Comments On the Prevalence of Linear Parameter Varying Systems

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Abstract
This note comments on a number of recent gain-scheduling approaches which assume that the plant to be controlled is in so-called linear or quasi-linear parameter-varying form. In the first part of the note it is shown that present theory does not support the reformulation of nonlinear systems into linear/quasi-linear parameter-varying form without, in general, considerable restrictions either on the class of nonlinear systems considered or on the allowable operating region. The shortcomings of a popular ad hoc reformulation approach are also demonstrated. In the second part of the note, it is shown that, by employing velocity-based formulation, a very general class of nonlinear systems can, indeed, be transformed into linear/quasi-linear parameter-varying form. It is emphasised that this transformation is global in nature, with no restriction to near equilibrium operation.

1. Introduction

Gain-scheduled controllers are linked by the design approach employed, whereby a nonlinear controller is constructed by interpolating, in some manner, between the members of a family of linear time-invariant controllers. In the conventional, and most common, gain-scheduling design approach (see, for example, Astrom & Wittenmark 1989, Hyde & Glover 1993), each linear controller is typically associated with a specific equilibrium operating point of the plant and is designed to ensure that, locally to the equilibrium operating point, the performance requirements are met. This approach is, essentially, applicable to every nonlinear plant which can be linearised at its equilibrium operating points.

Recently, a number of interesting alternative approaches have been proposed in the context of gain-scheduling design (for example, Becker *et al*. 1993, Packard 1994, Apkarian & Adams 1998). Since these approaches employ various types of so-called linear parameter-varying (LPV) plant representation, they are commonly referred to as LPV gain-scheduling methods. The term "linear parameter-varying" is widely employed in the literature to refer to any system of the form $\dot{\mathbf{x}} = \mathbf{A}(\theta)\mathbf{x}+\mathbf{B}(\theta)\mathbf{r}$, $\mathbf{y}=\mathbf{C}(\theta)\mathbf{x}+\mathbf{D}(\theta)\mathbf{r}$ where θ is a parameter belonging to some class Ω. Shamma (1988), Shamma & Athans (1991) consider systems where the parameter, θ, is an exogenous time-varying quantity (strictly independent of the state $\mathbf{x}$ of the system) which takes values in some allowable set. Becker *et al.* (1993) consider plants where the parameter, θ, takes values in some allowable set but may otherwise vary *arbitrarily* with time and so may, for example, depend on $\mathbf{x}$ and $\mathbf{r}$. Similarly, the approaches of Packard (1994) and Apkarian & Gahinet (1995) require that the plant is in the form of a linear time-invariant system enclosed by a feedback loop with gains which are bounded but may otherwise vary arbitrarily. More recently, Wu *et al.* (1995), Lim & How (1997), Apkarian & Adams (1997,1998) consider systems where, in addition to requiring that the parameter θ belongs to some bounded set, it is also assumed that there exists an upper bound on the rate of variation of θ. Although superficially similar, it should be emphasised that the dynamic characteristics of such systems are strongly dependent on the class Ω to which the parameters belong. In particular, when θ is permitted to depend on the state, $\mathbf{x}$[1], (a situation widely considered in the literature, see examples in Shamma 1988, Shamma & Cloutier 1993, Coetsee 1994, Apkarian *et al*. 1995, Scherer *et al.* 1997, Apkarian & Adams 1998, Huzmezan & Maciejowski 1998, Lim & How 1998, Wu & Grigoriadis 1998, Johansen 1999) the dependence of the **A** and **B** matrices on the state introduces nonlinear feedback not present in linear time-invariant/time-varying systems. Use of the term *linear* parameter-varying to describe such nonlinear systems is, therefore, potentially misleading. In the context of the present paper, a generalisation of the terminology of Shamma (1988) is thus adopted and systems where θ may depend on the state, $\mathbf{x}$, are hereafter referred to as quasi-LPV systems while the term LPV is reserved for systems where θ is a strictly exogenous time-varying quantity (strictly independent of the state $\mathbf{x}$ of the system).

A considerable body of results now exists relating to the design of controllers for plants which are in LPV or quasi-LPV form. However, the literature typically takes the existence of a plant in LPV/quasi-LPV form as its starting point, thus largely neglecting the critical issue of whether a significant loss of generality is introduced by

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[1] It should be noted that, in order to apply LPV gain-scheduling methods in such circumstances, it is usually necessary to restrict the input and initial conditions of the state such that the solution $\mathbf{x}(t)$ is confined to some bounded operating region $X \subset \Re^n$ thereby ensuring that the "parameter" θ is bounded.

confining consideration to this class of plants. Indeed, on the face of it few nonlinear systems have dynamics which are of the required LPV/quasi-LPV form and it is far from clear whether this class of plants is sufficiently rich to include a class of nonlinear plants of comparable generality to alternative control design methods such as conventional gain-scheduling approaches. The aim of this note is, firstly, to highlight the existence of this fundamental deficiency in the existing body of results and, secondly, to consider constructive methods for resolving this issue.

The organisation of the note reflects these twin aims. The primary theoretical results relating to LPV and quasi-LPV formulations of nonlinear systems are largely scattered through a diverse literature and are therefore gathered together and critically reviewed in section 2. The role of the recently developed velocity-based framework (Leith & Leithead 1998a,b) is then discussed in section 3 with regard to the constructive reformulation of general nonlinear systems in LPV/quasi-LPV form and the conclusions are summarised in section 4.

2. Linear parameter-varying representations

Existing theoretical results relating to the formulation of dynamic systems in LPV/quasi-LPV form have largely been developed in specific contexts, often independently of one another. Despite this diversity, there is a notable absence in the literature of a formal critical survey which considers the relationships between these results and provides an overview of their role in the LPV gain-scheduling context. In view of the fundamental importance of this theory to LPV gain-scheduling methods, a somewhat extended review is, therefore, presented in this section. Before proceeding, however, a simple example is discussed which illustrates some of the potential pitfalls associated with employing *ad hoc* methods for formulating systems in LPV/quasi-LPV form and to help motivate the consideration of more soundly-based methods .

Example 1 Consider the nonlinear system with dynamics described by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}r+\begin{bmatrix}0\\ -|x_2|x_2-10\end{bmatrix},\quad y=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{1}$$

(It should be noted that systems with similar types of nonlinearity are frequently encountered in practice, see, for example, Nichols *et al.* 1993). The requirement is to design an output-feedback controller which ensures a step response settling time of less than 2 seconds with zero steady-state error (this is, of course, not a complete performance specification but is sufficient in the present context). Combining ideas from conventional and LPV gain-scheduling, the following hybrid control design procedure is similar to *ad hoc* approaches proposed in the literature (see, for example, Apkarian *et al.* 1995, Spillman *et al.* 1996, Fialho *et al.* 1997, Lee & Spillman 1997). The series expansion linearisation of (1) about an equilibrium point (x_{10}, x_{20}, r_o, y_o) is

$$\begin{bmatrix}\delta\dot{x}_1\\ \delta\dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 1 & -2|x_{2o}|\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}\delta r,\quad \delta y=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} \tag{2}$$

where

$$\delta x_1 = x_1\text{-}x_{10},\ \delta x_2 = x_2\text{-}x_{20},\ \delta r = r\text{-}r_0,\ \delta y = y\text{-}y_0 \tag{3}$$

When considering the design of an LPV gain-scheduled controller for the system (1) with linearisation family defined by (2)-(3) it is perhaps natural to consider the quasi-LPV system

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}=\mathbf{A}(\theta)\begin{bmatrix}z_1\\ z_2\end{bmatrix}+\mathbf{B}r,\quad y=\mathbf{C}\begin{bmatrix}z_1\\ z_2\end{bmatrix} \tag{4}$$

where

$$\mathbf{A}(\theta)=\begin{bmatrix}-1 & 0\\ 1 & -2\theta\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}1\\ 0\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}0 & 1\end{bmatrix} \tag{5}$$

and θ equals $|z_2|$. Assume, for the moment, that $0\le\theta\le10$ and θ may vary arbitrarily within this range (this assumption amounts to a restriction on the class of allowable initial conditions and inputs to the system such that $|z_2|\le10$). Using standard software from the MATLAB LMI toolbox (Gahinet *et al.* 1995) and a conventional L_2 objective function with performance weighting, w_1, and control weighting, w_2, transfer functions[2]

$$w_1(s)=\frac{0.5}{s+0.002},\quad w_2(s)=\frac{2.0\text{x}10^{-2}s}{s+1000} \tag{6}$$

the controller obtained for this system is

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[2] The performance requirement specifies zero steady-state error which implies that the magnitude of the transfer function of w_1 should be infinite at d.c. ; for example, by including an integrator term. However, the transfer functions here specify the actual values used in the numerical calculations, with approximate rather than exact integral action present in w_1.

