On structurally constrained $\mathcal{H}_2$ performance bounds for stable MIMO plant models

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December 3, 2006

Abstract

This paper is about optimal control problems in which the controller must satisfy sparsity structure constraints. Conditions are derived under which the optimal controller associated to an unconstrained quadratic performance index is naturally structured and, as a consequence, the sparsity constraint imposed on the controller has no impact on the optimal loop performance. The results are then applied to study the control of triangular plants that have to be controlled by triangular controllers. We derive explicit characterisations of the Youla parameter that defines an optimal triangular controller and of the performance loss associated to the structural restriction imposed on the controller.

1 Introduction

Over the years, many control system design techniques have been established to deal with multi-input multi-output (MIMO) systems [1, 2]. The vast majority of these methodologies lead to controllers that do not have any specific structure. However, due to practical issues such as implementation, tuning

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or insight, it is often required that the control law belongs to a specific class of restricted structure controllers. This structural restrictions include sparsity constraints [3, 4, 5, 6, 7, 8, 9], communication pattern constraints [10, 11] and controller complexity restrictions [12].

In this paper, we consider the control of square stable discrete-time linear MIMO plants. Within this framework, our focus is on sparse-constrained controllers. In this class of controllers, the pattern of dependencies of control signals on measurements is fixed, that is, each control input depends only on a fixed subset of the available measurements. This class of constrained controllers originates in two sources: either the plant model structure induces the controller constraints, or the control designer forces the structure to accommodate design and tuning strategies. In most cases those two aspects are related to the predominant dynamic interactions in the plant, as discussed in [7].

Constraining the controller structure has an impact on loop performance. This performance can be analysed from the perspective of performance bound computation [13]. The vast majority of the available results consider quadratic performance measures and assume no special structure for the controller, apart from forcing it to be linear and time-invariant (LTI), see e.g. [14, 15, 16, 17, 18] and the references therein. In those references, the main conclusions are that, unless control effort penalisation is considered, the achievable performance is limited exclusively by plant NMP zeros, unstable poles, delays and their directionality properties. If control effort penalisation is considered, then other plant features play relevant roles [16]. In all the mentioned references, the problem hinges on the Youla parameterisation (see [19, 1, 20]) of all stabilising controllers for the considered class of plants.

Performance assessment and sparse-restricted controllers can be brought together through the following question: *what are the achievable performance bounds in the control of a MIMO plant, when the controller must satisfy certain sparsity constraints?* We deal with this question in two stages: the first part of this paper focuses on the conditions under which the solution of a quadratic performance bound evaluation problem leads to sparse-restricted controllers *per se* and therefore, there is no additional performance penalisation when restricting the structure of the controller. In the second part of this paper, we use those results to effectively compute a performance bound for triangular plant models, and triangular sparsity constraints on the controller, for those cases in which the conditions that guarantee that the unrestricted controller is naturally triangular are not met.

The main difficulty to achieve our goal relates to the fact that, in general, imposing sparsity constraints on the controller leads to non convex optimisation problems [9] and hence, the use of numerical search algorithms [21, 22, 9] or merely approximations to the problem seems inescapable [23, 24, 25]. To the best of the authors' knowledge, the only reported analytical results regarding sparse constrained optimal controllers, are those in [26]. In that work, the author solves the minimum variance control problem subject to the condition of complete decentralised (i.e., block diagonal) controllers. There are, however, several limitations in that approach, since loop stability is neither ensured nor explored. Nevertheless, the work [3] studies several (not necessarily sparse) controller structures and it is shown that, for certain plant structures, the optimisation problem turns to be convex. It must be pointed out, however, that the approach considered in [3] assumes that the plant model has the desired controller structure. If this condition is met, then the Youla parameter inherits the structural constraints on the controller for several (not necessarily sparse) structures of interest. Further extensions can be found in [27, 28].

A more general framework is presented in [29, 4, 5, 11, 6]. In that work, the authors define the notion of Quadratic Invariance (QI) of a (not necessarily sparse) structural constraint under a given plant model. QI is a necessary and sufficient algebraic condition for the Youla parameter to inherit the desired controller structure, and is the most general known condition under which it is possible to recast structure restricted optimal control problems as convex optimisation problems.

The previous discussion leads us to the first contribution of this paper: we show that, in several cases of interest, QI is *equivalent* to considering plant models with the desired controller structure. In particular, we show that this holds if the structure restriction is invariant under inversion. This allows one to conclude that the condition of plant models having the desired controller structure, considered to be sufficient in [3] is, indeed, necessary for many cases of interest.

As a second contribution, we establish precise conditions under which the optimal controller, associated to a quadratic unconstrained performance bound evaluation problem, has the same sparse structure of the plant model. For this to happen, it turns out that the structure of the interactor matrix of the plant becomes a key element (see [30, 31, 32, 33, 17, 34] and the references therein). The second contribution has practical implications, since it allows one to assess, prior to controller tuning, control structures that could yield good performance control loops. This is illustrated through an example that considers serial processes (see, e.g., [35]).

A third major contribution of this paper relates to performance bounds evaluation for the case of stable triangular plants and controllers subject to triangular sparsity constraints. In the cases in which the plant does not satisfy the conditions that guarantee that the unrestricted optimal controller is naturally triangular, we provide an explicit characterisation of the Youla parameter that defines an optimal triangular controller. Moreover, we provide an explicit characterisation of the performance loss associated to the structural restriction imposed on the controller. This analytic characterisation shows that, for the considered functional, the best achievable performance is limited not only by the plant NMP zeros and delays (as in the unrestricted case), but also by certain directionality features that are not present in the unrestricted case.

The paper is organised as follows: Section 2 introduces the notation and main assumptions considered in this paper. Section 3 provides a precise definition of the problems we are interested in, and motivates the subsequent sections. Section 4 is about conditions under which the Youla parameter inherits the controller structure. Section 5 reviews properties of transfer matrix zeros and interactor matrices, that will prove useful for the rest of the paper. Section 6 deals with structural properties of an optimal performance controller. Section 7 focuses on performance bound evaluation for triangular plants and controller subject to triangular sparsity constraints. Finally, in Section 8 we present an application of the results to serial processes and in Section 9 conclusions are drawn.

2 Preliminaries

This section introduces the notation and main assumptions made throughout the paper.

2.1 Notation

In this paper we use **bold face** for vectors and matrices; normal face is used only for scalars.

We denote by $\mathcal{R}$ the set of $n \times n$ real rational matrices in the complex variable z. $\boldsymbol{A}(z) \in \mathcal{R}$ is said to be proper if and only if $\lim_{z\to\infty} \boldsymbol{A}(z)$ is well defined in $\mathbb{R}^{n\times n}$. If $\boldsymbol{A}(z) \in \mathcal{R}$ is not proper, it is called improper (and *vice versa*). $\boldsymbol{A}(z) \in \mathcal{R}$ is said to be biproper (resp. strictly proper) if and only if $\lim_{z\to\infty} \boldsymbol{A}(z)$ is non singular (resp. $\lim_{z\to\infty} \boldsymbol{A}(z) = \boldsymbol{0}$).

For any $\boldsymbol{A}(z) \in \mathcal{R}$, $[\boldsymbol{A}(z)]_{ij} \triangleq A_{ij}(z)$ denotes its $(i,j)^{th}$ element and $[\boldsymbol{A}(z)]_{i*}$ denotes its i^{th} row.

We also define its ℓ^{th} *submatrix* as $\boldsymbol{A_\ell}(z) \in \mathcal{R}^{(n-\ell+1)\times(n-\ell+1)}$ such that

$$[\boldsymbol{A_\ell}(z)]_{ij} = A_{(i+\ell-1)(j+\ell-1)}(z), \qquad \forall i \leq n-\ell+1\,, \forall j \leq n-\ell+1. \tag{1}$$

According to this definition, $\boldsymbol{A_1}(z) = \boldsymbol{A}(z)$ and $\boldsymbol{A_n}(z) = A_{nn}(z)$.

As usual (see, e.g., [36]), $\boldsymbol{A}(z) \in \mathcal{R}$ is said to be non singular almost everywhere (a.e.) if and only if $\boldsymbol{A}(c) \in \mathbb{C}^{n\times n}$ has full rank for all $c \in \mathbb{C}$, except at isolated points (these points are the zeros of $\boldsymbol{A}(z)$ [36, 37]). If $\boldsymbol{A}(z) \in \mathcal{R}$ is non singular a.e., then $\boldsymbol{A}(z)^{-1} \in \mathcal{R}$ is well defined except at the zeros of $\boldsymbol{A}(z)$. A relevant fact, that derives from the consideration of rational transfer matrices, is that if there exist one $c \in \mathbb{C}$ such that $\boldsymbol{A}(c)$ is non singular, then $\boldsymbol{A}(z)$ is non singular a.e. We stress that the fact that a matrix is non singular a.e. is a pure algebraic property that only guarantees that its inverse is well defined.

