Analysis of the Local Robustness of Stability for Flows

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Abstract

In this paper the problem of measuring the robustness of stability for a perturbed continuous time nonlinear system at a singular fixed point is studied. Various stability radii are introduced and their values for the nonlinear system and its linearization are compared. It is shown that they generically coincide. This result may also be used to show generic continuity of linear real stability radii. Some examples are presented showing that it is sometimes necessary to consider the nonlinear system directly, and not simply to rely on the information provided by the linearization.

1 Introduction

Robustness analysis has played a prominent role in the theory of linear systems. In particular the state-state approach via stability radii has received considerable attention, see the [HP90], [HP92] and references therein. In this approach a perturbation structure is defined for a realization of the system, and the robustness of the system is identified with the norm of the smallest destabilizing perturbation. In recent years there has been a great deal of work done on extending these results to more general perturbation classes, see for example the survey paper [PD93], and for recent results on stability radii with respect to real perturbations, see [QBR$^+$95]. In some cases this approach has even lead to algorithms for designing controllers for maximum robustness, [HP95]. To date, the problem of extending these results to nonlinear systems has received little attention, although local stability analysis for nonlinear systems based on the linearization around a fixed point is well known, see e.g. [Vid93]. One approach in this direction is presented in [CK95], where time-varying perturbations are considered.

In this paper local stability radii for continuous-time nonlinear systems are studied, the corresponding problem for discrete-time systems was considered in [PW97a]. The system is assumed to have an exponentially stable fixed point x^* and an affine perturbation structure which leaves the fixed point invariant. The stability radius is then the norm of the smallest perturbation such that the system is no longer stable at x^*. In this fashion, radii may then be defined with respect to exponential, asymptotic and Lyapunov stability. Results on the relationships between the sizes of these stability radii are presented. It is seen that the nonlinear stability radii are contained in the interval defined by the radii of the linearized system, and that generically all radii are equal.

This also answers a question posed in [BKST90], where it was pointed out that for real data, stability radii (or robustness margins) of linear systems with respect to real perturbations may not depend continuously on the data. It was also suggested that this discontinuity might be non-generic. It follows from results below that this is indeed the case. This study of genericity relies in a fundamental manner on the theory of semi-algebraic sets, that is sets defined by polynomial equalities and inequalities. As a basic reference to this theory we refer to [BCR87]. Semi-algebraic sets have often found application in control theory and the study of dynamical systems, starting with [ABJ75]. In [K90] and [BB96] it is for instance shown that some problems in the area of systems theory do not have semi-algebraic descriptions. In particular we use that the spectral abscissa is a semi-algebraic function on the set of square real matrices of order n, and that the set of $n \times n$ Hurwitz matrices is semi-algebraic. This is an easy consequence of the Tarski-Seidenberg principle, see [BCR87] Chapter 1. The genericity results obtained in the linear case translate to nonlinear systems in the form that the set of systems for which all stability radii coincide contains an open and dense set with respect to the coarse and fine C^1-topologies. This result shows that generically local robustness analysis can be performed by studying the linearization of a system. Furthermore, the linearization provides sufficient information to determine whether the local robustness properties of the system depend on its nonlinearities.

Examples are presented to illustrate when and how differences between the stability radii occur.

The paper is organized as follows: In Section 2 preliminary definitions are presented, defining stability radii for the nonlinear system and its linearization. The relationships between these stability radii are studied in Section 3. In particular the main result on generic equality of the stability radii are shown. In Section 4, examples are presented demonstrating systems where the stability radii differ. In Section 5 we study extended stability radii, which are stability radii defined with respect to guaranteed exponential growth rates. Conclusions are drawn in Section 6.

2 Preliminaries

Consider the perturbed nonlinear continuous time system:

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x)\,, \tag{1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $f_i(x^*) = 0, i = 0, 1 \ldots m$. We assume that standard assumptions guaranteeing existence and uniqueness of solutions of (1) are satisfied. A solution for a given initial condition $x(0) = x_0$ and a fixed parameter $u \in \mathbb{R}^m$ is denoted by $\varphi(t; x_0, u)$, where $\varphi(0; x_0, u) = x_0$. When the system is unperturbed, we denote the solution $\varphi(t; x_0) = \varphi(t; x_0, 0)$.

The standard definitions of local (Lyapunov) stability, exponential and asymptotic stability are used, see *e.g.*[Vid93]. For each stability definition we define a stability radius, or measure of stability.

$$r_{ex}(f_0; (f_i)) \;:=\; \inf\{\rho > 0 \mid \exists u \in \mathbb{R}^m, \|u\| \le \rho, s.t.\ \textit{(1) is not exp. stable for } u\} \tag{2}$$

$$r_{as}(f_0; (f_i)) \;:=\; \inf\{\rho > 0 \mid \exists u \in \mathbb{R}^m, \|u\| \le \rho, s.t.\ \textit{(1) is not as. stable for } u\} \tag{3}$$

$$r_{st}(f_0; (f_i)) \;:=\; \inf\{\rho > 0 \mid \exists u \in \mathbb{R}^m, \|u\| \le \rho, s.t.\ \textit{(1) is not stable for } u\} \tag{4}$$

Although the norm used in the definition could in principle be any norm on $\mathbb{R}^m$ the results obtained in this paper are valid for norms whose unit ball is a semi-algebraic set, for example the Euclidean norm. This will be assumed from now on. We denote the open ball of radius ρ in $\mathbb{R}^m$ by $B(0, \rho)$.