$$\dot{\mathbf{x}}_c = \mathbf{A}_c(\theta)\mathbf{x}_c + \mathbf{B}_c(y_{ref} - y), \quad r = \mathbf{C}_c\mathbf{x} \tag{7}$$

where $\mathbf{A_c}(\theta)=\alpha\mathbf{A_o}+(1-\alpha)\mathbf{A}_1$, $\alpha=(10-\theta)/10$,

$$\mathbf{A}_o = \begin{bmatrix} 5.6949e+003 & -2.7145e+000 & 2.7060e+001 & 7.6000e+004 \\ -3.5506e+004 & -1.5530e+001 & 2.4283e+002 & -4.7350e+005 \\ -1.9130e+005 & 2.8303e-001 & -5.6208e+000 & -2.5521e+006 \\ -2.0873e+004 & 3.1121e-001 & -1.6133e+000 & -2.7847e+005 \end{bmatrix}, \mathbf{A}_1 = \begin{bmatrix} 5.6777e+003 & -5.4523e-001 & 2.6029e+001 & 7.6032e+004 \\ -3.5504e+004 & -1.5777e+001 & 2.4295e+002 & -4.7350e+005 \\ -1.9130e+005 & 4.0036e-001 & -5.6766e+000 & -2.5521e+006 \\ -2.0869e+004 & -1.5741e-001 & -1.3905e+000 & -2.7848e+005 \end{bmatrix} \tag{8}$$

$$\mathbf{B}_c = \begin{bmatrix} 3.3565e-002 \\ -3.1706e-003 \\ -1.5648e-002 \\ 1.5715e+000 \end{bmatrix}, \mathbf{C}_c = \begin{bmatrix} 4.7249e+004 \\ 1.4691e+001 \\ -2.4353e+002 \\ 6.3016e+005 \end{bmatrix}^T$$

The response to step change in demand from –3.16 units to 0 units of the closed-loop system, consisting of the Jacobean-based quasi-LPV plant, (4)-(5), and controller, (7)-(8), is shown by the dashed-line in figure 1. (Note that θ (*i.e.* $|y|$) lies in the required range [0,10]). The settling time requirement is evidently satisfied. Nevertheless, when the same controller, (7)-(8), is applied to the original nonlinear system, (1) , the corresponding response is as shown by the solid line in figure 1. It can be seen that the performance requirement is clearly not met and, indeed, that the nonlinear closed-loop system appears to be unstable.

It is interesting to note that the performance requirement is in fact met for larger step demands; for example, a step change from –3.16 units to 6 units. This is perhaps unexpected, since larger steps are associated with excursions into operating regions further from equilibrium and with faster parameter variations, and clearly indicates that the behaviour observed is not associated with any restriction to near equilibrium operation arising from the use of equilibrium linearisations for the controller design. Indeed, the system in this example is intentionally selected to be benign in the sense that it satisfies the extended local linear equivalence condition of Leith & Leithead (1996,1998c); that is, the neighbourhood of validity of each equilibrium linearisation is unbounded and the union of these neighbourhoods covers the entire operating space. Hence, control design approaches based on the equilibrium linearisations are *not* a priori restricted to near equilibrium operation. Further loss of performance associated with deviations from equilibrium operation can, of course, be anticipated for systems which do not satisfy such a condition. See section 2.2.1 for further discussion and a second example of the *ad hoc* LPV formulation approach considered here.

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The poor performance achieved in the foregoing example is perhaps unsurprising since no direct relationship is established between the quasi-LPV system, (4), used for control design and the nonlinear system which is actually of interest, (1). It is emphasised that the family of linear systems defined by the equilibrium linearisations of (1), being a *collection* of individual dynamic systems (each with its own distinct state, input and output defined by the transformations (3)) rather than a *single* dynamic system, is conceptually quite different from the quasi-LPV system, (4). Of course, controllers designed by approaches similar to that here may sometimes achieve acceptable performance. Nevertheless, the foregoing example indicates that this is certainly not the case in general. The requirement is for soundly-based techniques for formulating systems in LPV/quasi-LPV form.

2.1 LPV Systems

Perhaps the earliest work on LPV systems in the context of gain-scheduling is that of Shamma (1988), Shamma & Athans (1991). Shamma (1988), Shamma & Athans (1991) consider systems

$$\dot{\mathbf{x}} = \mathbf{A}(\theta(t))\mathbf{x}+\mathbf{B}(\theta(t))\mathbf{r}, \quad \mathbf{y}=\mathbf{C}(\theta(t))\mathbf{x}+\mathbf{D}(\theta(t))\mathbf{r} \tag{9}$$

where the parameter, θ, is an exogenous time-varying quantity which takes values in some allowable set. Under these conditions, an LPV system is simply a particular form of linear time-varying system. Linear time-varying representations of nonlinear systems are largely associated with series expansion linearisation theory and this is, therefore, briefly reviewed in this section.

Consider the nonlinear system,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}), \quad \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}) \tag{10}$$

where $\mathbf{r} \in \Re^m$, $\mathbf{y} \in \Re^p$, $\mathbf{x} \in \Re^n$, $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are differentiable with bounded, Lipschitz continuous derivatives. The set of equilibrium operating points of the nonlinear system, (10), consists of those points, $(\mathbf{x_o}, \mathbf{r_o})$, for which

$$\mathbf{F}(\mathbf{x_o}, \mathbf{r_o}) = 0 \tag{11}$$

Let $\Phi:\Re^n\times\Re^m$ denote the space consisting of the union of the state, $\mathbf{x}$, with the input, $\mathbf{r}$. The set of equilibrium operating points of the nonlinear system, (10), forms a locus of points, $(\mathbf{x_o}, \mathbf{r_o})$, in Φ and the response of the system to a general time-varying input, $\mathbf{r}(t)$, is depicted by a trajectory in Φ.

Let ($\tilde{\mathbf{x}}$(t), $\tilde{\mathbf{r}}$(t), $\tilde{\mathbf{y}}$(t)) denote a specific trajectory of the nonlinear system, (10); that is,

$$\dot{\tilde{\mathbf{x}}} = \mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}), \ \tilde{\mathbf{y}} = \mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}) \tag{12}$$

The trajectory, ($\tilde{\mathbf{x}}$(t), $\tilde{\mathbf{r}}$(t), $\tilde{\mathbf{y}}$(t)), could be an equilibrium operating point of (10), in which case $\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})$ is identically zero and $\tilde{\mathbf{x}}$ is a constant. The nonlinear system, (10), may be reformulated, relative to the trajectory ($\tilde{\mathbf{x}}$(t), $\tilde{\mathbf{r}}$(t), $\tilde{\mathbf{y}}$(t)), as,

$$\delta\dot{\mathbf{x}} = \nabla_{\mathbf{x}}\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{x} + \nabla_{\mathbf{r}}\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{r} + \varepsilon_{\mathbf{F}} \tag{13}$$

$$\delta\mathbf{y} = \nabla_{\mathbf{x}}\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{x} + \nabla_{\mathbf{r}}\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{r} + \varepsilon_{\mathbf{G}} \tag{14}$$

$$\delta\mathbf{r} = \mathbf{r} - \tilde{\mathbf{r}}, \ \mathbf{y} = \delta\mathbf{y} + \tilde{\mathbf{y}}, \ \mathbf{x} = \delta\mathbf{x} + \tilde{\mathbf{x}} \tag{15}$$

where,

$$\varepsilon_{\mathbf{F}} = \mathbf{F}(\mathbf{x}, \mathbf{r}) - \mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}) - \nabla_{\mathbf{x}}\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{x} - \nabla_{\mathbf{r}}\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{r} \tag{16}$$

$$\varepsilon_{\mathbf{G}} = \mathbf{G}(\mathbf{x}, \mathbf{r}) - \mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}) - \nabla_{\mathbf{x}}\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{x} - \nabla_{\mathbf{r}}\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{r} \tag{17}$$

From Taylor series expansion theory

$$|\varepsilon_{\mathbf{F}}| \le \sigma(|\delta\mathbf{x}| + |\delta\mathbf{r}|)^2, \qquad |\varepsilon_{\mathbf{G}}| \le \sigma(|\delta\mathbf{x}| + |\delta\mathbf{r}|)^2 \tag{18}$$

where σ is a finite positive constant (see, for example Desoer & Vidyasagar 1975 p130). Hence, provided $|\delta\mathbf{x}|$ and $|\delta\mathbf{r}|$ are sufficiently small, the dynamics, (13)-(14), can be approximated by the linear time-varying system

$$\delta\dot{\hat{\mathbf{x}}} = \nabla_{\mathbf{x}}\mathbf{F}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t))\delta\hat{\mathbf{x}} + \nabla_{\mathbf{r}}\mathbf{F}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t))\delta\mathbf{r} \tag{19}$$

$$\delta\hat{\mathbf{y}} = \nabla_{\mathbf{x}}\mathbf{G}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t))\delta\hat{\mathbf{x}} + \nabla_{\mathbf{r}}\mathbf{G}(\tilde{\mathbf{x}}(t), \tilde{\mathbf{r}}(t))\delta\mathbf{r} \tag{20}$$

The system, (19)-(20), is in LPV form (the parameter is an exogenous time-varying quantity independent of the state) and is simply the first-order Taylor series expansion of the nonlinear system, (10), relative to the trajectory, ($\tilde{\mathbf{x}}$(t), $\tilde{\mathbf{r}}$(t), $\tilde{\mathbf{y}}$(t)). The system, (19)-(20), approximates (13)-(14), in the sense that the solution to (19)-(20) approximates the solution to (13)-(14) (see, for example, Khalil 1992 theorem 2.5). Moreover, (13)-(14) is exponentially stable, locally to the trajectory ($\tilde{\mathbf{x}}$(t), $\tilde{\mathbf{r}}$(t), $\tilde{\mathbf{y}}$(t)), if and only if (19)-(20) is stable (see, for example, Khalil 1992 theorem 4.6). Since consideration is confined to a specific trajectory, ($\tilde{\mathbf{x}}$(t), $\tilde{\mathbf{r}}$(t), $\tilde{\mathbf{y}}$(t)), the transformations, (15), are static. Hence, the solution to the nonlinear system, (10), is directly related to the solution to (13)-(14).