We denote the transpose and hermitian operators by $(\cdot)^T$ and $(\cdot)^H$, respectively.

$\boldsymbol{A}(z) \in \mathcal{R}$ is said to be unitary if and only if $\boldsymbol{A}(z^{-1})^T\boldsymbol{A}(z) = I$, for all z.

The spaces $\mathcal{L}_2$ and $\mathcal{H}_\infty$ are defined as usual [38]. To restrict each of this spaces to the $n \times n$ real rational case, we will add the prefix $\mathcal{R}$. Therefore, $\mathcal{RH}_\infty$ turns out to be the subset of $\mathcal{R}$ formed by all proper and stable transfer matrices. The norm in $\mathcal{L}_2$ is called 2-norm and is denoted by $||\cdot||_2$.

For any $x \in \mathbb{C}$, $|x|$ represents its module and $\bar{x}$ its conjugate. Given any constant vector, $||\cdot||$ denotes its Euclidean norm. We define $\boldsymbol{e_i^n}$ as the i^{th} elementary vector of the canonical base of $\mathbb{R}^n$. Additionally, we introduce the $n \times n$ null matrix $\boldsymbol{0_n} \in \mathbb{R}^{n\times n}$ and the special matrix $\boldsymbol{1_{ii}} \in \mathbb{R}^{n\times n}$ that has a one on its $(i,i)^{th}$ entry and a zero in all the others. For the sake of clarity, the dimensions of the identity matrix will be specified, when necessary, with a subscript n as in $\boldsymbol{I_n} \in \mathbb{R}^{n\times n}$.

2.2 General setup and assumptions

This paper focuses on square proper stable discrete time LTI plant models, i.e., we consider plants whose transfer matrix, $\boldsymbol{G}(z)$, satisfies $\boldsymbol{G}(z) \in \mathcal{RH}_\infty$. For the control of these plants, we consider the closed loop depicted in Figure 1. In that figure, $\boldsymbol{r}$ is the reference signal, $\boldsymbol{y}$ is the plant output, $\boldsymbol{e}$ is the tracking error and $\boldsymbol{C}(z)$ is the controller transfer function.

For $\boldsymbol{G}(z) \in \mathcal{RH}_\infty$, the Youla parameterisation of all stabilising controllers (see, e.g., [1, 2, 39]) states that $\boldsymbol{C}(z)$ is a stabilising controller for $\boldsymbol{G}(z)$, that defines a *well-possed* closed loop [39], if and only if

$$\boldsymbol{C}(z) = \boldsymbol{Q}(z)\left(\boldsymbol{I} - \boldsymbol{G}(z)\boldsymbol{Q}(z)\right)^{-1}, \tag{2}$$

where $\boldsymbol{Q}(z) \in \mathcal{RH}_\infty$ and $\boldsymbol{I} - \boldsymbol{G}(\infty)\boldsymbol{Q}(\infty)$ is non singular.

In most practical settings it is desirable to track arbitrary constant references or reject constant disturbances [2, 40, 41]. A necessary condition for this to be possible is that $\boldsymbol{G}(1)$ is non singular [1, 15]. As a consequence, $\boldsymbol{G}(z)$ must be *non singular a.e.*.

A sufficient (and given that $\boldsymbol{G}(z)$ is stable, also necessary) condition for tracking arbitrary constant references, is that the controller exhibits integral action. Since we consider square stable plants, integral action is equivalent to having [1]

$$\boldsymbol{Q}(1) = \boldsymbol{G}(1)^{-1}, \tag{3}$$

which is well defined whenever $\boldsymbol{G}(1)$ is non singular. This, in turn, implies that $\boldsymbol{Q}(z)$ must be *non singular a.e.*.

On the other hand, since well possedness of the closed loop requires that $\boldsymbol{I} - \boldsymbol{G}(\infty)\boldsymbol{Q}(\infty)$ is non singular, we have that $\boldsymbol{I} - \boldsymbol{G}(z)\boldsymbol{Q}(z)$ must be non singular a.e. This fact, when combined with the discussion in the previous paragraphs, implies that any controller with integral action that defines a well possed closed loop must be *non singular a.e.*.

In the rest of this paper we will consider, of course, consider only well possed closed loops. In addition, we will restrict ourselves to the application-relevant case of closed loops whose controllers have integral action. Therefore, in what follows we will make the following assumptions:

Assumption 1 *(i) $\boldsymbol{G}(z) \in \mathcal{RH}_\infty$ and $\boldsymbol{G}(1)$ non singular (therefore, $\boldsymbol{G}(z)$ non singular a.e.),*

(ii) $\boldsymbol{Q}(1) = \boldsymbol{G}(1)^{-1}$ (therefore, $\boldsymbol{Q}(z)$ non singular a.e.), and that

(iii) the closed loop is well possed (therefore, $\boldsymbol{C}(z)$ non singular a.e.).

Assumption 1 will lead to significative technical simplifications in our subsequent discussion. In particular, Assumption 1 allows one to write (2) in the following form:

$$\boldsymbol{C}(z)^{-1} = \boldsymbol{Q}(z)^{-1} - \boldsymbol{G}(z). \tag{4}$$

3 Problem Definition

This paper constitutes a first step towards performance bounds evaluation in the control of a MIMO plant, when the controller must satisfy certain sparsity constraints. In particular, this paper focuses on the following questions:

- Which conditions guarantee that the optimal controller related to a performance bound evaluation problem has, *naturally*, the same sparse structure as the plant?
- If the conditions referred to in the previous question are not satisfied, which plant features influence the achievable optimal performance, when the controller must satisfy certain sparsity constraints?

The relevance of these questions is manyfold. A simple application of their answers, that aids MIMO controller design, will be discussed later in Section 8.

As in contemporary work related to performance bound evaluation [15, 16, 17, 18], we consider a quadratic measure of the loop error due to step references as performance index. In particular, we consider, for the one degree of freedom control loop in Figure 1, the cost function

$$J \triangleq \left|\left|\boldsymbol{S}(z)\frac{1}{z-1}\right|\right|_2^2, \tag{5}$$

where $\boldsymbol{S}(z)$ is the sensitivity function of the closed loop, i.e.,

$$\boldsymbol{S}(z) = \left(\boldsymbol{I} + \boldsymbol{G}(z)\boldsymbol{C}(z)\right)^{-1}. \tag{6}$$

This functional corresponds to the expected value, with respect to $\boldsymbol{v}$, of the 2-norm of the loop error due to a step change in direction $\boldsymbol{v}$, where $\boldsymbol{v}$ is a random variable that satisfies $\mathcal{E}_v\{\boldsymbol{v}\} = 0$, $\mathcal{E}_v\{\boldsymbol{v}\boldsymbol{v}^H\} = \boldsymbol{I}$, and $\mathcal{E}_v$ denotes expectation with respect to $\boldsymbol{v}$ [17]. In other words, J is the *energy of the tracking error* averaged over all possible reference directions. It thus relates to standard performance measures used in practice, as those discussed in [40, 42].

Using the parameterisation introduced in (2), it follows that

$$J = \left\| \frac{\boldsymbol{I} - \boldsymbol{G}(z)\boldsymbol{Q}(z)}{z-1} \right\|_2^2 . \tag{7}$$

We note that, in order to J to be well defined, the controller $\boldsymbol{C}(z)$ must have integral action and $\boldsymbol{G}(1)$ must be non singular. Therefore, Assumption 1 is natural in this setting.

We are interested in the minimisation of J subject to sparsity constraints on the controller. To that end, we introduce the following definition:

Definition 1 *A matrix $\boldsymbol{A}(z) \in \mathcal{R}$ is said to be* sparse constrained *if and only if some of its elements are constrained to be identically zero, i.e. $A_{ij}(z) \equiv 0$ for some specific pairs (i,j). The set that contains all matrices with a given sparsity constraint will be called a* sparse-constrained set*.*

It should be clear that sparse-constrained sets are subspaces of $\mathcal{R}$ and, as a consequence, they are convex.