We assume that the functions f_i are differentiable at x^*. The linearization of system (1) at x^* is of particular interest, and is defined by

$$\begin{aligned} \dot{x} &= \left(A_0 + \sum_{i=1}^{m} u_i A_i\right) x \\ &=: A(u)x \end{aligned} \tag{5}$$

where $A_i := \left.\frac{\partial f_i}{\partial x}\right|_{x^*}, i = 0, 1, \ldots m$. Stability radii for the linearized system (5) are now defined as follows:

$$r_{\mathbb{R}}(A_0; (A_i)) \;:=\; \inf\{\rho > 0 \mid \exists u \in \mathbb{R}^n, \|u\| \le \rho, s.t.\ \gamma(A(u)) \ge 0\} \tag{6}$$

$$\overline{r}_{\mathbb{R}}(A_0; (A_i)) \;:=\; \inf\{\rho > 0 \mid \exists u \in \mathbb{R}^n, \|u\| \le \rho, s.t.\ \gamma(A(u)) > 0\} \tag{7}$$

where $\gamma(M)$ denotes the spectral abscissa of $M \in \mathbb{R}^{n \times n}$. $r_{\mathbb{R}}(A_0; (A_i))$ measures the size of the smallest perturbation such that (5) is no longer exponentially stable. It corresponds to the stability radius usually studied for linear systems [HP90], and is motivated by the well known result that for linear systems, asymptotic and exponential stability are equivalent to each other and are also equivalent to $\gamma(A(u)) < 0$.

The radius $\overline{r}_{\mathbb{R}}(A_0; (A_i))$ is the infimum of the norms of the perturbations such that the system is exponentially unstable. To the best of our knowledge this is a new measurement of instability.

3 Relationships between the stability radii

In this section the relationships between the stability radii defined in the first section will be examined. For the moment we will not consider the problem of exactly how to calculate these quantities.

The following inequalities are immediate from the definitions.

Lemma 1 *Consider system (1) and its linearization (5), then*

$$r_{\mathbb{R}}(A_0; (A_i)) \leq \overline{r}_{\mathbb{R}}(A_0; (A_i)), \tag{8}$$

$$r_{ex}(f_0; (f_i)) \leq r_{as}(f_0; (f_i)) \leq r_{st}(f_0; (f_i)). \tag{9}$$

□

We now examine the relationship between the stability radii for a nonlinear system, and those of its linearization. These are direct consequences of the equivalence of exponential stability of a nonlinear system about an equilibrium point to that of its linearization. These ideas may also be well understood via the center manifold theorem, see *e.g.*[Car81].

Lemma 2 *Consider system (1) and its linearization (5), then*

$$r_{\mathbb{R}}(A_0; (A_i)) = r_{ex}(f_0; (f_i)). \tag{10}$$

$$r_{st}(f_0; (f_i)) \leq \overline{r}_{\mathbb{R}}(A_0; (A_i)). \tag{11}$$

□

Summarizing we have obtained that the three stability radii of interest for the nonlinear system are contained in an interval defined by the linearization, or are all equal if the two linear stability radii coincide. It is thus interesting to consider conditions under which

$$r_{\mathbb{R}}(A_0; (A_i)) \neq \overline{r}_{\mathbb{R}}(A_0; (A_i)).$$

Note that a significant characteristic of the equations is the movement of the roots of $\chi(A_0 + \sum u_i A_i)$ as the maximum allowable norm of u increases. Clearly, there exists u' with $||u'|| = r_{\mathbb{R}}(A_0; (A_i))$ such that $\gamma(A_0 + \sum u_i' A_i) = 0$. Now if $r_{\mathbb{R}}(A_0; (A_i)) < \overline{r_{\mathbb{R}}}(A_0; (A_i))$ this means that increasing perturbations do not push an eigenvalue across the imaginary axis. In the (generic) case of differentiable eigenvalues this implies that the derivative of the eigenvalue on the imaginary axis with respect to the parameters u is purely imaginary. Intuitively this should be a non-generic phenomenon. We will now investigate this problem. In order to prove a genericity result about this property we use the properties of semi-algebraic sets.

Proposition 3 *Let $n, m \in \mathbb{N}$ be fixed.*

(i) The sets

$$\begin{aligned} \mathcal{T}_\infty &:= \{(A_0, \dots, A_m) \mid r_{\mathbb{R}}(A_0, \dots, A_m) = \infty\}, \\ \overline{\mathcal{T}}_\infty &:= \{(A_0, \dots, A_m) \mid \overline{r}_{\mathbb{R}}(A_0, \dots, A_m) = \infty\} \end{aligned}$$

are semi-algebraic.

