) = 0$.
Assume $x^* = 0$ is asymptotically stable.

The domain of attraction of 0 is defined by

$$\mathcal{A}(0) := \{x \in \mathbb{R}^n \mid \varphi(t;x) \to 0, \text{ as } t \to \infty\} .$$

Here $\varphi(\cdot; x)$ denotes the solution of (1).

The domain of attraction: Properties

$\mathcal{A}(0)$ has the following properties:

- $\mathcal{A}(0)$ is diffeomorphic to $\mathbb{R}^n$,
- Conversely, any set, that is diffeomorphic to $\mathbb{R}^n$, is the domain of attraction of some asymptotically stable fixed point,
- dynamic properties:
  $\mathcal{A}(0)$ and $\mathrm{cl}\,\mathcal{A}(0)$ are invariant under the flow,

There is little general information available on the domain of attraction. On the other hand, in applications one is often interested in knowing, what it looks like.

Question:

How to compute (approximations of) the domain of attraction?

Zubov's result

V.I. Zubov:
Methods of A.M. Lyapunov and their application
Nordhoff, Groningen, The Netherlands, 1964
(Russian edition in 1957)

Zubov's result

$$\dot{x} = f(x), \quad t \in \mathbb{R} \tag{1}$$

Theorem

A set A containing 0 in its interior is the domain of attraction of (1) if and only if there exist continuous functions V, h such that

- $V(0) = h(0) = 0$,
  $0 < V(x) < 1$ *for* $x \in A \setminus \{0\}$, $h > 0$ *on* $\mathbb{R}^n \setminus \{0\}$
- *for every* $\gamma_2 > 0$ *there exist* $\gamma_1 > 0, \alpha_1 > 0$ *such that*
  $$V(x) > \gamma_1, \quad h(x) > \alpha_1, \quad \text{if } \|x\| \geq \gamma_2,$$
- $V(x_n) \to 1$ *for* $x_n \to \partial A$ *or* $\|x_n\| \to \infty$,
- $$\boxed{DV(x) \cdot f(x) = -h(x)(1 - V(x))\sqrt{1 + \|f(x)\|^2}}$$

Zubov's result II

By proper choice of h the function V can be made as smooth as f_0.
In particular:

Corollary

If f is continuously differentiable, then

$$DV(x) \cdot f(x) = -h(x)(1 - V(x))\sqrt{1 + \|f(x)\|^2}\,.$$

has at most one continuously differentiable solution satisfying $V(0) = 0$.
In order for the solution to exist it is sufficient that

$$\int_0^\infty h(\varphi(t, x_0))dt < \infty\,.$$

for all x_0 sufficiently small.

Zubov's result III: Series approximation

$$DV(x) \cdot f(x) = -h(x)(1 - V(x))\sqrt{1 + \|f(x)\|^2} \quad (Z)\,.$$

If f is real-analytic, then h, V may also be chosen real analytic. Series approximations of V may be calculated by expanding:

$$\begin{aligned} V(x) &= 0 + \ 0 \ + x^T Px + \mathcal{O}(3) \\ h(x) &= 0 + \ 0 \ + x^T Qx + \mathcal{O}(3) \\ f(x) &= 0 + Ax + \mathcal{O}(2) \end{aligned}$$

Then the second order terms in (Z) yield

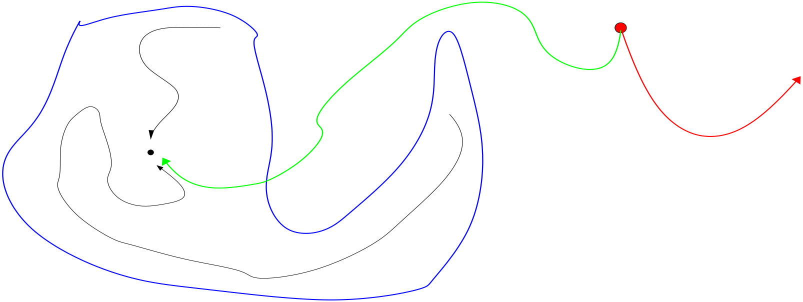
$$x^T PAx = -x^T Qx \quad \Leftrightarrow \quad A^T P + PA = -\frac{1}{2} Q\,.$$

Robust domains of attraction

Consider systems

$$\dot{x}(t) = f(x(t), d(t)), \quad t \in \mathbb{R}$$

with a perturbation term d. We are interested in robust stability properties. In particular, the robust domain of attraction.