Using the previous definition, it is possible to state the problem of performance bound evaluation under sparsity controller constraints as the problem of finding the Youla parameter, $\boldsymbol{Q}^{\boldsymbol{\mathcal{S}}}_{\boldsymbol{opt}}(z)$, defined as

$$\boldsymbol{Q}^{\boldsymbol{\mathcal{S}}}_{\boldsymbol{opt}}(z) \triangleq \arg \min_{\substack{\boldsymbol{Q}(z) \in \mathcal{RH}_\infty \\ \boldsymbol{C}(z) = \boldsymbol{Q}(z)(\boldsymbol{I} - \boldsymbol{G}(z)\boldsymbol{Q}(z))^{-1} \in \mathcal{S}}} J, \tag{8}$$

and the minimal value for J, $J^{\mathcal{S}}_{opt}$, given by

$$J^{\mathcal{S}}_{opt} \triangleq \min_{\substack{\boldsymbol{Q}(z) \in \mathcal{RH}_\infty \\ \boldsymbol{C}(z) = \boldsymbol{Q}(z)(\boldsymbol{I} - \boldsymbol{G}(z)\boldsymbol{Q}(z))^{-1} \in \mathcal{S}}} J, \tag{9}$$

where $\mathcal{S}$ is a sparse constrained set. The controller associated to $\boldsymbol{Q}^{\boldsymbol{\mathcal{S}}}_{\boldsymbol{opt}}(z)$ (see (2)) will be denoted by $\boldsymbol{C}^{\boldsymbol{\mathcal{S}}}_{\boldsymbol{opt}}(z)$.

The minimisation in (8) and (9) is far from trivial: although the functional J is convex in $\boldsymbol{Q}(z)$, the restrictions under which J is to be minimised are, in principle, non linear in $\boldsymbol{Q}(z)$. It is thus natural to ask whether there exist conditions under which it is possible to turn the structural restrictions on the controller into simple restrictions on the Youla parameter. Furthermore, if this restrictions turned to be convex, then the minimisation in (8) could be readily carried out. This is explored in Section 4.

In the trivial case in which $\mathcal{S} = \mathcal{R}$, it is well known (see, e.g., [17, 15]) that, if $\boldsymbol{G}(z)$ is non singular a.e. and does not have NMP zeros on the unit circle, then the optimal Youla parameter and the minimal functional value satisfy

$$\boldsymbol{Q_{opt}}(z) \triangleq \boldsymbol{Q^{\mathcal{R}}_{opt}}(z) = \left(\boldsymbol{\xi_G}(z)\boldsymbol{G}(z)\right)^{-1}, \tag{10}$$

and

$$J_{opt} \triangleq J^{\mathcal{R}}_{opt} = d + \sum_{i=1}^{r} \frac{|c_i|^2 - 1}{|1 - c_i|^2}, \tag{11}$$

where $\boldsymbol{\xi_G}(z)$ is a generalised left unitary interactor (GLUI) for $\boldsymbol{G}(z)$ [17] (a brief discussion on GLUI's is contained in Section 5), d is the number of zeros at infinity of $\boldsymbol{G}(z)$ (i.e, its relative degree) and $\{c_1, c_2, \ldots, c_r\}$ is the set of finite NMP zeros of $\boldsymbol{G}(z)$ counting multiplicities. In general, $\boldsymbol{C_{opt}}(z) \triangleq \boldsymbol{C^{\mathcal{R}}_{opt}}(z)$ does not have any specific structure, even if $\boldsymbol{G}(z)$ is structured. This brings back the first question to be addressed in this paper, namely conditions under which $\boldsymbol{C_{opt}}(z)$, i.e., the unrestricted optimal controller, belongs to a given sparse constrained set. From (10), it is apparent that the properties of GLUI's for $\boldsymbol{G}(z)$ will play a key role in the answer to this question. This will be explored in Section 5.

4 The structure of the Youla parameter

This section is about conditions under which the structural restriction on the controller can be recast as convex restrictions on the Youla parameter. The most general setting in which this is known to hold has been identified in [29]. In that work, the notion of quadratic invariance (QI) of a (not necessarily sparse) constrained set under a given plant model is defined and explored. This concept has been shown to be equivalent to the inheritance of the constraint on the controller by the Youla parameter, and *vice versa*. This implies that if $\mathcal{S}$ is sparse constrained (hence convex) and QI under $\boldsymbol{G}(z)$, then the minimisation problem in (8) can be written as the *convex* optimisation problem

$$\boldsymbol{Q}^{\boldsymbol{\mathcal{S}}}_{\boldsymbol{opt}}(z) = \arg \min_{\boldsymbol{Q}(z)\in\mathcal{RH}_\infty\cap\mathcal{S}} J, \tag{12}$$

where the convexity of $\mathcal{RH}_\infty \cap \mathcal{S}$ follows from the convexity of both $\mathcal{RH}_\infty$ and $\mathcal{S}$. The solution of this problem may be pursued using the methodologies in [3] or, after *vectorisation* (see [6, 23] or Section 22.6 in [1]), using standard unconstrained $\mathcal{H}_2$ optimisation methodologies.

In what follows, we present a sufficient condition under which QI is equivalent to enforcing the plant model to have the desired controller structure.

Before stating our result, we need the following definition:

Definition 2 *A sparse constrained set $\mathcal{S}$ is said to be* closed under inversion *if and only if for every non singular a.e. $\boldsymbol{X}(z) \in \mathcal{S}$, $\boldsymbol{X}(z)^{-1} \in \mathcal{S}$.*

It should be clear that sparse constrained sets that are closed under inversion are subspaces of $\mathcal{R}$ and, therefore, they are convex. Two examples of sparse restricted sets that are invariant under inversion are presented in the next example.

Example 1

- Block diagonal structures: $\boldsymbol{A}(z) \in \mathcal{R}$ *is said to be diagonal if and only if it belongs to the set*

$$\mathcal{S}_D = \{\boldsymbol{X}(z) \in \mathcal{R} : X_{ij}(z) \equiv 0 \, \forall i \neq j\}. \tag{13}$$

  Block diagonal models arise naturally when considering independent lower dimensional models as a whole.

- Triangular structures: $\boldsymbol{A}(z) \in \mathcal{R}$ *is said to be (lower) triangular if and only if it belongs to*

$$\mathcal{S}_t = \{\boldsymbol{X}(z) \in \mathcal{R} : X_{ij}(z) \equiv 0 \, \forall j > i, j \leq n, 1 \leq i \leq n\}. \tag{14}$$

  The models in $\mathcal{S}_t$ can be further classified as nested, chained, hierarchical *or* toeplitz *systems (see, e.g., [3, 35]). Examples of systems whose models are inherently triangular are cascaded chemical processes, platoons of aligned vehicles and networking architectures.*

We are now ready to present the first result in this paper:

Theorem 1 *Consider a sparse constrained set $\mathcal{S}$ that is closed under inversion and suppose that Assumption 1 holds. Then, the following statements are equivalent:*

(i) $\boldsymbol{G}(z) \in \mathcal{S}$.

(ii) $\boldsymbol{C}(z) \in \mathcal{S} \Leftrightarrow \boldsymbol{Q}(z) \in \mathcal{S}$.

(iii) $\mathcal{S}$ *is QI under* $\boldsymbol{G}(z)$.

Proof:

The equivalence between (ii) and (iii) follows from the results in [29].

The equivalence between (i) and (ii) is proved next:

- $((i) \Rightarrow (ii))$ Assume that $\boldsymbol{G}(z) \in \mathcal{S}$. Since $\mathcal{S}$ is closed under inversion, then it follows that if $\boldsymbol{C}(z) \in \mathcal{S}$, then $\boldsymbol{C}(z)^{-1} \in \mathcal{S}$. This in turn implies that $\boldsymbol{Q}(z)^{-1} = \boldsymbol{G}(z) + \boldsymbol{C}(z)^{-1} \in \mathcal{S}$ (see (4) and recall that sparse restricted sets are subspaces). Using again the fact that $\mathcal{S}$ is closed under inversion, it follows that $\boldsymbol{Q}(z) \in \mathcal{S}$.

  On the other hand, using the same arguments as above, if $\boldsymbol{Q}(z) \in \mathcal{S}$, then $\boldsymbol{Q}(z)^{-1} \in \mathcal{S}$. This in turn implies that $\boldsymbol{C}(z)^{-1} = \boldsymbol{Q}(z)^{-1} - \boldsymbol{G}(z) \in \mathcal{S} \Rightarrow \boldsymbol{C}(z) \in \mathcal{S}$.

- $((ii) \Rightarrow (i))$ Assume that $\boldsymbol{C}(z) \in \mathcal{S} \Leftrightarrow \boldsymbol{Q}(z) \in \mathcal{S}$. If $\boldsymbol{C}(z)$ or $\boldsymbol{Q}(z)$ belong to $\mathcal{S}$, then the fact that $\mathcal{S}$ is closed under inversion implies that $\boldsymbol{Q}(z)^{-1} \in \mathcal{S}$ and $\boldsymbol{C}(z)^{-1} \in \mathcal{S}$. Given (4) and recalling that $\mathcal{S}$ is a subspace, one haves that $\boldsymbol{G}(z) = \boldsymbol{Q}(z)^{-1} - \boldsymbol{C}(z)^{-1} \in \mathcal{S}$.