(ii) The functions

$$r_{\mathbb{R}} : \mathbb{R}^{n\times n\times(m+1)} \setminus \mathcal{T}_\infty \to \mathbb{R},$$
$$\overline{r}_{\mathbb{R}} : \mathbb{R}^{n\times n\times(m+1)} \setminus \overline{\mathcal{T}}_\infty \to \mathbb{R}$$

are semi-algebraic.

(iii) $r_{\mathbb{R}}$ *is upper semi-continuous.*

(iv) $\overline{r}_{\mathbb{R}}$ *is lower semi-continuous.*

□

Remark 4 Statement (iii) is mainly a rewording of a similar statement in [HP90], where different perturbation structures were considered. This, however, does not affect the argument. We include the proof as it shows how (iv) may be proved.

Proof.

(i) Note that the set of $n \times n$ Hurwitz matrices is semi-algebraic. It is easy to show that $\mathcal{T}_\infty$ and $\bar{\mathcal{T}}_\infty$ are the complements of sets which, via elimination of quantifiers (see [BCR87, Prop. 2.2.4]), are semi-algebraic, proving the result. *(ii)* Recall that the spectral abscissa γ is a semi-algebraic function. Now the graph of $r_{\mathbb{R}}$ is given by

$$\{ (A_0,\dots,A_m,t) \mid t \geq 0,\, \forall u \in B(0,t) : \gamma(A(u)) < 0$$
$$\text{and } \exists u' \in \mathbb{R}^m : \|u\| = t, \gamma(A(u')) = 0 \},$$

Applying (i) and using again that sets defined via polynomial equalities, inequalities and quantifiers over semi-algebraic sets are semi-algebraic, it may be seen that the graphs of r and $\bar{r}$ are semi-algebraic sets, proving the result.

(iii) If $r_{\mathbb{R}}(A_0;(A_i)) > c$ this implies that for all $u \in \mathbb{R}^m, \|u\| \leq c$ we have $\gamma(A(u)) \leq \rho < 0$ for a suitable constant ρ. By continuity of the spectral abscissa there exists a neighborhood V of $(A_0,\dots,A_m)$ such that $\gamma(B(u)) \leq \rho' < 0$ for all $(B_0,\dots,B_m) \in V$ and $u \in \mathbb{R}^m, \|u\| \leq c$. This implies $r_{\mathbb{R}}(B_0;(B_i)) > c$ for all elements of V, proving upper semi-continuity of $r_{\mathbb{R}}$. *(iv)* can be proved using a similar argument. ■

Theorem 5

Given $n, m \in \mathbb{N}$, the set $\mathcal{V}$ of matrices $A_0, A_1, \dots, A_m \in \mathbb{R}^{n\times n}$ for which

$$r_{\mathbb{R}}(A_0;(A_i)) < \overline{r}_{\mathbb{R}}(A_0;(A_i)) \tag{12}$$

is a nowhere dense, semi-algebraic subset of $\mathbb{R}^{n\times n\times(m+1)}$. □

Proof. By Proposition 3 it follows that the set

$$\mathcal{B} := \{(A_0,\dots,A_m) \mid r_{\mathbb{R}}(A_0;(A_i)) < \overline{r}_{\mathbb{R}}(A_0;(A_i))\}$$

is semi-algebraic. Thus it suffices to show that $\mathcal{B}$ is nowhere dense in $\mathbb{R}^{n\times n\times(m+1)}$. Because of the semi-algebraic structure of $\mathcal{B}$ we only have to show that its complement, which is also a semi-algebraic set, is dense. We first prove the following intermediate claim: For any open set $V \subset \mathbb{R}^{n\times n\times(m+1)}$ it holds that

$$\inf_{(A_0,\ldots,A_m)\in V} \overline{r}_{\mathbb{R}}(A_0;(A_i)) - r_{\mathbb{R}}(A_0;(A_i)) = 0\,.$$

To see this define $d := \inf_{(A_0,\ldots,A_m)\in V} r_{\mathbb{R}}(A_0;(A_i)) \geq 0$. If the infimum is ∞ there is nothing to show. Fix $\epsilon > 0$ and choose $(B_0;(B_i)) \in V$ such that $r_{\mathbb{R}}(B_0;(B_i)) - \epsilon < d$. Thus there exists $u, \|u\| \leq d+\epsilon$ such that $\gamma(B(u)) \geq 0$. In any neighborhood of $(B_0;(B_i))$ there exists a point $(C_0;(C_i))$ such that $\gamma(C(u)) > 0$ and hence $\overline{r}_{\mathbb{R}}(C_0;(C_i)) \leq d+\epsilon$ and so $\overline{r}_{\mathbb{R}}(C_0;(C_i)) - r_{\mathbb{R}}(C_0;(C_i)) \leq d+\epsilon-d = \epsilon$. This proves our claim.