Example 1 (cont) As noted above, the equilibrium linearisation relative to a specified equilibrium operating point of the nonlinear system, (1), is given by (2) together with the state, input and output transformations, (3).

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Owing to the requirement that $|\delta\mathbf{x}|$ and $|\delta\mathbf{r}|$ are sufficiently small, the series expansion linearisation, (19)-(20), of the nonlinear system, (10), is only valid within a small neighbourhood about ($\tilde{\mathbf{x}}$(t), $\tilde{\mathbf{r}}$(t), $\tilde{\mathbf{y}}$(t)). This limitation is inherent to the series expansion linearisation and is, of course, well known. Unfortunately, in the context of gain-scheduling it is rarely possible to restrict consideration to a single, isolated, equilibrium operating point or trajectory. Indeed, the existence of a *family* of equilibrium operating points which encompasses the envelope of plant operation is central to most conventional gain-scheduling arrangements (see, for example, Astrom & Wittenmark 1989, Hyde & Glover 1993).

2.2 Quasi-LPV systems

Conventional series expansion linearisation theory does *not* support the reformulation of general nonlinear systems in LPV form without strong restrictions on the operating region. However, following a similar approach to Helmersson (1995 chapter 10) (see also Scherer *et al.* (1997), Apkarian & Adams 1998 section IV), consider the nonlinear quasi-LPV system

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x},\mathbf{r})\mathbf{x} + \mathbf{B}(\mathbf{x},\mathbf{r})\mathbf{r}, \ \mathbf{y} = \mathbf{C}(\mathbf{x},\mathbf{r})\mathbf{x} + \mathbf{D}(\mathbf{x},\mathbf{r})\mathbf{r} \tag{21}$$

where $\mathbf{r} \in \Re^m$, $\mathbf{y} \in \Re^p$, $\mathbf{x} \in \Re^n$ and the input and initial conditions of the state are restricted such that

$\mathbf{r}\in R\subset\Re^m$ and the solution $\mathbf{x}(t)$ is confined to some operating region $X \subset \Re^n$. It is immediately evident that the solutions to the nonlinear system, (21), are a subset of the solutions to the system

$$\dot{\mathbf{x}} = \mathbf{A}(\theta)\mathbf{x}+\mathbf{B}(\theta)\mathbf{r},\ \ \mathbf{y}=\mathbf{C}(\theta)\mathbf{x}+\mathbf{D}(\theta)\mathbf{r} \tag{22}$$

with $\theta\in X\times R$. (Since the parameter θ can vary arbitrarily in (22), the solutions to (21) are just the solutions to (22) associated with particular parameter trajectories). Hence, whilst it is generally not possible to reformulate a nonlinear system as an LPV system, it is possible to over-bound the general nonlinear system, (21), by a quasi-LPV system, (22), in the sense that every solution to the nonlinear system is a solution to the quasi-LPV system (but not *vice versa*). Of course, some degree of conservativeness can be expected with such an approach. Moreover, it still remains to be established whether the class of quasi-LPV systems is sufficiently rich to include a reasonable range of gain-scheduling applications. The series-expansion, pseudo-linear and output-dependent quasi-LPV theory relating to the reformulation of nonlinear systems in quasi-LPV form are reviewed in the following sections.

2.2.1 Families of series expansion linearisations

Since the series expansion linearisation about an equilibrium operating point is only valid in a small operating region about that point, it might be expected that the allowable operating region may be enlarged by combining, in some sense, the series expansion linearisations about a number of equilibrium operating points; that is, it might be expected that, when the solution to the nonlinear system (10) remains within a neighbourhood about the locus of equilibrium operating points (but is no longer confined to the vicinity of a single equilibrium point), it is directly related to the solutions to the members of the series expansion linearisation family

$$\delta\dot{\hat{\mathbf{x}}} = \nabla_x\mathbf{F}(\mathbf{x_o},\mathbf{r_o})\delta\hat{\mathbf{x}} + \nabla_r\mathbf{F}(\mathbf{x_o},\mathbf{r_o})\delta\mathbf{r} \tag{23}$$

$$\delta\hat{\mathbf{y}} = \nabla_x\mathbf{G}(\mathbf{x_o},\mathbf{r_o})\delta\hat{\mathbf{x}} + \nabla_r\mathbf{G}(\mathbf{x_o},\mathbf{r_o})\delta\mathbf{r} \tag{24}$$

$$\delta\mathbf{r} = \mathbf{r} - \mathbf{r_o},\ \ \hat{\mathbf{y}} = \delta\hat{\mathbf{y}} + \mathbf{G}(\mathbf{x_o},\mathbf{r_o}),\ \ \hat{\mathbf{x}} = \delta\hat{\mathbf{x}} + \mathbf{x_o} \tag{25}$$

where $(\mathbf{x_o}, \mathbf{r_o})$ is an equilibrium operating point of the nonlinear system, (10).

However, as noted in the example at the beginning of section 2, it is important to make a clear distinction between an LPV system and the family of linear systems associated with the equilibrium points/trajectories of a nonlinear system. Clearly, the linearisation family is a collection of dynamic systems whilst the LPV system is a single dynamic system. The state, input and output of a series expansion linearisation are perturbation quantities which depend on the equilibrium point/trajectory considered. Hence, the members of the linearisation family each have different state, input and output in general and when the solution to (10) traces a trajectory which is not confined to a neighbourhood about a single equilibrium operating point, the relationship between the solution to the nonlinear system, (10), and the solutions to the linear systems, (23)-(24), is, in fact, no longer straightforward. It is emphasised that the input, output and state transformations, (25), are essential to the relationship between the nonlinear system and its series expansion linearisations and cannot be neglected. Taking account of the input, output and state transformations, the members of the series expansion linearisation family, (23)-(25), may be reformulated as,

$$\dot{\hat{\mathbf{x}}} = -\{\nabla_x\mathbf{F}(\mathbf{x_o},\mathbf{r_o})\mathbf{x_o} + \nabla_r\mathbf{F}(\mathbf{x_o},\mathbf{r_o})\mathbf{r_o}\} + \nabla_x\mathbf{F}(\mathbf{x_o},\mathbf{r_o})\,\hat{\mathbf{x}} + \nabla_r\mathbf{F}(\mathbf{x_o},\mathbf{r_o})\,\mathbf{r} \tag{26}$$

$$\hat{\mathbf{y}} = \{\mathbf{G}(\mathbf{x_o},\mathbf{r_o}) - \nabla_x\mathbf{G}(\mathbf{x_o},\mathbf{r_o})\mathbf{x_o} - \nabla_r\mathbf{G}(\mathbf{x_o},\mathbf{r_o})\mathbf{r_o}\} + \nabla_x\mathbf{G}(\mathbf{x_o},\mathbf{r_o})\,\hat{\mathbf{x}} + \nabla_r\mathbf{G}(\mathbf{x_o},\mathbf{r_o})\,\mathbf{r} \tag{27}$$

The members, of the family of first-order representations, (26)-(27), have the same input, output and state. However, the members of the family, (26)-(27), are *affine* rather than linear. It is emphasised that affine systems are quite distinct from linear ones. In particular, affine systems do not satisfy superposition and, since the inhomogeneous term may be very large, cannot generally be treated as linear systems subject to a small disturbance. The corresponding combined representation is neither an LPV system nor a quasi-LPV system.

Example 1 (cont) Reformulating the equilibrium linearisation, (2)-(3) as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0\\ 2|x_{2o}|x_{2o}\end{bmatrix} + \begin{bmatrix}-1 & 0\\ 1 & -2|x_{2o}|\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}r,\quad y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{28}$$

the input, state and output are now the same at every equilibrium point. However, the system, (6), is affine rather than linear.