$\square\square\square$

Last result gives a sufficient condition under which the fact that the controller structure is inherited by the Youla parameter is equivalent to the condition of $\boldsymbol{G}(z)$ having the desired controller structure. Under this condition, the most general characterisation of sparse restricted control problems that are amenable to convex synthesis, namely those involving sparse constraints that are QI under $\boldsymbol{G}(z)$, reduce to the (restrictive) condition of considering a plant model with the desired controller structure. We note that for the special case of diagonal sparsity constraints this results was already known [29].

It is also worth mentioning that, in a recent work [3], it is proved for several (not necessarily sparse) structures of interest that, if the plant turns out to have the desired controller structure, *then* the structure of the controller is inherited by the Youla parameter. Our result complete those observations since it gives a characterisation of sparse constraint for which the fact that the plant has the desired controller structure is *not only sufficient*, but *also necessary* for the Youla parameter to inherit the controller structure.

Given the fact that we are working under Assumption 1, the cases in which $\mathcal{S}$ is QI under $\boldsymbol{G}(z)$ but $\mathcal{S}$ has only identically singular elements is left out of the discussion. Therefore, a key question that is left unanswered in the above discussion is *whether there exist non-identically singular sparse constraints that are QI under $\boldsymbol{G}(z)$, but are not invariant under inversion.* If the answer is affirmative, then the notion of QI would have significative advantages over our characterisation, because it would be able to deal with cases in which our results fail. On the contrary, a negative answer to this question would imply a debate on what advantages has the notion of QI over the results in this paper, for the considered framework (i.e, square stable plants and non-singular controller sparsity constraints). This is certainly a starting point for further research. We note that if we remove the constraint of the sparsity restriction being non-identically singular, then there exists a positive answer to the aforementioned question: Section 5.2 in [4] proves that any lower triangular *skyline* sparse structure is QI under a lower triangular plant model (a lower triangular *skyline* matrix is singular everywhere, unless it is lower triangular).

Finally, it must pointed out that the previous discussion (as well as the notion of QI), are not only useful for quadratic loss functionals. Every optimal control problem that, through the Youla parameterisation, can be recast as an optimal problem with a convex functional, and is subject to sparsity constraints on the controller, can benefit from the previous circle of ideas (see, e.g., [3]).

5 Properties of Transfer Matrix Zeros and Interactor Matrices

Last section identifies conditions under which structure restrictions on the controller can be recast as the same restrictions on the associated Youla parameter. This conditions, however, shed no light on the characterisation of the situations in which the unrestricted optimal controller has a given structure, *per se*. As already discussed in Section 3, this turns out to be related to properties of the GLUI's for the plant model. Properties of zeros of a transfer matrices and their implications on GLUI properties are discussed next (see also the Appendix).

Given $\boldsymbol{A}(z) \in \mathcal{R}$, we define its poles and zeros (both finite and infinite ones), as well as their directions, in the usual way [37, 43, 1]. A zero at $z = c$ is said to be of non-minimum phase (NMP) if and only if $|c| \geq 1$ (note that c may be infinite). If a transfer matrix has no NMP zeros, it is said to be minimum phase (MP), otherwise it is said to be NMP.

Given a zero at $z = c$ of $\boldsymbol{A}(z)$, we define its algebraic multiplicity, α_c, as usual [37], i.e., as the total number of zeros of $\boldsymbol{A}(z)$ at $z = c$. The total number of NMP zeros of $\boldsymbol{A}(z)$ equals the sum of the algebraic multiplicities of each NMP zero. To avoid ambiguities, we define two sets associated to the NMP zeros of $\boldsymbol{A}(z)$: $\mathcal{C} = \{c_i\}_{i=1,\cdots,n_c}$ is the set of (arbitrarily ordered) NMP zeros of $\boldsymbol{A}(z)$, counting their multiplicities (i.e., for every i, c_i is zero of $\boldsymbol{A}(z)$ and the number appearances of c_i in $\mathcal{C}$ equals its algebraic multiplicity). On the other hand, $\mathcal{C}^\dagger = \{c_i^\dagger\}_{i=1,\cdots,n_c^\dagger}$ is the set of (arbitrarily ordered) NMP of $\boldsymbol{A}(z)$ not counting their multiplicities (i.e., for every i, c_i is zero of $\boldsymbol{A}(z)$ and appears only once in $\mathcal{C}^\dagger$).

We next introduce the notion of left-canonical zero, which will be useful in the characterisation of matrices whose GLUI's are diagonal.

Definition 3 *A zero at $z = c$ of a matrix $\boldsymbol{A}(z) \in \mathcal{R}$ is said to be left-canonical if and only if its algebraic multiplicity, α_c, satisfies*

$$\alpha_c = \sum_{j=1}^{n} m_j^c, \tag{15}$$

where m_j^c is such that

$$[\boldsymbol{A}(z)]_{j*} = \begin{cases} (z-c)^{m_j^c} \boldsymbol{F_j^c}(z) & \text{if } |c| < \infty \\ \\ \frac{1}{z^{m_j^c}} \boldsymbol{F_j^c}(z) & \text{if } c = \infty \end{cases}, \tag{16}$$

and $0 < \left|\left| \boldsymbol{F_j^c}(c)^T \right|\right| < \infty$.

The notion of right canonical zero can be defined in an analogous way, but considering columns instead of rows. A necessary condition for a zero to be canonical is that it is *concentrated*, that is, that it makes either columns or rows of transfer matrices equal zero.

Given any nonsingular a.e. $\boldsymbol{A}(z) \in \mathcal{RH}_\infty$, we will refer to $\boldsymbol{\xi}_{\boldsymbol{A}}(z) \in \mathcal{R}$ as a generalised left unitary interactor (GLUI) for $\boldsymbol{A}(z)$ if and only if $\boldsymbol{\xi}_{\boldsymbol{A}}(z)$ is non singular a.e., unitary, MP, has unity DC-gain (i.e., $\boldsymbol{\xi}_{\boldsymbol{A}}(1) = \boldsymbol{I}$) and is such that $\boldsymbol{\xi}_{\boldsymbol{A}}(z)\boldsymbol{A}(z)$ is biproper, stable and MP (see [17] and the references therein). A GLUI for $\boldsymbol{A}(z)$ exists only if $\boldsymbol{A}(z)$ does not have zeros on the unit circle and can be proven to be unique [44].

Lemma 1 *Consider $\boldsymbol{A}(z) \in \mathcal{RH}_\infty$ non singular a.e. and without zeros on the unit circle. Then, the GLUI for $\boldsymbol{A}(z)$ is diagonal if and only if every NMP zero of $\boldsymbol{A}(z)$ is left-canonical.*

Proof:

See Appendix.

$\square\square\square$

To the best of the authors' knowledge, Lemma 1 is the first result in the literature providing an analytical condition for a generalised interactor matrix to be diagonal. Previous work on this subject [45, 46] considered characterisations of diagonal interactor matrices for the case of NMP zeros at infinity, only. Moreover, those results do not lead to explicit characterisations of diagonal interactors in terms of the *plant features*. Our result provides a significative and simple to check sufficient and necessary condition.

The relevance of diagonal interactor matrices for a given transfer matrix, relies in the fact that its existence implies that the *cost* of diagonal decoupling is zero (see, e.g., [47]). Diagonal interactors also play a role in the context of autocorrelation tests for minimum variance control in MIMO systems, as explored in [48]. For the purpose of this paper, the relevance of last result will become apparent in the next section.

6 Structure of an $\mathcal{H}_2$ Optimal Performance Controller

In this section we state the second contribution of this paper: conditions under which there is no performance penalisation, as measured by J (see (5)), when restricting the controller to be sparse. To that end, we restrict ourselves to control problems that satisfy Assumption 1 and for which the sparsity structure of the controller is invariant under inversion. In these cases, as shown in the previous sections, the Youla parameter inherits the structure of the controller if and only if the plant is itself structured, with the same sparse structure as the controller. Therefore, we further restrict our attention to plants with the desired controller structure.

Theorem 2 *Assume that the conditions of Theorem 1 are satisfied. If, in addition, $\mathcal{S}$ is closed under the standard product in $\mathcal{R}$ and $\boldsymbol{G}(z) \in \mathcal{S}$ without zeros on the unit circle, then $\boldsymbol{C_{opt}}(z) \in \mathcal{S} \Leftrightarrow \boldsymbol{\xi_G}(z) \in \mathcal{S}$.*

Proof:

Given Assumption 1 and (10), it is possible to write

$$\boldsymbol{\xi_G}(z) = \boldsymbol{Q_{opt}}(z)^{-1}\boldsymbol{G}(z)^{-1}. \tag{17}$$

Since $\boldsymbol{G}(z) \in \mathcal{S}$, Theorem 1 allows one to conclude that $\boldsymbol{C_{opt}}(z) \in \mathcal{S} \Rightarrow \boldsymbol{Q_{opt}}(z) \in \mathcal{S}$. Moreover, the fact that $\mathcal{S}$ is closed under inversion also implies that both $\boldsymbol{Q_{opt}}(z)^{-1}$ and $\boldsymbol{G}(z)^{-1}$ belong to $\mathcal{S}$. These observations, jointly with the fact that $\mathcal{S}$ is closed under the standard product in $\mathcal{R}$ and (17), imply that $\boldsymbol{\xi_G}(z) \in \mathcal{S}$.