Thus for every $n \in \mathbb{N}$ the following set is dense:

$$W_n := \{(A_0,\ldots,A_m) \mid \overline{r}_{\mathbb{R}}(A_0;(A_i)) - r_{\mathbb{R}}(A_0;(A_i)) < 1/n\}\,.$$

As $\overline{r}_{\mathbb{R}} - r_{\mathbb{R}}$ is lower semi-continuous W_n is also open and Baire's theorem ([Ped89] Proposition 2.2.2) shows that $\bigcap_{n\in\mathbb{N}} W_n$ is dense, i.e. the set of points where $r_{\mathbb{R}}$ and $\overline{r}_{\mathbb{R}}$ coincide is dense. By our previous considerations this completes the proof. ■

To paraphrase the previous result is to say that *generically* it does not happen that $r_{\mathbb{R}}(A_0;(A_i))$ and $\overline{r}_{\mathbb{R}}(A_0;(A_i))$ differ. This statement has a strength that is far superior to the interpretation of open and dense subsets as being generic, as we have really shown that the set where the linear stability radii differ is contained in a lower dimensional algebraic set.

As $r_{\mathbb{R}}$ is upper semi-continuous and $\overline{r}_{\mathbb{R}}$ is lower semi-continuous, it follows that on the interior of $\complement\mathcal{V}$ the (linear) real stability radius depends continuously on the data. As the interior of a semi-algebraic set is semi-algebraic we have also obtained the following corollary, that answers a problem posed in [BKST90].

Corollary 6 *There exists an open and dense semi-algebraic set $\mathcal{C} \subset \mathbb{R}^{(n\times n)\times(m+1)}$ such that on $\mathcal{C}$ the stability radii $r_{\mathbb{R}}(A_0;(A_i)), \overline{r}_{\mathbb{R}}(A_0;(A_i))$ coincide and are continuous.* □

The result for the linear stability radii extends to the case of nonlinear systems as follows. First, denote by $C^1(\mathbb{R}^n,\mathbb{R}^n,x^*)$ the set of continuously differentiable maps from $\mathbb{R}^n$ to itself satisfying $f(x^*) = 0$. This space may be endowed with the C^1 topology inherited from the topologies on the space $C^1(\mathbb{R}^n,\mathbb{R}^n)$. We briefly recall the construction of these topologies, see [Die72] Chapter 17. Fix a countable, locally finite family of compact sets K_i covering $\mathbb{R}^n$. For any $f \in C^1(\mathbb{R}^n,\mathbb{R}^n)$, $\epsilon > 0$ and $i \in \mathbb{N}$ let

$$W(f,\epsilon,i) := \left\{ g \in C^1(\mathbb{R}^n,\mathbb{R}^n) \mid \|f-g\|_{\infty,i} < \epsilon, \left\| \frac{\partial f}{\partial x_j} - \frac{\partial g}{\partial x_j} \right\|_{\infty,i} < \epsilon, j = 1,\ldots,n \right\}$$

where $\|\cdot\|_{\infty,i}$ denotes the uniform norm on K_i. The coarse C^1-topology is defined as the topology in which finite intersections of sets of the form $W(f,\epsilon,i)$ form a fundamental neighborhood system of f for any $f \in C^1(\mathbb{R}^n,\mathbb{R}^n)$. The fundamental neighborhood systems in the fine C^1-topology are obtained by taking all strictly positive sequences (ϵ_i) and considering the intersections

$$\bigcap_{i=1}^{\infty} W(f, \epsilon_i, i)\,.$$

Note that the topologies are independent of the family $\{K_i\}$.

Corollary 7 *Given $n, m \in \mathbb{N}$, the set $\mathcal{W}$ of functions $(f_0, f_1, \dots, f_m) \in C^1(\mathbb{R}^n, \mathbb{R}^n, x^*)^{m+1}$ for which*

$$r_{ex}(f_0; (f_i)) = r_{as}(f_0; (f_i)) = r_{st}(f_0; (f_i)) \tag{13}$$

contains an open and dense subset of $C^1(\mathbb{R}^n, \mathbb{R}^n, x^*)^{m+1}$ with respect to both the coarse and the fine C^1 topologies. □

Proof. As the fine C^1 topology is a refinement of the coarse, it suffices to show that the set $\mathcal{W}$ contains a nonempty coarse open set that is dense in the fine topology. Without loss of generality we may assume that x^* is contained only in $K_1 = \operatorname{cl} B(x^*, \delta)$, for some $\delta > 0$. Note that in order to establish (13), it is sufficient to show that (12) does not hold for the linearization. Let $(f_0, \dots, f_m) \in \mathcal{W}$ have linearizations $(A_0, \dots, A_m) \in \operatorname{int} \complement\mathcal{V}$, which is possible by Theorem 5. By definition there is an $\epsilon > 0$ such that for all

$$(g_0, \dots, g_m) \in [W(f_0, \epsilon, 1) \times \dots \times W(f_m, \epsilon, 1)] \cap C^1(\mathbb{R}^n, \mathbb{R}^n, x^*)^{m+1}$$

the corresponding linearization in x^* is contained in $\operatorname{int} \complement\mathcal{V}$. This proves openness of a nonempty subset of $\mathcal{W}$ in the coarse C^1 topology.