Example 2 Consider the nonlinear system

$$\dot{x} = f(x - 10r),\quad y = x \tag{29}$$

where f is twice differentiable with $f(0) = 0$. Systems with dynamics of the form, (29), are encountered in a number of application areas; for example, automotive control, and the system, (29), is employed as a second running example in the sections which follow. The series expansion linearisation associated with the equilibrium point (x_o,r_o) is

$$\delta\dot{x} = \nabla f(0)\delta x - 10\nabla f(0)\delta r,\quad \delta y = \delta x \tag{30}$$

with

$$\delta r = r - r_o,\ x = x_o + \delta x,\ y = x_o + \delta y \tag{31}$$

It should be noted that there exist infinitely many equilibrium points of this system; namely, $\{(x_o,r_o):x_o-10r_o=0,\ r_o\in\Re\}$. Despite the strong nonlinearity of the system, the linearised dynamics, (30), are the *same* at every equilibrium point although the input, state and output transformations, (31), vary. Evidently, the equilibrium linearisations are unable to capture the nonlinear character of this system.

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2.2.2 Higher-order series expansions

The series expansion linearisation of a nonlinear system is obtained by truncating the Taylor series expansion of $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ after the first two terms. Owing to this truncation, the series expansion linearisation representation is only valid locally to a specific trajectory or equilibrium operating point. Provided $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are differentiable sufficiently many times, the neighbourhood within which the series expansion representation is valid can be enlarged by truncating the series expansion later thereby retaining more higher-order terms. For example, Banks & Al-Jurani (1996) consider the unforced nonlinear system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{32}$$

where $\mathbf{x} \in \Re^n$, $\mathbf{F}(\cdot)$ is analytic, and $\mathbf{F}(0) = 0$. By retaining infinitely many higher-order terms in the Taylor series expansion of $\mathbf{F}(\cdot)$ about the origin, the nonlinear system (32) may be reformulated as the quasi-LPV system (referred to by Banks & Al-Jurani (1996) as a pseudo-linear system)

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} \tag{33}$$

provided the initial condition of $\mathbf{x}$ is restricted such that the components of $\mathbf{x}(t)$ are uniformly bounded. Whilst Banks & Al-Jurani (1996) restrict consideration to unforced systems, the extension to forced systems, (13), is straightforward provided $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are analytic and $\mathbf{F}(0,0)=0=\mathbf{G}(0,0)$ (see, for example, Helmersson 1995 chapter 10). The assumption that $\mathbf{F}(0,0)=0=\mathbf{G}(0,0)$ is not restrictive, provided the system has at least one equilibrium operating point, since this requirement can always be satisfied by adding/subtracting a constant from the state, input and output. The requirement that $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are analytic (and therefore infinitely differentiable) is rather stronger. Furthermore, considerable practical difficulties may be associated with this approach. Owing to the difficulty of evaluating the higher-order derivatives of a non-linear function and the sensitivity of polynomial series expansions to errors in the coefficients of the higher order terms, the infinite series approach of Banks & Al Jurani (1996) is impractical and it is almost always necessary, in practice, to employ a truncated series with only a finite number of terms. Indeed, it is frequently necessary to truncate the infinite series after only a relatively small number of terms, particularly for high-order systems with a large number of states, inputs and outputs. Hence, the quasi-LPV representation obtained by this method is, in practice, only valid within a neighbourhood about the origin. Whilst this neighbourhood subsumes that within which the series expansion linearisation is valid, it may nevertheless still be small.

Example 1 (cont) The right-hand side of the differential equation, (1), is only once differentiable: the first derivative is not differentiable at operating points where x_2 equals zero. Hence, conventional higher-order Taylor series expansions do not exist for this system. Moreover, owing to the offset term on the right-hand side of (1), the nonlinearity cannot simply be directly factored as- $(|x_2|+10/x_2)x_2$ since $10/x_2$ is unbounded at the origin.

2.2.3 Reformulation by mean value theorem

In an approach which is closely related to that considered in the previous section, the mean value theorem can be employed to reformulate a general nonlinear system in quasi-LPV form. Consider the nonlinear system,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}) \tag{34}$$

where $\mathbf{r} \in \Re^m$, $\mathbf{x} \in \Re^n$, $\mathbf{F}(\cdot,\cdot)$ is differentiable with bounded, Lipschitz continuous derivatives. It follows from the mean value theorem (see, for example, Boyd *et al.* 1994) that

$$\mathbf{c}^T\left(\mathbf{F}(\mathbf{x},\mathbf{r}) - \mathbf{F}(\bar{\mathbf{x}},\bar{\mathbf{r}})\right) = \mathbf{c}^T\nabla_x\mathbf{F}(\mathbf{z_x},\mathbf{z_r})(\mathbf{x}-\bar{\mathbf{x}}) + \mathbf{c}^T\nabla_\mathbf{r}\mathbf{F}(\mathbf{z_x},\mathbf{z_r})(\mathbf{r}-\bar{\mathbf{r}}) \tag{35}$$

where $\mathbf{c}^T\in\Re^n$ and $(\mathbf{z_x},\mathbf{z_r})$ is a point lying on the line segment in $\Re^n\times\Re^m$ joining $(\mathbf{x},\mathbf{r})$ and $(\bar{\mathbf{x}},\bar{\mathbf{r}})$. Hence, assuming without loss of generality that $\mathbf{F}(0,0)=0$, then

$$\mathbf{c}^T\mathbf{F}(\mathbf{x},\mathbf{r}) = \mathbf{c}^T\nabla_\mathbf{x}\mathbf{F}(\mathbf{z_x},\mathbf{z_r})\mathbf{x} + \mathbf{c}^T\nabla_\mathbf{r}\mathbf{F}(\mathbf{z_x},\mathbf{z_r})\mathbf{r} \tag{36}$$

Since $\mathbf{c}$ can take any value in $\Re^n$, it follows that

$$\dot{\mathbf{x}} = \begin{bmatrix} \left(\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{z}_{\mathbf{x}_1}, \mathbf{z}_{\mathbf{r}_1})\right)_1 \\ \vdots \\ \left(\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{z}_{\mathbf{x}_n}, \mathbf{z}_{\mathbf{r}_n})\right)_n \end{bmatrix}\mathbf{x} + \begin{bmatrix} \left(\nabla_{\mathbf{r}}\mathbf{F}(\mathbf{z}_{\mathbf{x}_1}, \mathbf{z}_{\mathbf{r}_1})\right)_1 \\ \vdots \\ \left(\nabla_{\mathbf{r}}\mathbf{F}(\mathbf{z}_{\mathbf{x}_n}, \mathbf{z}_{\mathbf{r}_n})\right)_n \end{bmatrix}\mathbf{r} \tag{37}$$

where $\left(\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{z}_{\mathbf{x}_i}, \mathbf{z}_{\mathbf{r}_i})\right)_i$ denotes the i^{th} row of $\nabla_{\mathbf{x}}\mathbf{F}(\mathbf{z}_{\mathbf{x}_i}, \mathbf{z}_{\mathbf{r}_i})$ and the $(\mathbf{z}_{\mathbf{x}_i}, \mathbf{z}_{\mathbf{r}_i})$, i=1,..n, are points which lie on the line segment in $\Re^n \times \Re^m$ from (**x**,**r**) to the origin. It should be noted that the points $(\mathbf{z}_{\mathbf{x}_i}, \mathbf{z}_{\mathbf{r}_i})$, i=1,..n, strongly depend, in general, on the value of (**x**,**r**) and so vary as the solution (**x**(t),**r**(t)) to the system evolves. Evidently, the dynamics, (37), are in quasi-LPV form with parameter $\theta = \left[\left[\mathbf{z}_{\mathbf{x}_1}{}^T \;\; \mathbf{z}_{\mathbf{r}_1}{}^T\right] \;\cdots\; \left[\mathbf{z}_{\mathbf{x}_n}{}^T \;\; \mathbf{z}_{\mathbf{r}_n}{}^T\right]\right]^T$.

Unfortunately, whilst applicable to a general class of systems, the reformulation, (37), is essentially an existence result and, in particular, provides little insight regarding a general means for determining the specific value of θ associated with an operating point, (**x**,**r**). LPV control synthesis techniques require that a measurement or estimate of the varying parameter, θ, is available to the controller. In addition, owing to the conservativeness of controllers designed to accommodate arbitrary rates of parameter variation, it is often required that an upper bound can be placed on $\dot{\theta}$. Hence, it is necessary to explicitly establish a mapping from (**x**,**r**) to θ; that is, from (**x**,**r**) to $\left[\left[\mathbf{z}_{\mathbf{x}_1}{}^T \;\; \mathbf{z}_{\mathbf{r}_1}{}^T\right] \;\cdots\; \left[\mathbf{z}_{\mathbf{x}_n}{}^T \;\; \mathbf{z}_{\mathbf{r}_n}{}^T\right]\right]^T$. Unfortunately, this is highly non-trivial in general, particularly for high-order multivariable systems, which greatly diminishes the utility of the formulation, (37). Moreover, it is noted that the interpretation of the linear system obtained by "freezing" the parameter θ in (37) at a particular value is unclear in the sense that no direct relationship exists between the solution to the frozen-parameter linear system and the solution to the nonlinear quasi-LPV system. Indeed, the frozen-parameter system associated with an equilibrium point is quite different from the conventional series expansion linearisation at that point.