Robust domains of attraction

$$\dot{x}(t) = f(x(t), d(t)), \quad t \in \mathbb{R}$$

Assumptions:

- $f : \mathbb{R}^n \times D \to \mathbb{R}^n$ continuous, locally Lipschitz continuous in x, uniformly in d
- $D \subset \mathbb{R}^m$ compact, convex, $d(t) \in D$ a.e.
- $f(0, d) = 0$ for all $d \in D$
- 0 is locally uniformly asymptotically stable

Notation: $\mathcal{D} := \{d : \mathbb{R} \to D \,;\, d \text{ Lebesgue measurable}\}$

Question:

Can the approach of Zubov be used to determine Lyapunov functions on the robust domain of attraction $\mathcal{A}_D(0)$?

A robust version of Zubov's theorem

Theorem:[Zubov's theorem for perturbed systems]
Under suitable conditions on g there is a unique viscosity solution of

$$\begin{cases} \inf_{d\in D} \{-Dv(x)f(x,d) - (1-v(x))g(x,d)\} = 0 \\ v(0) = 0 \end{cases}$$

The robust domain of attraction satisfies

$$\mathcal{A}_D(0) = v^{-1}([0,1))\,.$$

This is a straightforward generalization of Zubov's original equation

$$DV(x)\cdot f(x) = -h(x)(1-V(x))\sqrt{1+\|f(x)\|^2}\,.$$

for which we had

$$\mathcal{A} = V^{-1}([0,1))\,.$$

A robust version of Zubov's theorem: Outline of proof

- choose g continuous, positive definite, such that

$$\int_0^\infty g(\phi(t,x_0,d),d(t))dt < \infty$$

  if and only if $\phi(t,x_0,d) \to 0$.
  To be able to do this information on the local convergence rate near $x^* = 0$ is needed.
- define v via the optimal control problem

$$v(x) = \sup_{d\in\mathcal{D}} \left[1 - \exp\left(\int_0^\infty g(\phi(t,x_0,d))dt\right)\right] .$$

- Use relation between Bellman principle and viscosity solutions to show that v solves the robust Zubov equation.
- Prove uniqueness.

Example

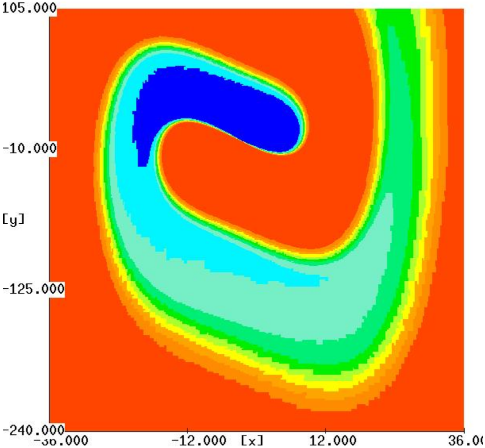
$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3
\end{aligned}
$$

Fixed points:
$[0, 0]$, unstable
$[-2.5505, -2.5505]$, asymptotically stable
$[-7.4495, -7.4495]$, asymptotically stable.

Example

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = 0.1x_1 - 2x_2 - x_1^2 - (0.1 + d(t))\, x_1^3$$

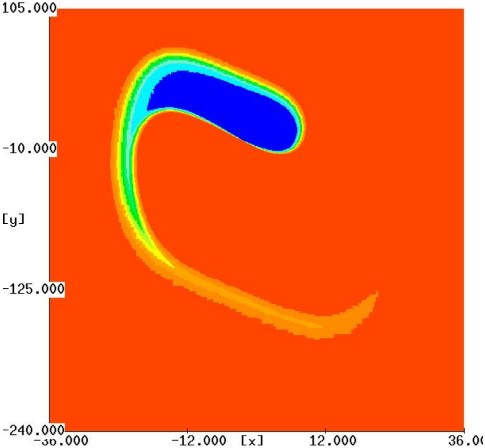
$$D = \{0\}$$



Example

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = 0.1x_1 - 2x_2 - x_1^2 - (0.1 + d(t))\, x_1^3$$