Now let $(f_0, \dots, f_m) \notin \mathcal{W}$, so that its linearization $(A_0, \dots, A_m)$ is in $\mathcal{V}$. By Theorem 5 we may choose matrix sequences Δ_i^k $i = 0, \dots, m$, $k \in \mathbb{N}$ such that $\Delta_i^k \to 0$ $i = 0, \dots, m$ and $(A_0, \dots, A_m) + (\Delta_0^k, \dots, \Delta_m^k) \in \operatorname{int} \complement\mathcal{V}$ for all $k \in \mathbb{N}$. Then we may construct C^∞-functions δ_i^k such that

$$\delta_i^k(x) = 0 \text{ for } x \in \{x^*\} \cup \bigcup_{j=2}^{\infty} K_j, i = 0, \dots, m\,, k \in \mathbb{N},$$

$$\left.\frac{\partial \delta_i^k}{\partial x}\right|_{x^*} = \Delta_i^k, \quad i = 0, \dots, m\,, k \in \mathbb{N}.$$

and $\delta_i^k \to 0$ in the coarse and thus also in the fine C^1 topology. It follows that $(f_0, \dots, f_m) + (\delta_0^k, \dots, \delta_m^k) \in \mathcal{W}$ and $(f_0, \dots, f_m) + (\delta_0^k, \dots, \delta_m^k) \to (f_0, \dots, f_m)$ in the fine topology. Furthermore, by the first part of the proof $(f_0, \dots, f_m) + (\delta_0^k, \dots, \delta_m^k)$ is contained in the coarse interior of $\mathcal{W}$. ■

Remark 8 It is immediate that for the case $n = 1$

$$r_{ex}(f_0; (f_i)) = r_{as}(f_0; (f_i)) = r_{st}(f_0; (f_i))\,. \tag{14}$$

4 Examples

In this section we present examples demonstrating when the inequalities proven in the previous section are strict inequalities. We begin with a two-dimensional example, which also shows that for $n \geq 2$ the sets $\mathcal{T}_\infty, \overline{\mathcal{T}}_\infty$ considered in Proposition 3 (i) differ.

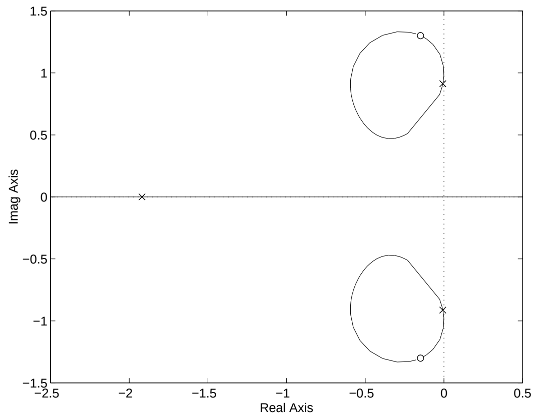

Figure 1: Root loci for $\chi(A_0 + uA_1)$.

Example 9 Let $n = 2$ and

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 1 \\ -9/8 & -1 \end{bmatrix}.$$

With these data it is easy to see that the eigenvalues of $A_0 + uA_1$ are given by $-3/2 \pm \sqrt{4 + 16u - 2u^2}/4$. This shows that 0 is an eigenvalue of $A_0 + 4A_1$, while for all $u \neq 4$ the matrix $A(u)$ is Hurwitz stable. Thus $r_{\mathbb{R}}(A_0, A_1) = 4$ while $\overline{r}_{\mathbb{R}}(A_0, A_1) = \infty$. □

Example 10 To demonstrate what may happen in a situation where the inequality (12) holds, consider the following system

$$\dot{x} \;=\; A_0 x + uA_1 x\,, \tag{15}$$

where

$$A_0 = \begin{bmatrix} -29/15 & -31/36 & -8/5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad A_1 = \begin{bmatrix} -7/15 & -5/36 & -4/5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we use the standard norm, $||u|| = |u|$.

Due to the special form of the matrices, the characteristic equation of the system (15) may be seen to be

$$s^3 + \frac{29\,s^2}{15} + \frac{31\,s}{36} + \frac{8}{5} + \left(\frac{7\,s^2}{15} + \frac{5\,s}{36} + \frac{4}{5}\right) u = 0\,. \tag{16}$$

Note that $A_0 + uA_1$ is Hurwitz for $u \in (-2, \infty)\backslash\{1\}$. The matrix $A_0 + A_1$ has eigenvalues $i, -i, -12/5$, and the matrix $A_0 - 2A_1$ has eigenvalues $0, -1/2 + i/\sqrt{3}, -1/2 - i/\sqrt{3}$. See Fig. 1 for a root locus diagram for the system. It is thus clear that the system (15) has the following stability radii: $r_{\mathbb{R}}(A_0,(A_1)) = 1$, and $\overline{r}_{\mathbb{R}}(A_0,(A_1)) = 2$. Furthermore, system (15) is an example where

$$r_{\mathbb{R}}(A_0,(A_1)) = r_{ex}(A_0,(A_1)) = r_{as}(A_0,(A_1)) < r_{st}(A_0,(A_1)) = \overline{r}_{\mathbb{R}}(A_0,(A_1)).$$

If we consider the system in $\mathbb{R}^6$ given by

$$\begin{aligned} \dot{x} &= \left(\begin{bmatrix} A_0 & I \\ 0 & A_0 \end{bmatrix} + u \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix} \right) x \\ &=: B_0 x + u B_1 x \end{aligned}$$

then of course the values of $r_{\mathbb{R}}, \overline{r_{\mathbb{R}}}$ are the same as for system (15), but again we have an example where

$$r_{\mathbb{R}}(B_0,(B_1)) = r_{ex}(B_0,(B_1)) = r_{as}(B_0,(B_1)) = r_{st}(B_0,(B_1)) < \overline{r}_{\mathbb{R}}(B_0,(B_1)).$$

□

A difference between the exponential and the asymptotic stability radius is a truly nonlinear phenomenon. To exhibit this we study a nonlinear system which has (15) as its linearization.