On the other hand, since, given the hypothesis of the present lemma, $\boldsymbol{G}(z) \in \mathcal{S}$, (17) holds and $\mathcal{S}$ is closed under inversion, it follows that if $\boldsymbol{\xi_G}(z) \in \mathcal{S}$, then $\boldsymbol{Q_{opt}}(z)^{-1} \in \mathcal{S} \Rightarrow \boldsymbol{Q_{opt}}(z) \in \mathcal{S} \Rightarrow \boldsymbol{C_{opt}}(z) \in \mathcal{S}$ (here we used the fact that $\mathcal{S}$ is closed under inversion and under the standard product in $\mathcal{R}$, as well as Theorem 1).

□□□

Theorem 2 states that, as anticipated in Section 3, $\boldsymbol{C_{opt}}(z)$ can result to be full MIMO even if the plant is sparse constrained. When the plant is structured (and the other conditions of the theorem are satisfied), it is sufficient and necessary that the NMP zero structure of the plant has a special feature, namely that it is such that the GLUI has also the desired controller structure. In these cases, there is *no performance deterioration*, as measured by J, due to the controller structure restriction.

Last result can be particularised to the control structures defined in Example 1:

Corollary 1 *Assume that Assumption 1 holds and that $\boldsymbol{G}(z)$ has no zeros on the unit circle. Then:*

(i) If $\boldsymbol{G}(z) \in \mathcal{S}_D$*, then* $\boldsymbol{C_{opt}}(z) \in \mathcal{S}_D$*.*

(ii) If $\boldsymbol{G}(z) \in \mathcal{S}_t$*, then* $\boldsymbol{C_{opt}}(z) \in \mathcal{S}_t$ *if and only if* $\boldsymbol{G}(z)$ *has only left-canonical NMP zeros.*

Proof:

We note that $\mathcal{S}_D$ and S_t are closed under inversion and under the standard product in $\mathcal{R}$. Since, in addition, Assumption 1 holds and $\boldsymbol{G}(z)$ has no zeros on the unit circle, we have that Theorem 2 applies in this case. Therefore:

(i) $\boldsymbol{C_{opt}}(z) \in \mathcal{S_D} \Leftrightarrow \boldsymbol{\xi_G}(z) \in \mathcal{S_D}$. But, since $\boldsymbol{G}(z) \in \mathcal{S}_D$ and matrices in $\mathcal{S}_D$ have interactors in $\mathcal{S}_D$, the result follows.

(ii) $\boldsymbol{C_{opt}}(z) \in \mathcal{S}_t \Leftrightarrow \boldsymbol{\xi_G}(z) \in \mathcal{S}_t$. The only unitary triangular matrices in $\mathcal{R}$ are diagonal ones and therefore, $\boldsymbol{C_{opt}}(z) \in \mathcal{S}_t \Leftrightarrow \boldsymbol{\xi_G}(z)$ is diagonal. Using Lemma 1 the result follows.

□□□

Remark 1 *It is worth mentioning that both Theorem 2 and Corollary 1 can be extended to sparsity constraints that, via appropriate permutations, satisfy the conditions of the respective results (e.g., upper triangular structures, etc.).*

The first part of last Corollary is quite obvious, given the independence of the blocks of a block-diagonal plant model. The second part is more relevant. It reinforces the known result that states that the structure of the *non invertible part* of a plant, i.e., its NMP zero structure, is crucial in the structure of the resulting controller. In particular, it states that if the *non invertible part* of a triangular plant is sufficiently simple, i.e., if all its NMP zeros are left-canonical, then the optimal controller has the same structure of the plant.

If the conditions of Theorem 2 or Corollary 1 are not met, then it is necessary to consider further developments to evaluate a structure restricted optimal controller. This is accomplished in the next section for the case of triangular plants.

7 Best Achievable Performance of Triangular Controllers for Triangular Plants

We now apply the results in the previous sections to the case of triangular sparsity controller constraints. In particular, we focus on the second question raised in Section 3. For the same reasons given at the beginning of Section 6, we will focus on triangular plant models. Specifically, we are interested in finding the optimal Youla parameter $\boldsymbol{Q}_{\boldsymbol{opt}}^{\boldsymbol{\mathcal{S}_t}}(z)$ (see (8)) and the best achievable performance $J_{opt}^{\mathcal{S}_t}$ (see (9)) when $\boldsymbol{G}(z) \in \mathcal{S}_t$. The main result of this section is an analytical expression for both $\boldsymbol{Q}_{\boldsymbol{opt}}^{\boldsymbol{\mathcal{S}_t}}(z)$ and $J_{opt}^{\mathcal{S}_t}$. These are given in the next theorem:

Theorem 3 *Suppose that Assumption 1 holds and that $\boldsymbol{G}(z) \in \mathcal{S}_t$ has no zeros on the unit circle. Then:*

1. *The optimal parameter $\boldsymbol{Q}_{\boldsymbol{opt}}^{\boldsymbol{\mathcal{S}_t}}(z)$ is given by*

$$\boldsymbol{Q}_{\boldsymbol{opt}}^{\boldsymbol{\mathcal{S}_t}}(z) = \sum_{i=1}^{n} \mathrm{diag}\left\{\boldsymbol{0_{i-1}}, \tilde{\boldsymbol{G}}_{\boldsymbol{i}}(z)^{-1}\right\} \boldsymbol{1_{ii}}, \tag{18}$$

where $\tilde{\boldsymbol{G}}_{\boldsymbol{i}}(z) = \boldsymbol{\xi}_{\boldsymbol{G_i}}(z)\boldsymbol{G_i}(z)$ and $\boldsymbol{G_i}(z)$ is the i^{th} submatrix of $\boldsymbol{G}(z)$, defined as in (1).

2. *Define the ordered set of NMP zeros of $G_{\ell\ell}(z)$ as*

$$\mathcal{C}^{\ell\ell} = \left\{c_1^{\ell\ell}, c_2^{\ell\ell}, \ldots, c_{n_c^{\ell\ell}}^{\ell\ell}\right\}, \tag{19}$$

and, recalling that the NMP zeros of a triangular matrix are the NMP zeros of the elements of its diagonal, define the ordered set of NMP zeros of $\boldsymbol{G_k}(z)$ as

$$\begin{aligned} \mathcal{C}^{k} &= \left\{c_1^{k}, c_2^{k}, \ldots, c_{n_c^{k}}^{k}\right\} && (20)\\ &= \bigcup_{\ell=k}^{n} \mathcal{C}^{ll}, && (21) \end{aligned}$$

where the union preserves the ordering. Then, the restricted minimal cost $J_{t\,opt}$ is given by

$$J_{t\,opt} = J_{opt} + \sum_{k=1}^{n} \sum_{i=n_c^{kk}+1}^{n_c^k} h(c_i^k) \left|\boldsymbol{\eta_i^k}^H \boldsymbol{e_1^{n-k+1}}\right|^2, \tag{22}$$

where J_{opt} *is defined in* (11),

$$h(c_i^k) = \begin{cases} 1 & \textit{if } c_i^k = \infty \\ \\ \frac{|c_i^k|^2-1}{|1-c_i^k|^2} & \textit{if } |c_i^k| < \infty \end{cases}, \tag{23}$$

and $\boldsymbol{\eta_i^k}$ *is defined as in* (75) *with* $\boldsymbol{A}(z) = \boldsymbol{G_k}(z)$.