2.2.4 Output dependent quasi-LPV systems

LPV synthesis techniques require that a measurement or estimate of the varying parameter, θ, is available and this naturally leads to consideration of systems where θ is one of the outputs. In particular, Shamma & Athans (1992) consider nonlinear systems

$$\begin{bmatrix} \dot{\mathbf{y}} \\ \dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\mathbf{y}}(\mathbf{y}) \\ \mathbf{f}_{\mathbf{x}}(\mathbf{y}) \end{bmatrix} + \begin{bmatrix} \mathbf{A}_{\mathbf{yy}}(\mathbf{y}) & \mathbf{A}_{\mathbf{yx}}(\mathbf{y}) \\ \mathbf{A}_{\mathbf{xy}}(\mathbf{y}) & \mathbf{A}_{\mathbf{xx}}(\mathbf{y}) \end{bmatrix}\begin{bmatrix} \mathbf{y} \\ \mathbf{x} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{\mathbf{y}} \\ \mathbf{B}_{\mathbf{x}} \end{bmatrix}\mathbf{r} \tag{38}$$

where $\mathbf{y} \in \Re^m$, $\mathbf{x} \in \Re^{n-m}$ and the input, $\mathbf{r} \in \Re^m$, has the same dimension as the subset of state, **y**. It is assumed that **r** is uniformly zero at every equilibrium operating point and, in addition, it is assumed that the family of equilibrium operating points are smoothly parameterised by **y**. Under these conditions, (38) may be transformed into the quasi-LPV system

$$\dot{\xi} = \mathbf{A}(\mathbf{y})\xi + \mathbf{B}(\mathbf{y})\mathbf{r} \tag{39}$$

where the matrices **A**(**y**) and **B**(**y**) are appropriately defined, ξ equals $[\mathbf{y} \;\; \mathbf{x}\text{-}\mathbf{x}_o(\mathbf{y})]^T$ and $\mathbf{x}_o(\mathbf{y})$ is the equilibrium value of **x** corresponding to **y** (Shamma & Athans 1992).

Of course, few systems are of the form, (38): in particular, it is restrictive to require that the nonlinearity is dependent only on the quantity, **y**, which parameterises the equilibrium operating points and that the input is uniformly zero at every equilibrium operating point. However, with regard to the former restriction, consider the nonlinear system

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}, \mathbf{r}) \tag{40}$$

where $\mathbf{r} \in \Re^m$, $\mathbf{z} = [\mathbf{y} \;\; \mathbf{x}]^T$ with $\mathbf{y} \in \Re^m$, $\mathbf{x} \in \Re^{n-m}$ and $\mathbf{F}(\cdot,\cdot) = [\mathbf{F}_y(\cdot,\cdot) \;\; \mathbf{F}_x(\cdot,\cdot)]^T$ is differentiable with bounded, Lipschitz continuous derivatives. (It should be noted that, when the output function $\mathbf{G}(\cdot,\cdot)$ does not depend on the input **r**, the nonlinear system, (13), can be reformulated as in (2)). Assume that the family of equilibrium operating points are smoothly parameterised by **y**. Adopting a similar approach to Shamma & Athans (1992) and employing a partial first-order series expansion about the equilibrium operating point, $(\mathbf{x}_o(\mathbf{y}),\mathbf{r}_o(\mathbf{y}))$, the nonlinear system may be reformulated as

$$\begin{bmatrix} \dot{\mathbf{y}} \\ \delta\dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \nabla_x\mathbf{F}_y([\mathbf{y} \;\; \mathbf{x}_o(\mathbf{y})]^T, \mathbf{r}_o(\mathbf{y})) \\ \mathbf{0} & \nabla_x\mathbf{F}_x([\mathbf{y} \;\; \mathbf{x}_o(\mathbf{y})]^T, \mathbf{r}_o(\mathbf{y})) - \nabla\mathbf{x}_o(\mathbf{y})\nabla_x\mathbf{F}_y([\mathbf{y} \;\; \mathbf{x}_o(\mathbf{y})]^T, \mathbf{r}_o(\mathbf{y})) \end{bmatrix}\begin{bmatrix} \mathbf{y} \\ \delta\mathbf{x} \end{bmatrix} + \begin{bmatrix} \nabla_r\mathbf{F}_y([\mathbf{y} \;\; \mathbf{x}_o(\mathbf{y})]^T, \mathbf{r}_o(\mathbf{y})) \\ \nabla_r\mathbf{F}_x([\mathbf{y} \;\; \mathbf{x}_o(\mathbf{y})]^T, \mathbf{r}_o(\mathbf{y})) - \nabla\mathbf{x}_o(\mathbf{y})\nabla_r\mathbf{F}_y([\mathbf{y} \;\; \mathbf{x}_o(\mathbf{y})]^T, \mathbf{r}_o(\mathbf{y})) \end{bmatrix}\delta\mathbf{r} + \begin{bmatrix} \varepsilon_y \\ \varepsilon_x \end{bmatrix} \tag{41}$$

$$\delta\mathbf{x} = \mathbf{x} - \mathbf{x}_o(\mathbf{y}), \; \delta\mathbf{r} = \mathbf{r} - \mathbf{r}_o(\mathbf{y}) \tag{42}$$

where

$$\varepsilon_{\mathbf{y}}=\mathbf{F}_{\mathbf{y}}([\mathbf{y}\ \ \mathbf{x}]^T, \mathbf{r}) - \nabla_x\mathbf{F}_{\mathbf{y}}([\mathbf{y}\ \ \mathbf{x_o}(\mathbf{y})]^T, \mathbf{r_o}(\mathbf{y}))\delta\mathbf{x}-\nabla_r\mathbf{F}_{\mathbf{y}}([\mathbf{y}\ \ \mathbf{x_o}(\mathbf{y})]^T, \mathbf{r_o}(\mathbf{y}))\delta\mathbf{r} \tag{43}$$

$$\begin{aligned}\varepsilon_{\mathbf{x}}=&\mathbf{F}_{\mathbf{x}}([\mathbf{y}\ \ \mathbf{x}]^T, \mathbf{r}) - \nabla\mathbf{x_o}(\mathbf{y})\mathbf{F}_{\mathbf{y}}([\mathbf{y}\ \ \mathbf{x}]^T, \mathbf{r}) - \nabla_x\mathbf{F}_{\mathbf{x}}([\mathbf{y}\ \ \mathbf{x_o}(\mathbf{y})]^T, \mathbf{r_o}(\mathbf{y}))\delta\mathbf{x}-\nabla_r\mathbf{F}_{\mathbf{x}}([\mathbf{y}\ \ \mathbf{x_o}(\mathbf{y})]^T, \mathbf{r_o}(\mathbf{y}))\delta\mathbf{r}\\ &+\nabla\mathbf{x_o}(\mathbf{y})\nabla_x\mathbf{F}_{\mathbf{y}}([\mathbf{y}\ \ \mathbf{x_o}(\mathbf{y})]^T, \mathbf{r_o}(\mathbf{y}))\delta\mathbf{x}+\nabla\mathbf{x_o}(\mathbf{y})\nabla_r\mathbf{F}_{\mathbf{y}}([\mathbf{y}\ \ \mathbf{x_o}(\mathbf{y})]^T, \mathbf{r_o}(\mathbf{y}))\delta\mathbf{r}\end{aligned} \tag{44}$$

Assume that $\mathbf{r}$ is uniformly zero at every equilibrium operating point; that is, $\mathbf{r_o}(\mathbf{y})=0$ and $\delta\mathbf{r} = \mathbf{r}$. It follows that

$$\begin{bmatrix}\dot{\mathbf{y}}\\ \delta\dot{\mathbf{x}}\end{bmatrix}=\begin{bmatrix}\mathbf{0} & \nabla_{\mathbf{x}}\mathbf{F}_y([\mathbf{y}\ \mathbf{x}_o(\mathbf{y})]^T,0)\\ \mathbf{0} & \nabla_{\mathbf{x}}\mathbf{F}_x([\mathbf{y}\ \mathbf{x}_o(\mathbf{y})]^T,0)-\nabla\mathbf{x}_o(\mathbf{y})\nabla_{\mathbf{x}}\mathbf{F}_y([\mathbf{y}\ \mathbf{x}_o(\mathbf{y})]^T,0)\end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \delta\mathbf{x}\end{bmatrix}+\begin{bmatrix}\nabla_{\mathbf{r}}\mathbf{F}_y([\mathbf{y}\ \mathbf{x}_o(\mathbf{y})]^T,0)\\ \nabla_{\mathbf{r}}\mathbf{F}_x([\mathbf{y}\ \mathbf{x}_o(\mathbf{y})]^T,0)-\nabla\mathbf{x}_o(\mathbf{y})\nabla_{\mathbf{r}}\mathbf{F}_y([\mathbf{y}\ \mathbf{x}_o(\mathbf{y})]^T,0)\end{bmatrix}\mathbf{r}+\begin{bmatrix}\varepsilon_y\\ \varepsilon_x\end{bmatrix} \tag{45}$$