Example 11 Consider the system

$$\dot{x} = A_0 x + u_1 A_1 x - x^T x\, x + u_2 x^T x\, x \tag{17}$$

with the Euclidean norm $||u|| = (u_1^2 + u_2^2)^{\frac{1}{2}}$. Note that the linearization of (17) is (15), thus exponential stability or instability of the system will be determined by the parameter u_1.

Consider now that $u_1 = 1$, so that the linearization is stable, but not asymptotically stable. We examine the stability of the system using the center manifold theorem and a Lyapunov argument. First note that $(A_0 + A_1)$ is similar to the matrix

$$B := \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -12/5 \end{bmatrix}.$$

Let S be the transformation matrix, and let $Q = \left(S^T\right)^{-1} S^{-1}$, note $Q > 0$. In this basis, and using $u_1 = 1$ the system(17) has the special form:

$$\dot{x} = Bx - (1 - u_2)\, x^T Q x\, x$$

Due to the center manifold theorem, we need only consider what happens on the center manifold to determine stability of the system. In this case the subspace $(x_1, x_2, 0)$ is a center manifold for the system. For $x = (x_1, x_2, 0)^T$, $\dot{x}_3 = 0$, and

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} - (1 - u_2) x^T Q x \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Consider the function $V(x) = x_1^2 + x_2^2$. Then

$$\begin{aligned} \dot{V}(x) &:= 2\dot{x}_1 x_1 + 2\dot{x}_2 x_2 \\ &= -2x^T x\, x^T Q x (1 - u_2) \end{aligned}$$

Clearly for $u_2 < 1$, $\dot{V} < 0$, and so the system is asymptotically stable; for $u_2 = 1$, $\dot{V} = 0$, and so the system is Lyapunov stable; and for $u_2 > 1$, $\dot{V} > 0$, and so the system is unstable. It is thus clear that

$$1 = r_{ex}(f_0; (f_i)) < r_{as}(f_0; (f_i)) = r_{st}(f_0; (f_i)) = \sqrt{2}\,.$$

□

Note that using combinations of Examples 10 and 11 it is easy to construct a higher dimensional example where indeed

$$r_{\mathbb{R}} = r_{ex} < r_{as} < r_{st} < \overline{r}_{\mathbb{R}}\,.$$

5 Extended Stability Radii

In some situations it may be interesting to consider an extended version of the stability radius for the linearized system.

$$r_{\mathbb{R}}^c(A_0; (A_i)) := \inf\{\rho > 0 \mid \exists u \in \mathbb{R}^n, \|u\| \le \rho, s.t.\ \gamma(A(u)) \ge c\} \tag{18}$$

$$\overline{r}_{\mathbb{R}}^c(A_0; (A_i)) := \inf\{\rho > 0 \mid \exists u \in \mathbb{R}^n, \|u\| \le \rho, s.t.\ \gamma(A(u)) > c\} \tag{19}$$

This allows measurement of the robustness of the system with respect to a guaranteed level of exponential convergence or divergence. These new stability radii may be linked to those of (6), (7) in a straightforward fashion, as follows.

Proposition 12 *Let* $(A_0, \ldots, A_m) \in \mathbb{R}^{(n\times n)\times(m+1)}$, *then*

$$r_{\mathbb{R}}^c(A_0; (A_i)) = r_{\mathbb{R}}(A_0 - cI; (A_i)) \tag{20}$$

$$\overline{r}_{\mathbb{R}}^c(A_0; (A_i)) = \overline{r}_{\mathbb{R}}(A_0 - cI; (A_i)) \tag{21}$$

□

Using this proposition, the question of whether the equality $r_{\mathbb{R}}^c(A_0; (A_i)) = \overline{r}_{\mathbb{R}}^c(A_0; (A_i))$ holds generically may be answered for every c. A more interesting question regards the genericity of the property:

$$\forall c \in \mathbb{R}, \qquad r_{\mathbb{R}}^c(A_0; (A_i)) = \overline{r}_{\mathbb{R}}^c(A_0; (A_i))\,. \tag{22}$$

We begin by noting the following property.