Proof:

1. Using elementary properties of the $\mathcal{L}_2$ norm, J can be written in vectorised form as

$$J = \left\| \frac{\boldsymbol{\Lambda} - \boldsymbol{G_A}(z)\boldsymbol{Q_{vec}}(z)}{z-1} \right\|_2^2, \tag{24}$$

where

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{e_1^n} \\ \boldsymbol{e_1^{n-1}} \\ \vdots \\ \boldsymbol{e_1^1} \end{bmatrix}, \qquad \boldsymbol{Q_{vec}} = \begin{bmatrix} Q_{11}(z) \\ Q_{21}(z) \\ \vdots \\ Q_{n1}(z) \\ Q_{22}(z) \\ Q_{32}(z) \\ \vdots \\ Q_{n2}(z) \\ \vdots \\ Q_{nn}(z) \end{bmatrix}, \tag{25}$$

$$\boldsymbol{G_A}(z) = \operatorname{diag}\left\{\boldsymbol{G_1}(z), \boldsymbol{G_2}(z), \ldots, \boldsymbol{G_n}(z)\right\}. \tag{26}$$

Since $\boldsymbol{G_A}(z)$ is block diagonal (with square blocks), then the matrix

$$\boldsymbol{\xi_A}(z) = \operatorname{diag}\left\{\boldsymbol{\xi_{G_1}}(z), \boldsymbol{\xi_{G_2}}(z), \ldots, \boldsymbol{\xi_{G_n}}(z)\right\} \tag{27}$$

is a GLUI of $\boldsymbol{G_A}(z)$, so that we can introduce $\boldsymbol{\xi_A}(z)$ in (24) without affecting the norm as follows

$$J = \left\| \frac{\boldsymbol{\xi_A}(z)\boldsymbol{\Lambda} - \boldsymbol{\tilde{G}_A(z)Q_{vec}(z)}}{z-1} \right\|_2^2 \tag{28}$$

$$= \left\| \frac{\boldsymbol{\xi_A}(z)\boldsymbol{\Lambda} - \boldsymbol{\Lambda}}{z-1} + \frac{\boldsymbol{\Lambda} - \boldsymbol{\tilde{G}_A}(z)\boldsymbol{Q_{vec}}(z)}{z-1} \right\|_2^2, \tag{29}$$

where $\boldsymbol{\tilde{G}}_A(z) = \boldsymbol{\xi_A}(z)\boldsymbol{G_A}(z)$. Given that $\boldsymbol{\xi_A}(1) = \boldsymbol{I}$ and that we require integral action on the controller, it must hold that $\boldsymbol{Q}(1) = \boldsymbol{G}(1)^{-1}$ and hence, both terms in last expression belong to $\mathcal{L}_2$. Moreover, from the definition of GLUI it follows that $\boldsymbol{\xi_A}(z) \in \mathcal{H}_2^{\perp}$ and the stability of $\boldsymbol{G}(z)$ ensures that $\left(\boldsymbol{\Lambda} - \boldsymbol{\tilde{G}_A}(z)\boldsymbol{Q_{vec}}(z)/(z-1)\right) \in \mathcal{H}_2$. This statements allow us to perform an orthogonal decomposition as

$$J = \left\| \frac{\boldsymbol{\xi_A}(z)\boldsymbol{\Lambda} - \boldsymbol{\Lambda}}{z-1} \right\|_2^2 + \left\| \frac{\boldsymbol{\Lambda} - \boldsymbol{\tilde{G}}_A(z)\boldsymbol{Q_{vec}}(z)}{z-1} \right\|_2^2. \tag{30}$$

Then, the optimal vector $\boldsymbol{Q_{vec\,opt}}(z)$ that minimises J also minimises the second term in (30), i.e.

$$\boldsymbol{Q_{vec\,opt}}(z) = \boldsymbol{\tilde{G}}_A(z)^{-1}\boldsymbol{\Lambda}. \tag{31}$$

This choice for $\boldsymbol{Q_{vec\,opt}}$ sets the second term in (30) to zero and, since $\boldsymbol{\tilde{G}_A}(z)$ is minimum phase and biproper, then $\boldsymbol{Q_{vec\,opt}}(z) \in \mathcal{RH}_\infty$. $\boldsymbol{Q_{opt}^{\mathcal{S}_t}}(z)$ can then be reconstructed as

$$\boldsymbol{Q_{opt}^{\mathcal{S}_t}}(z) = \sum_{i=1}^{n} \operatorname{diag}\left\{\boldsymbol{0_{i-1}}, \boldsymbol{\tilde{G}_i}^{-1}(z)\right\} \boldsymbol{1_{ii}}. \tag{32}$$

2. From (30) it follows that

$$J_{t\,opt} = \left\| \frac{1}{z-1} \left\{\boldsymbol{\xi_A}(z) - \boldsymbol{I}\right\} \boldsymbol{\Lambda} \right\|_2^2. \tag{33}$$

Therefore, using (25) and (27) the optimal cost can be rewritten as

$$J_{t\,opt} = \sum_{k=1}^{n} \underbrace{\left\| \frac{1}{z-1} \left\{ \boldsymbol{\xi_{G_k}}(z) - \boldsymbol{I_{n-k+1}} \right\} \boldsymbol{e_1^{n-k+1}} \right\|_2^2}_{J_{t\,opt}^k} \tag{34}$$

$$= \sum_{k=1}^{n} J_{t\,opt}^k . \tag{35}$$

Given that $\boldsymbol{\xi_{G_k}}(z)$ can be constructed with the NMP zeros ordered as in $\mathcal{C}^k$ defined in (21), and using Theorem 3 in [17], the k^{th} cost in (35) can be computed as

$$J_{t\,opt}^k = \sum_{i=1}^{n_c^k} h(c_i^k) \left| \boldsymbol{\eta_i^k}^H \boldsymbol{e_1^{n-k+1}} \right|^2 , \tag{36}$$

where $h(c_i^k)$ is defined in (23). Isolating the first n_c^{kk} terms in (36), i.e. all the terms involving the NMP zeros that belong to $\mathcal{C}^{kk}$ (NMP zeros of $G_{kk}(z)$), it follows that:

$$J_{t\,opt}^k = \sum_{i=1}^{n_c^{kk}} h(c_i^k) \left| \boldsymbol{\eta_i^k}^H \boldsymbol{e_1^{n-k+1}} \right|^2 + \sum_{i=n_c^{kk}+1}^{n_c^k} h(c_i^k) \left| \boldsymbol{\eta_i^k}^H \boldsymbol{e_1^{n-k+1}} \right|^2 , \tag{37}$$

where the second summation is well defined since when $n_c^{kk} = n_c^k$, then $i > n_c^k$ and the sum is empty. It should be noticed that the ordering introduced in $\mathcal{C}^k$ can be interpreted as a zero factorisation in which we firstly extract the NMP zeros of $\boldsymbol{G_k}(z)$ that are also zeros of $[\boldsymbol{G_k}(z)]_{11} = G_{kk}(z)$, then we sequentially extract the NMP zeros of $\boldsymbol{G_k}(z)$ that are also zeros of $[\boldsymbol{G_k}(z)]_{22}$, $[\boldsymbol{G_k}(z)]_{33}$, …, $[\boldsymbol{G_k}(z)]_{(n-k+1)(n-k+1)}$. This implies that the first n_c^{kk} zeros in $\mathcal{C}^k$ are such that $\left[\boldsymbol{G_k}(c_i^{kk})\right]_{1*} = \left[\boldsymbol{G_k}(c_i^k)\right]_{1*} = 0$, $\forall i = 1, 2, \ldots, n_c^{kk}$. Hence, it holds that $\boldsymbol{\eta_i^k} = \boldsymbol{e_1^{n-k+1}}$, $\forall i = 1, 2, \ldots, n_c^{kk}$ and (37) yields

$$J_{t\,opt}^k = \sum_{i=1}^{n_c^{kk}} h(c_i^k) + \sum_{i=n_c^{kk}+1}^{n_c^k} h(c_i^k) |\boldsymbol{\eta_i^k}^H \boldsymbol{e_1^{n-k+1}}|^2 . \tag{38}$$

Substituting (38) in (35), the minimal total cost becomes

$$J_{t\,opt} = \sum_{k=1}^{n} \sum_{i=1}^{n_c^{kk}} h(c_i^k) + \sum_{k=1}^{n} \sum_{i=n_c^{kk}+1}^{n_c^k} h(c_i^k) |\boldsymbol{\eta_i^k}^H \boldsymbol{e_1^{n-k+1}}|^2 . \tag{39}$$

Using the definition of $h(c_i^k)$ in (23) and recalling that $c_i^k = c_i^{kk}$, $\forall i = 1, 2, \ldots, n_c^{kk}$, it follows that

$$\sum_{i=1}^{n_c^{kk}} h(c_i^k) = d^{kk} + \sum_{i \in \mathbb{I}^{kk}} \frac{|c_i^{kk}|^2 - 1}{|1 - c_i^{kk}|^2}, \tag{40}$$

where d^{kk} is number of zeros at infinity of G_{kk} and the index set is $\mathbb{I}^{kk} = \left\{i : |c_i|^{kk} < \infty\right\}$. Finally, the unrestricted cost J_{opt} is clearly recognised in (39)

$$J_{t\,opt} = \sum_{k=1}^{n} \left(d^{kk} + \sum_{i \in \mathbb{I}^{kk}} \frac{|c_i^{kk}|^2 - 1}{|1 - c_i^{kk}|^2} \right) + \sum_{k=1}^{n} \sum_{i=n_c^{kk}+1}^{n_c^k} h(c_i^k) |\boldsymbol{\eta_i^k}^H \boldsymbol{e_1^{n-k+1}}|^2 \tag{41}$$

$$= \underbrace{d + \sum_{i \in \mathbb{I}} \frac{|c_i|^2 - 1}{|1 - c_i|^2}}_{J_{opt}} + \sum_{k=1}^{n} \sum_{i=n_c^{kk}+1}^{n_c^k} h(c_i^k) |\boldsymbol{\eta_i^k}^H \boldsymbol{e_1^{n-k+1}}|^2, \tag{42}$$

where the index set is $\mathbb{I} = \{i : |c_i| < \infty\}$, and d and c_i are defined as in (11).