Neglecting the perturbation terms $\varepsilon_{\mathbf{y}}$ and $\varepsilon_{\mathbf{x}}$, it is evident that (3) is of the quasi-LPV form (39). From Taylor series expansion theory, the perturbation terms $\varepsilon_{\mathbf{y}}$ and $\varepsilon_{\mathbf{x}}$ can be made arbitrarily small provided the magnitudes of $\mathbf{r}$ and $\mathbf{x}$-$\mathbf{x_o}(\mathbf{y})$ are sufficiently small. Hence, provided $|\mathbf{r}|$ and $|\mathbf{x}\text{-}\mathbf{x_o}(\mathbf{y})|$ are sufficiently small, the solution to the nonlinear system, (2), is approximated by the solution to quasi-LPV system obtained by neglecting $\varepsilon_{\mathbf{y}}$ and $\varepsilon_{\mathbf{x}}$ in (3). It should be noted that the matrices, $\mathbf{A}(\mathbf{y})$ and $\mathbf{B}(\mathbf{y})$, in the quasi-LPV system do *not* correspond to the series expansion linearisations, about the family of equilibrium operating points, of the nonlinear system, (2).

The foregoing analysis relaxes the requirement that the nonlinearity is dependent solely on the quantity, $\mathbf{y}$, which parameterises the equilibrium operating points. However, this is achieved at the cost of restricting consideration to the class of inputs and initial conditions for which $|\mathbf{r}|$ and $|\mathbf{x}\text{-}\mathbf{x_o}(\mathbf{y})|$ are sufficiently small. Hence, any analysis based on this quasi-LPV formulation is strictly local in nature. It should be noted that, from (3), restricting $|\mathbf{r}|$ and $|\mathbf{x}\text{-}\mathbf{x_o}(\mathbf{y})|$ imposes an implicit constraint on the rate of variation of the states, $\mathbf{y}$ and $\mathbf{x}$. This constraint may, in general, be extremely restrictive since the region of validity can be vanishingly small.

Moreover, the requirement that $\mathbf{r}$ is uniformly zero at the equilibrium operating points is still necessary since it ensures that the input transformation, (42), associated with the series expansion is trivial and, in particular, does not vary with the equilibrium operating point. Indeed, this condition is central to both the approach of Shamma & Athans (1992) and to the foregoing quasi-LPV reformulation. (When this condition is not satisfied, the quasi-LPV approximation is only accurate about the specific equilibrium operating point at which $\mathbf{r_o}$ is zero). In the specific control design situation where the nonlinear system to be reformulated is the plant and the controller output is the only input to the plant and the controller contains pure integral action, this condition can be satisfied by formally including the controller integral action within the plant so that the input, $\mathbf{r}$, to the augmented plant is zero in equilibrium. It should be noted, however, that in a more general context the requirement that the input is uniformly zero in equilibrium is rather restrictive; for example, when the input, $\mathbf{r}$, consists of command signals and/or disturbances.

It should be noted that the requirement in the foregoing analysis that there exists a sub-set, $\mathbf{y}$, of the state which parameterises the equilibrium points can be relaxed. However, the quasi-LPV system obtained is then only valid in the vicinity of a *single* equilibrium point; see example 2 below.

Example 2 (cont) Consider the nonlinear system, (29). The input, r, is generally not zero at the equilibrium points. Nevertheless, following the approach of Shamma (1988), Shamma & Athans (1992), assume that there exists a controller containing pure integral action associated with (29). Formally augmenting (29) with the pure integrator from the controller, it follows that the input, z, to the augmented system is zero at every equilibrium point. In addition, further augmenting the system with an output, $\tilde{y}$, having dynamics $\dot{\tilde{y}} = \dot{x} - 10\dot{r}$ and initial condition $\tilde{y}(0)=x(0)-10r(0)$, it follows that (29) may be reformulated as

$$\begin{bmatrix}\dot{\tilde{y}}\\ \dot{x}\\ \dot{r}\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\tilde{y}\\ x\\ r\end{bmatrix} + \begin{bmatrix}f(\tilde{y})\\ f(\tilde{y})\\ 0\end{bmatrix} + \begin{bmatrix}-10\\ 0\\ 1\end{bmatrix}z \tag{46}$$

It can be seen that the dynamics, (46), are similar in form to (38) and the input to the system is zero in equilibrium. However, the output state $\tilde{y}$ is uniformly zero at the equilibrium points and the equilibrium points are, consequently, *not* parameterised by $\tilde{y}$ as required in the approach of Shamma (1988), Shamma & Athans (1992). It is clear that (29) cannot be reformulated in output-dependent quasi-LPV form except perhaps in a region about the equilibrium points within which the magnitude of $\tilde{y}$ is sufficiently small. This region may, depending on the characteristics of the function f, be vanishingly small.

■

3. Velocity-based representations

Nonlinear systems of the particular form, (38), and nonlinear systems belonging to the specific class for which the infinite series expansion can be expressed in closed-form, may be reformulated as quasi-LPV systems. However, these classes of nonlinear system are rather restrictive in comparison to those permitted in other control design approaches. Representations of more general nonlinear systems, (10), in either LPV or quasi-LPV form are strictly local in nature and confined to a small neighbourhood about a specific trajectory, equilibrium operating point or family of equilibrium operating points. Hence, present theory does *not* support the representation of general nonlinear dynamic systems in LPV or quasi-LPV form without, in general, substantial restrictions either on the class of nonlinear systems considered or on the allowable operating region. The potential significance of this observation is obviously considerable in the context of LPV gain-scheduling design methods. It should be noted, however, that it is unclear whether such restrictions are simply associated with the limitations of the present theory or whether they are inherent to the LPV/quasi-LPV formulation.

Consider the reformulation of the nonlinear system, (10), in quasi-LPV form. The restrictions on the operating region essentially arise from the limitations of series expansion linearisation theory. In particular, the conventional series expansion of a nonlinear system is, in general, linear only when the expansion is carried out relative to a specific equilibrium operating point (or trajectory). Recently, Leith & Leithead (1998a,b) have developed a velocity-based analysis framework which associates a linear system with every operating point of a nonlinear system, rather than just the equilibrium operating points. In the present context, the velocity-based linearisation approach therefore clearly has the potential to provide insight into the representation of nonlinear systems in quasi-LPV form.

Before proceeding, it is useful to reformulate the nonlinear system, (10), as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{r} + \mathbf{f}(\rho), \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{r} + \mathbf{g}(\rho) \tag{47}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are appropriately dimensioned constant matrices, $\mathbf{f}(\bullet)$ and $\mathbf{g}(\bullet)$ are nonlinear functions and $\rho(\mathbf{x},\mathbf{r})\in\Re^q$, $q\le m+n$, embodies the nonlinear dependence of the dynamics on the state and input with $\nabla_{\mathbf{x}}\rho$, $\nabla_{\mathbf{r}}\rho$ functions of ρ alone. Trivially, this reformulation can always be achieved by letting $\rho = [\mathbf{x}^T \ \ \mathbf{r}^T]^T$, in which case $q=m+n$. However, the nonlinearity of the system is frequently dependent on only a subset of the states and inputs, in which case the dimension, q, of ρ is less than $m+n$. Since $\nabla_{\mathbf{x}}\rho$, $\nabla_{\mathbf{r}}\rho$ are functions of ρ alone, the variable, $\rho(\mathbf{x},\mathbf{r})$, equals the constant value, ρ_1, upon a surface of co-dimension q in Φ and $\nabla_{\mathbf{x}}\rho$ and $\nabla_{\mathbf{r}}\rho$ are constant over each surface. Hence, the normal to each surface is identical at every point on the surface and each surface is, therefore, affine. Moreover, to ensure that ρ is a unique function of $\mathbf{x}$ and $\mathbf{r}$, these surfaces must be parallel for all ρ. Consequently, it may in fact be assumed, without loss of generality, that $\nabla_{\mathbf{x}}\rho$ and $\nabla_{\mathbf{r}}\rho$ are constant.