Proposition 13 *Suppose that* $r^c_{\mathbb{R}}(A_0;(A_i)) < \overline{r}^c_{\mathbb{R}}(A_0;(A_i))$ *for some* $c \in \mathbb{R}$. *Then for all* $\alpha > 0$ *and* $0 \neq \beta \in \mathbb{R}$, $r^{\alpha c}_{\mathbb{R}}(\alpha A_0;(\beta A_i)) < \overline{r}^{\alpha c}_{\mathbb{R}}(\alpha A_0;(\beta A_i))$. *Furthermore,* $r^{\alpha c}_{\mathbb{R}}(\alpha A_0;(\beta A_i)) = \frac{\alpha}{|\beta|} r^c_{\mathbb{R}}(A_0;(A_i))$ *and* $\frac{\alpha}{|\beta|}\overline{r}^c_{\mathbb{R}}(A_0;(A_i)) = \overline{r}^{\alpha c}_{\mathbb{R}}(\alpha A_0;(\beta A_i))$. □

Proof. Note that $\forall \alpha > 0,\ \gamma(\alpha A) = \alpha\gamma(A)$.

For a given $u \in \mathbb{R}^m$ and $\alpha, \beta > 0$, consider the scaled perturbation $u' = \frac{\beta}{\alpha} u$. Then:

$$\begin{aligned} \alpha A_0 + \sum_{i=0}^{m} \beta A_i u_i &= \alpha A_0 + \sum_{i=0}^{m} \alpha A_i u'_i \\ &= \alpha A(u'). \end{aligned}$$

Thus it follows that

$$\gamma\left(\alpha A_0 + \sum_{i=0}^{m} \beta A_i u_i\right) = \alpha\gamma\left(A(\frac{\beta}{\alpha}u)\right).$$

Thus from the definition it follows that $r^{\alpha c}_{\mathbb{R}}(\alpha A_0;(\beta A_i)) = \frac{\beta}{\alpha} r^c_{\mathbb{R}}(A_0;(A_i)) < \frac{\beta}{\alpha}\overline{r}^c_{\mathbb{R}}(A_0;(A_i)) = \overline{r}^{\alpha c}_{\mathbb{R}}(\alpha A_0;(\beta A_i))$.

Given $\beta < 0$, considering the scaling $u' = -\frac{\beta}{\alpha} u$ then proves the result. ■

This shows that the set

$$\{(A_0,\ldots,A_m) \mid \exists c \in \mathbb{R} : r^c_{\mathbb{R}}(A_0;(A_i)) < \overline{r}^c_{\mathbb{R}}(A_0;(A_i))\}$$

has the structure of a cone. Furthermore it is semi-algebraic. We will now show that this cone has interior points. To this end we introduce the following sufficient condition.

Lemma 14 *Let* $(A_0,\ldots,A_m)$ *satisfy*

(i) $r^c_{\mathbb{R}}(A_0;(A_i)) = \rho_0$.

(ii) There exists an $\epsilon > 0$ *such that for all* $\rho_0 < \rho < \rho_0 + \epsilon$

$$\max\{\gamma(A(u)) \mid \|u\| = \rho\} < c,$$

then

(i) $r^c_{\mathbb{R}}(A_0;(A_i)) = \rho_0 < \rho_0 + \epsilon \le \overline{r}^c_{\mathbb{R}}(A_0;(A_i))$.

(ii) There exists an open neighborhood U *of* $(A_0,\ldots,A_m)$ *such that for all* $(B_0,\ldots,B_m) \in U$ *there exists a* c' *satisfying*

$$r^{c'}_{\mathbb{R}}(B_0;(B_i)) < \overline{r}^{c'}_{\mathbb{R}}(B_0;(B_i)).$$

□

Proof. Note that the spectrum of a matrix is continuous with respect to its entries. Thus the spectral abscissa is continuous with respect to its arguments. Choose $\rho \in (\rho_0, \rho_0+\epsilon)$. By assumption there exists $\delta > 0$ such that for all $u \in \mathbb{R}^m$ satisfying $||u|| = \rho$ we have $\gamma(A(u)) \le c - \delta < c$. By continuity there exists a neighborhood U_1 of $(A_0,\ldots,A_m)$ such that for all $(B_0,\ldots,B_m) \in U_1$ we have

$$||u|| = \rho \Rightarrow \gamma(B(u)) \le c - 2\delta/3.$$

On the other hand there exists a neighborhood U_2 of $(A_0, \ldots, A_m)$ such that

$$|\max\{\gamma(B(u)) \mid \|u\| \leq \rho_0\} - c| < \delta/3$$

for all $(B_0, \ldots, B_m) \in U_2$. It follows for all $(B_0, \ldots, B_m) \in U_1 \cap U_2$ that for some c' with $|c - c'| < \delta/3$ (depending on $(B_0, \ldots, B_m)$)

$$r_{\mathbb{R}}^{c'}(B_0; (B_i)) < \overline{r_{\mathbb{R}}}^{c'}(B_0; (B_i)) .$$

This shows the assertion. ■

These results may now be summarized as follows, showing that (22) is not a generic property.