□□□

The result stated in Theorem 3 is strong, since it provides an analytic expression for the triangular optimal Youla parameter and hence, the triangular optimal controller. Additionally, the performance bound is explicitly quantified in terms of the directions $\boldsymbol{\eta_i^k}$ and the location of the NMP zeros of the plant. We next present an example to illustrate the construction of the optimal triangular Youla parameter.

Example 2 *Consider $\boldsymbol{G}(z) \in \mathcal{RH}_\infty \cap \mathcal{S}_t$ defined as*

$$\boldsymbol{G}(z) = \begin{bmatrix} G_{11}(z) & 0 & 0 \\ G_{21}(z) & G_{22}(z) & 0 \\ G_{31}(z) & G_{32}(z) & G_{33}(z) \end{bmatrix}. \tag{43}$$

The Youla parameter which guarantees that $\boldsymbol{C}(z)$ is a stabilising controller with the desired structure *is given by*

$$\boldsymbol{Q}(z) = \begin{bmatrix} Q_{11}(z) & 0 & 0 \\ Q_{21}(z) & Q_{22}(z) & 0 \\ Q_{31}(z) & Q_{32}(z) & Q_{33}(z) \end{bmatrix} \in \mathcal{RH}_\infty. \tag{44}$$

The cost function J can be written as

$$J = \left\| \left(\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{\Lambda}} - \underbrace{\begin{bmatrix} G_{11}(z) & 0 & 0 & 0 & 0 & 0 \\ G_{21}(z) & G_{22}(z) & 0 & 0 & 0 & 0 \\ G_{31}(z) & G_{32}(z) & G_{33}(z) & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{22}(z) & 0 & 0 \\ 0 & 0 & 0 & G_{32}(z) & G_{33}(z) & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{33}(z) \end{bmatrix}}_{\boldsymbol{G_A}} \underbrace{\begin{bmatrix} Q_{11}(z) \\ Q_{21}(z) \\ Q_{31}(z) \\ Q_{22}(z) \\ Q_{32}(z) \\ Q_{33}(z) \end{bmatrix}}_{\boldsymbol{Q_{vec}}(z)} \right) \frac{1}{z-1} \right\|_2^2 . \tag{45}$$

Using (31) *it follows that*

$$\boldsymbol{Q_{vec\,opt}}(z) = \Bigg(\underbrace{\begin{bmatrix} \boldsymbol{\xi_{G_1}}(z) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\xi_{G_2}}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\xi_{G_3}}(z) \end{bmatrix} \boldsymbol{G_A}(z)}_{\tilde{\boldsymbol{G}}_A(z)} \Bigg)^{-1} \boldsymbol{\Lambda} \tag{46}$$

$$= \begin{bmatrix} \tilde{\boldsymbol{G}}_{\mathbf{1}}(z)^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\boldsymbol{G}}_{\mathbf{2}}(z)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{\boldsymbol{G}}_{\mathbf{3}}(z)^{-1} \end{bmatrix} \boldsymbol{\Lambda}, \tag{47}$$

where $\mathbf{0}$ *denotes the zero matrix of appropriate dimensions and* $\boldsymbol{\xi_{G_1}}(z)$, $\boldsymbol{\xi_{G_2}}(z)$ *and* $\boldsymbol{\xi_{G_3}}(z)$ *are GLUI* *of the submatrices*

$$\boldsymbol{G_1}(z) = \boldsymbol{G}(z), \qquad \boldsymbol{G_2}(z) = \begin{bmatrix} G_{22}(z) & 0 \\ G_{32}(z) & G_{33}(z) \end{bmatrix}, \qquad \boldsymbol{G_3}(z) = G_{33}(z), \tag{48}$$

respectively, and $\tilde{\boldsymbol{G}}_{\boldsymbol{i}}(z) = \boldsymbol{\xi}_{\boldsymbol{G_i}}(z)\boldsymbol{G_i}(z)$, $\forall i = 1,2,3$. *Finally, the optimal Youla parameter is reconstructed using* (18)

$$\boldsymbol{Q_{t\,opt}}(z) = \tilde{\boldsymbol{G}}_{\boldsymbol{1}}(z)^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \tilde{\boldsymbol{G}}_{\boldsymbol{2}}(z)^{-1} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{\boldsymbol{G}}_{\boldsymbol{3}}(z)^{-1} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{49}$$

$\square$

The form of the optimal triangular solution given in Theorem 3 exposes sequentiality features on its construction, although different to those arising from classical sequential design methods for MIMO systems (see, e.g., [49]). In particular, from (18) it can be noticed that

$$[\boldsymbol{Q_{t\,opt}}(z)]_{nn} = \left(\boldsymbol{\xi}_{\boldsymbol{G_n}}(z)\boldsymbol{G_n}(z)\right)^{-1}, \tag{50}$$

and since $\boldsymbol{G_n}(z) = G_{nn}(z)$ is always a scalar transfer function, then this result shows that $[\boldsymbol{Q_{t\,opt}}(z)]_{nn}$ must be chosen to minimise the sensitivity of a SISO loop around $G_{nn}(z)$. This property suggests that the sequence is different from the one usually considered in sequential design techniques, i.e. choosing $[\boldsymbol{Q_{t\,opt}}(z)]_{11}$ in terms of a SISO design for $G_{11}(z)$. This difference arises from the fact that in a triangular system, the n^{th} input affects only the n^{th} output, so that the only portion of the controller that can be chosen without affecting the overall dynamics is the $(n,n)^{th}$ entry of $\boldsymbol{Q_{t\,opt}}(z)$.

Additionally, from Corollary 1, it follows that if every NMP zero of $\boldsymbol{G}(z)$ is left-canonical, then the optimal unrestricted solution belongs naturally to $\mathcal{S}_t$. This statement relates directly to the restricted structure optimal cost in (22) as shown in the next corollary.

Corollary 2 *Consider the conditions and notation of Theorem 3. Then:*

1. *The extra cost due to constraining* $\boldsymbol{C}(z) \in \mathcal{S}_t$ *is given by*

$$\Delta J_t = J_{t\,opt} - J_{opt} \tag{51}$$

$$= \sum_{k=1}^{n} \sum_{i=n_c^{kk}+1}^{n_c^k} h(c_i^k) \left| \boldsymbol{\eta_i^k}^H \boldsymbol{e_1^{n-k+1}} \right|^2. \tag{52}$$

2. $\Delta J_t = 0$ *if and only if every NMP zero of* $\boldsymbol{G}(z)$ *is left-canonical.*

Proof:

1. Immediate from Theorem 3.

2. From (52) it can be noticed that $\Delta J_t = 0$ if and only if $\boldsymbol{\eta_i^k} \perp \boldsymbol{e_1^{n-k+1}}$, $\forall k$ and $\forall i \geq n_c^{kk} + 1$. Using Lemma 2 in the Appendix the result follows.

□□□

The quantity ΔJ_t in Corollary 2 measures the loop performance deterioration due to the structural restriction on the controller. This measure, as part 2 of this corollary shows, depends only on the non left-canonical zeros of $\boldsymbol{G}(z)$ and is consistent with the result in Corollary 1, i.e. when all the NMP zeros of the plant are left-canonical, then the unrestricted optimal solution $\boldsymbol{Q_{opt}}(z)$ is triangular *per se* and hence, the loop performance degradation reduces to zero.

Additional insight into the dependence of ΔJ_t on the plant dynamical features can be gained by noting that:

(a) The only zeros that affect the performance loss are the non left-canonical ones. In fact, if c_i is any left-canonical NMP zero of $\boldsymbol{G}(z)$, then it is also a left-canonical zero of $\boldsymbol{G_k}(z)$. Then, from Lemma 2, and using the fact that in (52) the only NMP zeros involved are those of $\boldsymbol{G_k}(z)$ that are not zeros of $G_{kk}(z)$, it is always possible to choose an ordering in $\mathcal{C}^k$ such that $\boldsymbol{\eta_i^k}$ belongs to the elementary basis of $\mathbb{R}^{n-k+1}$ and $\boldsymbol{\eta_i} \perp \boldsymbol{e_1^{n-k+1}}$, hence its contribution to the extra cost is zero.