Differentiating (47), an alternative representation of the nonlinear system is

$$\dot{\rho} = \nabla_{\mathbf{x}}\rho\mathbf{w} + \nabla_{\mathbf{r}}\rho\ \dot{\mathbf{r}} \tag{48}$$

$$\dot{\mathbf{w}} = (\mathbf{A}+\nabla\mathbf{f}(\rho)\,\nabla_{\mathbf{x}}\rho\,)\mathbf{w} + (\mathbf{B}+\nabla\mathbf{f}(\rho)\,\nabla_{\mathbf{r}}\rho\,)\,\dot{\mathbf{r}} \tag{49}$$

$$\dot{\mathbf{y}} = (\mathbf{C}+\nabla\mathbf{g}(\rho)\,\nabla_{\mathbf{x}}\rho\,)\mathbf{w} + (\mathbf{D}+\nabla\mathbf{g}(\rho)\,\nabla_{\mathbf{r}}\rho\,)\,\dot{\mathbf{r}} \tag{50}$$

Dynamically, (48)-(50), with appropriate initial conditions, and (10) are equivalent (have the same solution, $\mathbf{x}$). While (47) and (51)-(53) are equivalent in the sense that they both embody the dynamics of the nonlinear system, they are not equivalent in other respects. In particular, the velocity representation, (48)-(50), may be trivially reformulated as the quasi-LPV system

$$\dot{\rho} = \nabla_{\mathbf{x}}\rho\mathbf{w} + \nabla_{\mathbf{r}}\rho\,\mathbf{z}, \quad \dot{\mathbf{r}} = \mathbf{z} \tag{51}$$

$$\dot{\mathbf{w}} = (\mathbf{A}+\nabla\mathbf{f}(\rho)\,\nabla_{\mathbf{x}}\rho\,)\mathbf{w} + (\mathbf{B}+\nabla\mathbf{f}(\rho)\,\nabla_{\mathbf{r}}\rho\,)\mathbf{z} \tag{52}$$

$$\dot{\mathbf{y}} = (\mathbf{C}+\nabla\mathbf{g}(\rho)\,\nabla_{\mathbf{x}}\rho\,)\mathbf{w} + (\mathbf{D}+\nabla\mathbf{g}(\rho)\,\nabla_{\mathbf{r}}\rho\,)\mathbf{z} \tag{53}$$

where $\mathbf{z}$ is the input to the transformed system. Hence, it follows immediately that every nonlinear system, (47), (and so every nonlinear system, (10)) can be reformulated as an quasi-LPV system, (51)-(53) to which the developed LPV control design methods may be brought to bear. When ρ depends only on the input, $\mathbf{r}$, to the system (51)-(53) is an LPV rather than quasi-LPV system. When $\mathbf{w} = \mathbf{A}\mathbf{x}+\mathbf{B}\,\mathbf{r}+\mathbf{f}(\rho)$, $\mathbf{y} = \mathbf{C}\mathbf{x}+\mathbf{D}\mathbf{r}+\mathbf{g}(\rho)$ is invertible for every $(\mathbf{x}, \mathbf{r})$, so that $\mathbf{x}$ may be expressed as a function of $\mathbf{w}$, $\mathbf{r}$ and $\mathbf{y}$, then the transformation relating (51)-(53) to (47) is *algebraic*. In contrast to previous approaches, the reformulation, (51)-(53), is valid for a very general class of nonlinear systems. Moreover, it is emphasised that the velocity-based LPV/quasi-LPV representation is valid globally with no restriction whatsoever to a neighbourhood of the equilibrium operating points.

The relationship between (48)-(50) and (47) is evidently direct and it is argued that this directness is, in fact, a significant strength of the approach. Moreover, the directness of the relationship extends rather more deeply than might initially be expected. Consider the linear system, obtained by "freezing" (48)-(50) at an operating point at which ρ equals ρ_1,

$$\dot{\hat{\mathbf{w}}} = (\mathbf{A}+\nabla\mathbf{f}(\rho_1)\,\nabla_{\mathbf{x}}\rho\,)\,\hat{\mathbf{w}} + (\mathbf{B}+\nabla\mathbf{f}(\rho_1)\,\nabla_{\mathbf{r}}\rho\,)\,\dot{\mathbf{r}} \tag{54}$$

$$\dot{\hat{\mathbf{y}}} = (\mathbf{C}+\nabla\mathbf{g}(\rho_1)\,\nabla_{\mathbf{x}}\rho\,)\,\hat{\mathbf{w}} + (\mathbf{D}+\nabla\mathbf{g}(\rho_1)\,\nabla_{\mathbf{r}}\rho\,)\,\dot{\mathbf{r}} \tag{55}$$

The system (54)-(55) is referred to as the velocity-based linearisation of (47) associated with the operating point. It may be shown that when $\hat{\mathbf{w}}(t_1) = \mathbf{w}(t_1)$, $\hat{\mathbf{y}}(t_1) = \mathbf{y}(t_1)$ then the solutions to the linear system (54)-(55) are an accurate approximation to the solutions of the nonlinear system, (47), locally to the operating point (Leith & Leithead 1998a). Furthermore, while the solution to an individual velocity-based linearisation is only a locally accurate approximation, there exists a velocity-based linearisation, (54)-(55), for every operating point (**x**,**r**) and thus a velocity-based linearisation family, with members defined by (54)-(55), can be associated with the nonlinear system, (47). The solutions to the members of the family of velocity-based linearisations may be pieced together to approximate the solution to the nonlinear system (47) to an arbitrary degree of accuracy (Leith & Leithead 1998a). It is emphasised that, unlike conventional series expansion linearisation approaches, no restriction to near equilibrium operation is involved. This direct relationship between the nonlinear quasi-LPV systems and the linear system obtained by simply "freezing" the system at a particular parameter value is an important aspect of the velocity-based quasi-LPV formulation in the context of gain-scheduling.

Example 1 (cont) The nonlinear system, (1), can be reformulated by differentiating, as

$$\dot{\rho} = w_2, \quad \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2|\rho| \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dot{r}, \quad \dot{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{56}$$

The system, (56), is in quasi-LPV form and existing LPV control design methods may be brought to bear. Of course, practical issues associated with the increased order of (56) in comparison to (1) and the presence of the derivative operator at the input to the reformulated plant remain to be adequately resolved. With regard to the order of the quasi-LPV representation, it is noted from (1) that

$$\begin{bmatrix} w_1 - r \\ w_2 \end{bmatrix} = T(x_1, x_2) = \begin{bmatrix} -1 & 0 \\ 1 & -|x_2| \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{57}$$

Hence, the **x** and **w** states are related by a non-singular *algebraic* mapping with $x_2=T_2^{-1}(w_1\text{-}r,w_2)$. The velocity-based system, (56), may therefore be reformulated as the reduced-order system

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2|T_2^{-1}(w_1 - r, w_2)| \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dot{r}, \quad \dot{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{58}$$

With regard to the derivative operator at the plant input, it is noted that the steady-state performance specification requires that the controller contain pure integral action. Hence, by explicitly partitioning the controller into a pure integral term plus additional dynamics, say C_o, the velocity-based control loop depicted in figure 2a may, for design purposes, be reformulated as shown in figure 2b. The design task may now proceed by determining quasi-LPV controller dynamics, C_o, which achieve the required performance when applied with the reformulated plant augmented with an integrator at the output. The controller for the original nonlinear system, (1), is realised as the dynamics, C_o, followed by a pure integrator (see figure 2a). Using standard design software from the MATLAB LMI toolbox and an L_2 objective function with performance weighting w_1 and weighting w_2 on the control input r, as defined by (6), the controller dynamics, C_o, obtained for this system is

$$\dot{\mathbf{x}}_c = \mathbf{A}_c(\theta)\mathbf{x}_c + \mathbf{B}_c(y_{ref} - y), \quad r = \mathbf{C}_c(\theta)\mathbf{x} \tag{59}$$

where $\mathbf{A_c}(\theta)=\alpha\mathbf{A}_o+(1\text{-}\alpha)\mathbf{A}_1$, $\mathbf{C_c}(\theta)=\alpha\mathbf{C}_o+(1\text{-}\alpha)\mathbf{C}_1$, $\alpha=(10\text{-}\theta)/10$,