Theorem 15

Let $n \geq 2, m \geq 1$. The set

$$\{(A_0, \ldots, A_m) \mid \exists c \in \mathbb{R} : r_{\mathbb{R}}^c(A_0; (A_i)) < \overline{r}_{\mathbb{R}}^c(A_0; (A_i))\}$$

is a semi-algebraic cone with interior points. □

Proof. The first two properties are shown in Proposition 13. To construct interior points consider the matrices A_0 and A_1 defined in Example 9 and let

$$A_0' = \begin{bmatrix} A_0 & C_0 \\ 0 & D_0 \end{bmatrix} \quad A_1' = \begin{bmatrix} A_1 & C_1 \\ 0 & 0 \end{bmatrix}, \quad A_j' = \begin{bmatrix} 0 & C_j \\ 0 & 0 \end{bmatrix},$$

where $j = 2, \ldots, m$ $C_0, C_1, \ldots, C_m \in \mathbb{R}^{2\times(n-2)}$ are arbitrary and D_0 is an arbitrary stable matrix. By Example 9 $(A_0', \ldots, A_m')$ satisfies the assumption of Lemma 14. This completes the proof. ■

In light of this result, the question of characterizing the set

$$\{(A_0, \ldots, A_m) \mid \forall c \in \mathbb{R}, \quad r_{\mathbb{R}}^c(A_0; (A_i)) = \overline{r}_{\mathbb{R}}^c(A_0; (A_i))\} \tag{23}$$

becomes an interesting one. It is clear that $(A_0, \ldots, A_m)$ will belong to this set if the function $\rho \to \max_{||u||=\rho} \gamma(A(u))$ satisfies a monotonicity property, but a further characterization, based on the entries of the A_i is currently unknown. Nevertheless, for given $(A_0, \ldots, A_m)$, the following properties of its extended stability radii may be proven. Most significantly, $r_{\mathbb{R}}^c$ and $\overline{r}_{\mathbb{R}}^c$ may only differ at a finite number of points.

Proposition 16 *Let $m, n \in \mathbb{N}$ and $(A_0, \ldots, A_m) \in \mathbb{R}^{(n\times n)\times m}$ be fixed. For the maps*

$$h : c \mapsto r_{\mathbb{R}}^c(A_0; (A_i)) , \quad \overline{h} : c \mapsto \overline{r}_{\mathbb{R}}^c(A_0; (A_i)) ,$$

the following statements hold:

(i) $h, \overline{h}$ are semi-algebraic functions, when the domain of definition is restricted to those $c \in \mathbb{R}$ where they take real values.

(ii) $h, \overline{h}$ *have at most finitely many discontinuities.*

(iii) h *is upper semi-continuous,* $\overline{h}$ *is lower semi -continuous.*

(iv) If $h(c_0) < \overline{h}(c_0)$ *then* $h, \overline{h}$ *are discontinuous at* c_0*.*

(v) $h(c) = \overline{h}(c)$ *for all* $c \in \mathbb{R}$ *with the exception of at most finitely many points.*

□

Proof. Note that the domains where $h, \overline{h}$ take real values are intervals. Items (i) and (iii) follow from Proposition 3 and Proposition 12 respectively, while (ii) follows from the fact that a semi-algebraic function may only have finitely many discontinuities on any given interval.

(iv) By definition and continuity of the spectral abscissa $h(c_0 + \epsilon) \geq \overline{h}(c_0)$ for any $\epsilon > 0$. Thus the assumption implies discontinuity of h at c_0. Discontinuity of $\overline{h}$ at c_0 follows from $h(c_0) \geq \overline{h}(c_0 - \epsilon)$.

(v) This is an immediate consequence of (iv) and (ii). ■

6 Conclusions

In this paper we have shown that for a nonlinear system (1) and its linearization (5) the stability radii are related in the following way:

$$r_{\mathbb{R}}(A_0;(A_i)) = r_{ex}(f_0;(f_i)) \leq r_{as}(f_0;(f_i)) \leq r_{st}(f_0;(f_i)) \leq \overline{r}_{\mathbb{R}}(A_0;(A_i)),$$

where $A_i = \left.\frac{\partial f_i}{\partial x}\right|_{x=x^*}$. Examples have been presented to show that the inequalities may not be replaced by equalities, however for systems of dimension 1, and generically for systems of higher dimension

$$r_{\mathbb{R}}(A_0;(A_i)) = r_{ex}(f_0;(f_i)) = r_{as}(f_0;(f_i)) = r_{st}(f_0;(f_i)) = \overline{r}_{\mathbb{R}}(A_0;(A_i)).$$

Extended stability radii have been introduced, with which the robustness of the system with respect to a guaranteed rate of convergence may be measured. It is shown that $\{(A_0,\ldots,A_m)\,|\,\exists c \in \mathbb{R}, r^c_{\mathbb{R}}(A_0;(A_i)) \neq \overline{r}^c_{\mathbb{R}}(A_0;(A_i))\}$ has the structure of a cone with interior points. Nevertheless, for a given system, the extended stability radii may only differ at a finite number of isolated points.

In this paper we only consider real, time invariant perturbations of the system. In further work we intend extending these results to include complex or time-varying perturbations.

The results of Sections 2 to 4 may be extended to discrete time systems, as explored in [PW97a]. Time-varying perturbations of discrete-time systems are considered in [PW97b], but it is not expected that the techniques applied there will translate straightforwardly to the continuous time setting.

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