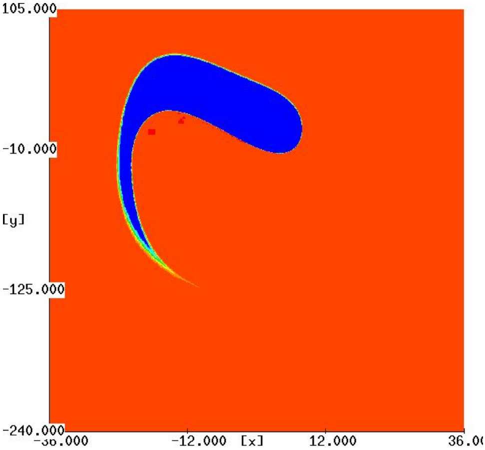
$$D = [-0.02, 0.02]$$



Example

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= 0.1x_1 - 2x_2 - x_1^2 \\
&\quad - (0.1 + d(t))\, x_1^3
\end{aligned}
$$

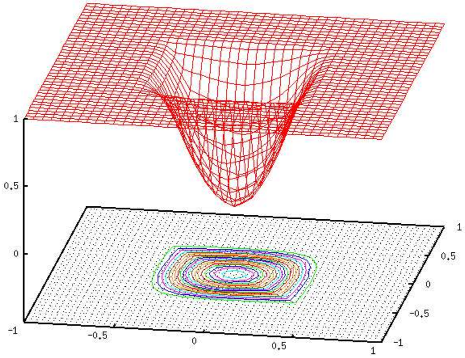
$$D = [-0.02, 0.02]$$



A toy example

$$\dot{x}_1 = -x_1 + d(t)x_1^2$$
$$\dot{x}_2 = -x_2 + d(t)x_2^2$$

$$D = [-2, 2]$$



A toy example

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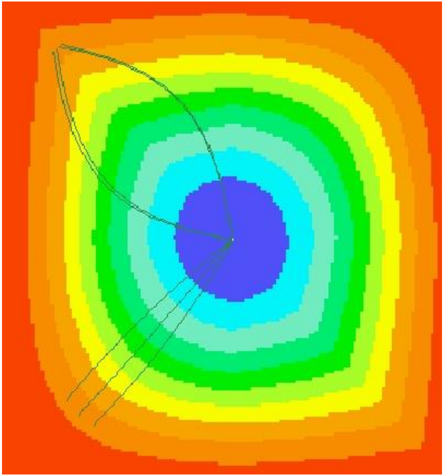


Another toy example

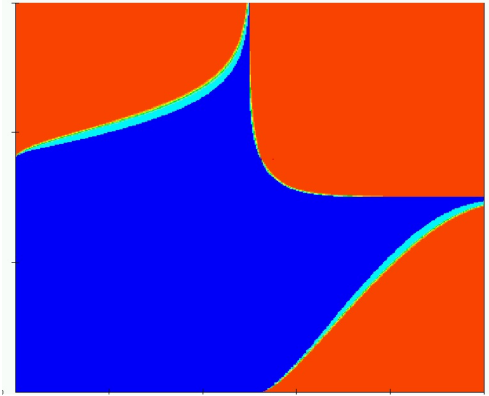
Khalil, Nonlinear Systems, 2nd ed., p. 190

$$\begin{aligned}\dot{x}(t) &= -2x(t) + x(t)y(t)\\ \dot{y}(t) &= -y(t) + (1 + d(t))x(t)y(t)\,.\end{aligned}$$

Another toy example

$$\dot{x} = -2x + xy$$
$$\dot{y} = -y + (1 + d(t))xy\,.$$

$$D = [-1, 1]$$



In reality, the whole lower half plane is contained in the robust domain of attraction. The discrepancy to the picture is caused by calculating on a finite domain.