$$\mathbf{A}_o = \begin{bmatrix} 1.7058e+000 & 5.2402e-002 & -4.4428e-001 & 9.5434e-001 & 3.1461e+001 \\ -2.3313e+004 & -2.3724e+001 & 3.6040e+003 & -1.0898e+002 & -4.0914e+005 \\ 2.8731e+003 & 1.0562e+001 & -1.4210e+003 & 1.6039e+002 & 5.0429e+004 \\ 7.6071e+002 & -1.3892e+000 & 2.9756e+001 & -1.8557e+001 & 1.3349e+004 \\ -4.2678e+001 & 1.9619e-002 & -1.3072e+000 & -1.2245e-001 & -7.5074e+002 \end{bmatrix}, \mathbf{A}_1 = \begin{bmatrix} 1.7056e+000 & -2.3312e+004 & 2.8729e+003 & 7.6071e+002 & -4.2678e+001 \\ 5.6886e-002 & -5.5308e+001 & 1.4298e+001 & -1.3594e+000 & 1.9330e-002 \\ -4.3108e-001 & 3.5110e+003 & -1.4100e+003 & 2.9842e+001 & -1.3079e+000 \\ 1.0419e+000 & -7.2559e+002 & 2.3332e+002 & -1.7976e+001 & -1.2810e-001 \\ 3.1461e+001 & -4.0914e+005 & 5.0429e+004 & 1.3349e+004 & -7.5074e+002 \end{bmatrix} \tag{60}$$

$$\mathbf{B}_c = \begin{bmatrix} 2.7342e-002 \\ -1.5454e-005 \\ -3.6988e-005 \\ -2.8850e-003 \\ -4.9850e+000 \end{bmatrix}, \mathbf{C}_0 = \begin{bmatrix} -2.3321e+004 \\ -2.2619e+001 \\ 3.5972e+003 \\ -1.0787e+002 \\ -4.0927e+005 \end{bmatrix}^T, \mathbf{C}_1 = \begin{bmatrix} -2.3319e+004 \\ -5.4161e+001 \\ 3.5044e+003 \\ -7.2368e+002 \\ -4.0927e+005 \end{bmatrix}^T$$

The response of the nonlinear closed-loop system to a step change in demand from –3.16 units to 0 units is shown in figure 3. In contrast to the results obtained with an *ad hoc* quasi-LPV reformulation approach, it can be seen that the closed-loop system is stable and achieves the required performance (similar responses are also obtained for other magnitudes of step demand).

Example 2 (cont) Consider the nonlinear system, (29), which can be globally reformulated, by differentiating, as

$$\begin{aligned} \dot{w} &= \nabla f(y - r)w - \nabla f(y - r)\dot{r} \\ \dot{y} &= w \end{aligned} \tag{61}$$

The velocity-based formulation, (61), is evidently in quasi-LPV form.

■

4. Conclusion

The study of LPV gain-scheduling methods is currently the subject of great interest in the literature. These methods are related by the use of various types of LPV/quasi-LPV representation. However, on the face of it, few nonlinear systems are of the required form. Quite a number of approaches have been proposed in the literature for transforming a nonlinear system into a suitable parameter-varying form, including first-order series expansions, families of first-order series expansions, higher-order expansions/pseudo-linear systems, approaches based on the mean-value theorem, reformulation of output-dependent quasi-LPV systems. However, in the first part of this note it is demonstrated here that the results obtained with these approaches do *not* generally support the representation of nonlinear dynamic systems in LPV/quasi-LPV form without considerable restrictions (i) on the class of nonlinear systems considered and/or (ii) on the allowable operating region. The potential significance of this observation in the context of LPV gain-scheduling design methods is clear.

In the second part of this note, it is shown that every smooth nonlinear system of the form $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r})$, $= \mathbf{G}(\mathbf{x},\mathbf{r})$ can, by adopting the velocity-based framework of Leith & Leithead (1998a), indeed be transformed into LPV/quasi-LPV form. This velocity-based formulation resolves the deficiencies of previous approaches in the sense that

- the reformulation is valid for a very general class of nonlinear systems
- the parameter-varying representation obtained is valid *globally* with no restriction whatsoever to a neighbourhood of the equilibrium operating points.

The velocity-based quasi-LPV reformulation does not, of course, alter the dynamics of the system concerned. However, by transforming a nonlinear system into quasi-LPV, it enables established LPV gain-scheduling design methods to be brought to bear on the nonlinear control design task (a simple design example is also presented). Since the velocity-based quasi-LPV representation is valid globally, controllers synthesised by LPV methods can be designed to be valid globally. Hence, the inherent restriction to near equilibrium operation associated with conventional gain-scheduling methods (see, for example, Leith & Leithead 1998b) is avoided. Furthermore, an important aspect of the velocity-based quasi-LPV formulation in the context of gain-scheduling is that a direct relationship exists between the nonlinear quasi-LPV systems and the linear system obtained by simply "freezing" the system at a particular parameter value. Specifically, the solution to the linear system approximates, to second order, the solution to the nonlinear system locally to the operating point associated with the parameter value considered. This relationship is valid at *every* operating point, including those far from equilibrium, not just the equilibrium points and provides a rigorous basis for inferring the dynamic characteristics of velocity-based quasi-LPV system from those of its associated family of frozen-parameter linear systems.

The utility of velocity-based methods is, of course, not confined to LPV gain-scheduling approaches but also extends, for example, to other gain-scheduling approaches.

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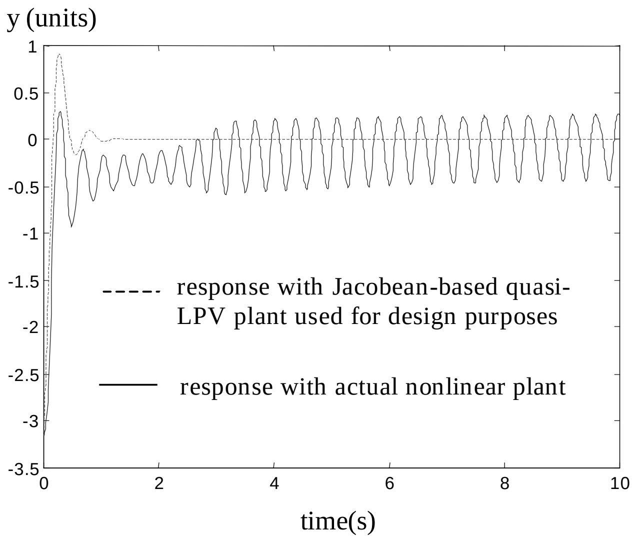

Figure 1 Nonlinear step responses for Jacobean-based quasi-LPV controller.

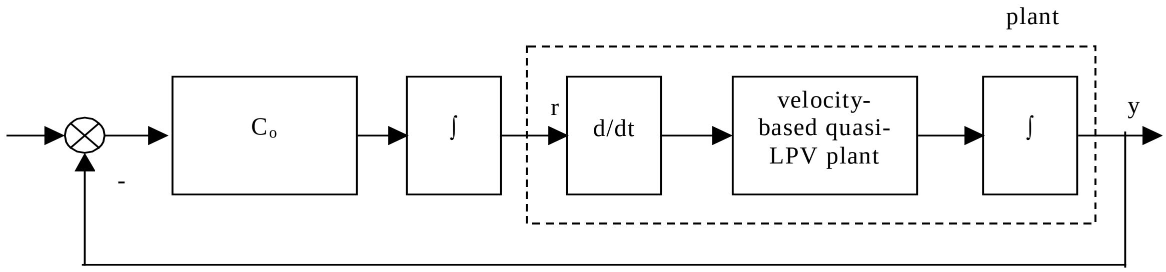

Figure 2a Velocity-based quasi-LPV control loop

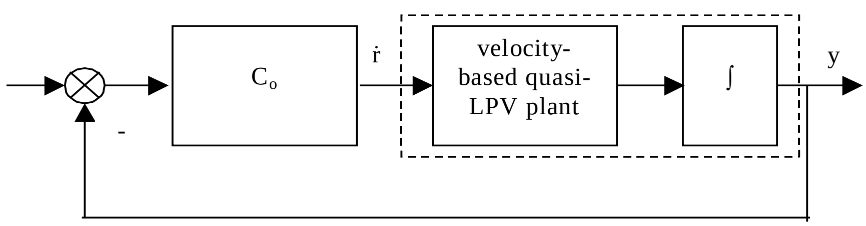

Figure 2b Reformulated velocity-based quasi-LPV control loop

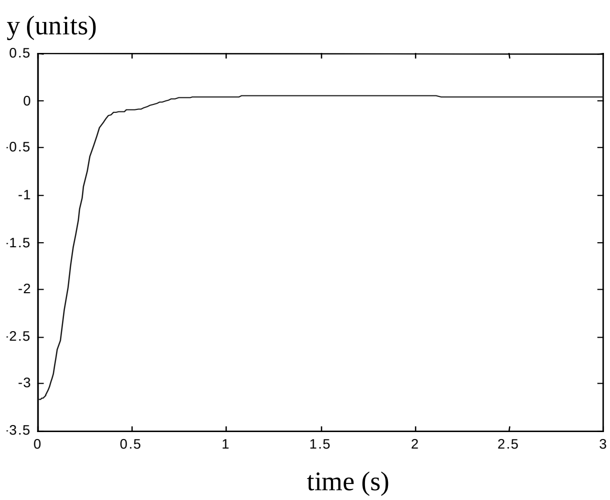

Figure 3 Nonlinear step response with velocity-based quasi-LPV controller.