Asymptotic nullcontrollability

Consider the control system

$$\dot{x} = f(x, u)\,,$$

where $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is sufficiently regular (to be made precise) and $U \subset \mathbb{R}^m$ is closed.
Assume that $f(0,0) = 0$.
We are now interested in asymptotic nullcontrollability.
The admissible controls are given by

$$\mathcal{U} := \{u : \mathbb{R}_+ \to U \mid \text{ measurable and essentially bounded } \}\,.$$

The domain of asymptotic nullcontrollability is defined by

$$\mathcal{A}_U(0) := \{x \in \mathbb{R}^n \,|\, \text{there exists } u \in \mathcal{U} \text{ with } \varphi(t,x,u) \to 0 \text{ for } t \to \infty\}\,.$$

Control Lyapunov functions

$$\dot{x} = f(x,u)\,, \quad f(0,0) = 0\,.$$

$$\mathcal{A}_U(0) := \{x \in \mathbb{R}^n \,|\, \text{there exists } u \in \mathcal{U} \text{ with } \varphi(t,x,u) \to 0 \text{ for } t \to \infty\}\,.$$

It is known that asymptotic nullcontrollability is closely tied to the existence of a control Lyapunov function:
Consider a continuous, positive definite[1], proper[2] function $v : \mathbb{R}^n \to \mathbb{R}_+$.
v is called a control Lyapunov function on $\mathcal{A}_U(0)$ if there is a positive definite function W such that v is a viscosity supersolution on $\mathcal{A}_U(0)$ of

$$\max_{u \in U, \|u\| \leq \kappa(\|x\|)} -Dv(x)f(x,u) \geq W(x)\,.$$

[1] $v(x) \geq 0$ and $v(x) = 0 \Leftrightarrow x = 0$
[2] preimages of compact sets are compact

A Zubov type theorem

Theorem:[Zubov's theorem for controlled systems]
For suitable g there exists a unique viscosity v solution of

$$\begin{cases} \sup_{u\in U}\{-Dv(x)f(x,u) - (1-v(x))g(x,u)\} = 0 \\ v(0) = 0 \end{cases}$$

In particular, v is a control Lyapunov function on

$$\mathcal{A}_U(0) = v^{-1}([0,1))\,.$$

Again this is a straightforward generalization of Zubov's original equation

$$Dv(x)\cdot f(x) = -h(x)(1-v(x))\sqrt{1+\|f(x)\|^2}\,.$$

Comments on proof:

Again proof relies on the analysis of an optimal control problem, namely

$$v(x) = \inf_{u \in \mathcal{U}} \left[1 - \exp \left(\int_0^\infty g(\phi(t, x_0, u)) \right) \right] .$$

The choice of g is a bit more complicated this time, because we have to deal with the unboundedness of u.
Still explicit conditions, that are easily checkable are available.

Example

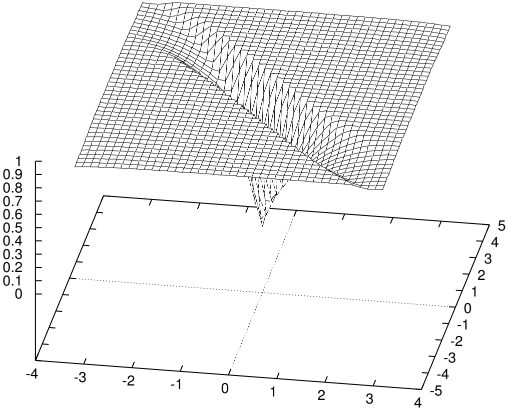
Consider the inverted pendulum

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin(x_1+\pi) - x_2 + u\end{aligned}$$

Example

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin(x_1+\pi) - x_2 + u$$

$$U = [-0.7, 0.7]$$



Example

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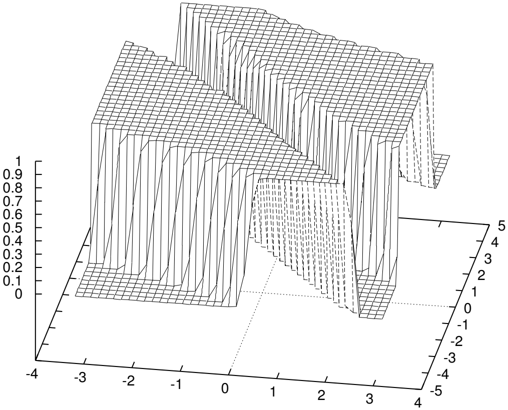
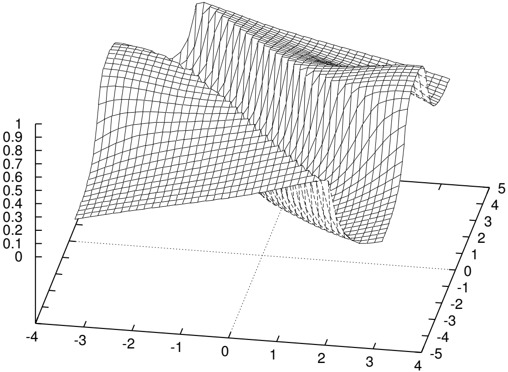


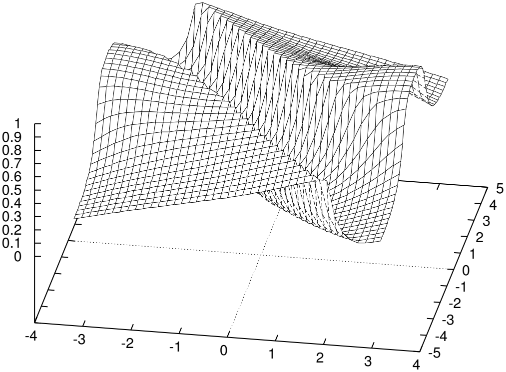
Example

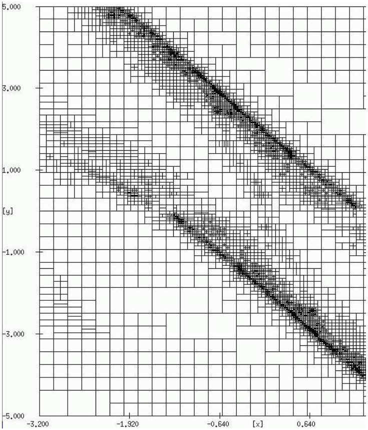
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin(x_1 + \pi) - x_2 + u$$

$$U = [-1, 1]$$



Example


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-4 -3 -2 -1 0 1 2 3 4
5 4 3 2 1 0 -1 -2 -3 -4 -5


5.000
3.000
1.000
[y]
-1.000
-3.000
-5.000
-3.200
-1.920
-0.640
[x]
0.640

Numerical treatment

Consider the perturbed case:
Zubov's PDE has a singularity at 0.

$$\begin{cases} \inf_{d \in D} \{-Dv(x)f(x,d) - (1 - v(x))g(x,d)\} = 0 \\ v(0) = 0 \end{cases}$$

To make the numerical treatment rigorous we consider the function

$$g_\varepsilon(x,d) = \max\{g(x,d), \varepsilon\}$$

for fixed $\varepsilon > 0$ and the approximate equation

$$\begin{cases} \inf_{d \in D} \{-Dv(x)f(x,d) - g(x,d) + v(x)g_\varepsilon(x,d)\} = 0 \\ v(0) = 0 \end{cases}$$

Numerical treatment II

Theorem Let v be the unique solution of Zubov's equation. Then for each $\varepsilon > 0$ the approximate equation has a unique continuous viscosity solution v_ε with the following properties.

(i) $v_\varepsilon(x) \leq v(x)\,, \quad \forall x \in \mathbb{R}^n$

(ii) $v_\varepsilon \to v$ uniformly in $\mathbb{R}^n$ as $\varepsilon \to 0$

(iii) If $\varepsilon < g_0$ then we have the characterization

$$\mathcal{A}_D(0) = \{x \in \mathbb{R}^n \,|\, v_\varepsilon(x) < 1\}$$

(iv) If $f(\cdot, d)$ and $g(\cdot, d)$ are uniformly Lipschitz (uniformly in D with Lipschitz constants L_f and L_g) and further technical assumptions on g are satisfied then v_ε is uniformly Lipschitz on $\mathbb{R}^n$.