(b) The effect of the non left-canonical NMP zero location is very clear, since as (52) shows, the effect is more deleterious as these zeros approach to $z = 1$. This is consistent with known results in [17, 18, 15].

(c) The directions $\boldsymbol{\eta_i^k}$ limit the performance depending on how aligned they are with $\boldsymbol{e_1^{n-k+1}}$. Moreover, since $\boldsymbol{\eta_i^k}$ are directions associated with the k^{th} submatrix of the plant, it becomes evident that the interactions in each subsystem $\boldsymbol{G_k}(z)$ can impact the best achievable performance when dealing with a triangular controller.

(d) The dependence of ΔJ_t on the directions $\boldsymbol{\eta_i^k}$ is a feature of major importance within the scope of this paper. Since J_{opt} in (11) depends only on the NMP zero location, it is evident that the effect on $\boldsymbol{\eta_i^k}$ arises exclusively from the structural constraints imposed on the controller. This is a distinguishing feature of triangular control that is not present in the unrestricted case when the functional given in (5) is minimised.

To illustrate our findings, we present an example where the extra cost is explicitly quantified in terms of plant parameters and clarifies the relation between the NMP zero location, left-canonical zeros and the extra cost.

Example 3 *Assume that the best achievable performance with a triangular controller is to be computed for a* 2×2 *plant modelled by*

$$\boldsymbol{G}(z) = \begin{bmatrix} \frac{1}{z} & 0 \\ \frac{z-\beta}{z^2} & \frac{z-\alpha}{z^2} \end{bmatrix}, \tag{53}$$

where $\alpha > 1$ *and* β *is an arbitrary constant. Following the notation of Theorem 3 we have that* $\mathcal{C}^{11} = \{\infty\}$, $\mathcal{C}^{1} = \{\infty, \infty, \alpha\}$, $\mathcal{C}^{22} = \{\infty, \alpha\}$ *and* $\mathcal{C}^{2} = \{\infty, \alpha\}$. *Hence, this system has one NMP zero at infinity (with algebraic multiplicity 2) and a single finite NMP zero at* $z = \alpha$. *We wish to apply the main results of this paper to this simple example. The reader is invited to verify that the zero at infinity is left-canonical and hence, its effect on the extra cost vanishes. Moreover, a direct use of* (52) *gives*

$$\Delta J_t = \frac{|\alpha|^2 - 1}{|1-\alpha|^2} |\boldsymbol{\eta_3^1}^H \boldsymbol{e_1^2}|^2, \tag{54}$$

where the direction $\boldsymbol{\eta_3^1}$ *is a member of an orthonormal base of the left null space of* $\boldsymbol{G_1}(\alpha)$, *i.e. it* *satisfies*

$$\boldsymbol{\eta_3^1}^H \boldsymbol{G_1}(\alpha) = 0, \tag{55}$$

$$||\boldsymbol{\eta_3^1}|| = 1. \tag{56}$$

One possible choice for $\boldsymbol{\eta_3^1}$ *is*

$$\boldsymbol{\eta_3^1} = \begin{bmatrix} 1 \\ -\dfrac{\alpha}{\alpha-\beta} \end{bmatrix} \frac{(\alpha-\beta)}{\sqrt{(\alpha-\beta)^2+\alpha^2}}. \tag{57}$$

The extra cost can then be computed from

$$\Delta J_t = \frac{|\alpha|^2-1}{|1-\alpha|^2} \cdot \frac{(\alpha-\beta)^2}{(\alpha-\beta)^2+\alpha^2}. \tag{58}$$

On the other hand, from (11) *we know that the unrestricted performance bound for* $\boldsymbol{G}(z)$ *is given by*

$$J_{opt} = 2 + \frac{|\alpha|^2-1}{|1-\alpha|^2}. \tag{59}$$

The plot of the percent relative extra cost, i.e the plot of $100 \cdot \Delta J_t / J_{opt}$*, is shown in Fig. 2 as a function of* $1 < \alpha < 10$ *and* $-5 < \beta < 10$*. Some conclusions can be drawn from the results in Fig. 2, namely:*

(a) Both the extra cost and the unrestricted cost can become arbitrarily large when the zero approaches $z = 1$*. However, the relative extra cost* $\Delta J_t / J_{opt}$ *grows but remains bounded. This can verified from* (58) *and* (59)*.*

(b) If $\alpha = \beta$*, the zero at* $z = \alpha$ *is left-canonical, and hence there is no extra cost when constraining the controller structure. In fact, when* $\alpha = \beta$ *it holds that* $\boldsymbol{\eta_3^1} = \boldsymbol{e_2^2}$*.*

(c) As we move away from the condition $\alpha = \beta$*, the relative extra cost tends to increase because the zero is farther from being left-canonical. This is related with the fact that, as we move away from the line* $\alpha = \beta$*, the angle between* $\boldsymbol{\eta_3^1}$ *and* $\boldsymbol{e_1^2}$ *tends to decrease, which in turn implies that the*

quantity $|{\boldsymbol{\eta}_{\mathbf{3}}^{\mathbf{1}}}^H \boldsymbol{e}_{\mathbf{1}}^{\mathbf{2}}|^2$ *grows. Indeed, from* (54) *and* (58) *it can be noticed that*

$$\lim_{\beta \to \infty} \Delta J_t = \frac{|\alpha|^2 - 1}{|1 - \alpha|^2}, \tag{60}$$

$$\lim_{\alpha \to \infty} \Delta J_t = \frac{1}{2}, \tag{61}$$

from where it is clear that the deleterious effect on the optimal performance depends critically on the alignment of $\boldsymbol{\eta}_{\mathbf{3}}^{\mathbf{1}}$ *and* $\boldsymbol{e}_{\mathbf{1}}^{\mathbf{2}}$*. As expected, this situation is worsened if, additionally, the NMP zero is near* $z = 1$.

□

8 An Application to Serial Processes

A common class of triangular systems that can be found on real applications are those known as *serial processes* (see, e.g., [50, 35]). Those systems are composed by several subunits that are connected in series, so that their linear models have lower (block-) triangular transfer matrices in which each (block-) row has a specific structure. If each subsystem that composes the serial process is SISO, then the resulting transfer matrix from the process input to the process output is square and its i^{th} row equals

$$[\boldsymbol{G}(z)]_{i*} = \boldsymbol{C_i} \left(z\boldsymbol{I} - \boldsymbol{A_{ii}} \right)^{-1} \begin{bmatrix} \boldsymbol{N_{i-1}}(z)\boldsymbol{B_1} & \boldsymbol{N_{i-2}}(z)\boldsymbol{B_2} & \cdots & \boldsymbol{N_1}(z)\boldsymbol{B_{i-1}} & \boldsymbol{B_i} & \boldsymbol{0} \end{bmatrix}, \tag{62}$$

where $\boldsymbol{C_i}$, $\boldsymbol{B_i}$, $\boldsymbol{N_{i-j}}(z)$ and $\boldsymbol{A_{ii}}$ ($i = 1, \cdots, n$; $j = 1, \cdots, i-1$) are matrices of appropriate dimensions that depend upon the parameters of each subsystem (for details see [35]). For the purpose of this paper, it suffices to note that $\boldsymbol{N_{i-j}}(z)$ is a strictly proper matrix and that all the poles and zeros of every subsystem are also poles and zeros of $\boldsymbol{G}(z)$.

Consider the case of a serial process in which all its subsystems are stable. Then, the results in this paper apply. In particular, we first note that all the NMP zeros at infinity (i.e., the NMP zeros related to the plant delay structure) are left-canonical. This follows from (62) upon noting that, since every $\boldsymbol{N_{i-j}}(z)$ is strictly proper, the relative degree of every element in a row, not lying on the main diagonal, is greater than the relative degree of the corresponding diagonal element. In [35] this property is stated, equivalently, as the plant model having a *diagonal structure at infinity.* As a consequence, if the subsystems do not have other NMP zeros than those at infinity, or the finite NMP zeros of the triangular transfer matrix are left-canonical, then the optimal controller for the serial process, $\boldsymbol{C_{opt}}(z)$, will be lower triangular. If there is one finite NMP that is non left-canonical, then the optimal controller will not be lower triangular.

The previous facts imply that, if each subsystem is MP, or there are only left-canonical NMP zeros, then there exist strong reasons (the discussion in [35] and the properties of the controller that minimises J - recall the observations regarding J in Section 3) to choose to exploit the benefits of feedforward and selecting a triangular controller for the serial process at hand. Moreover, in those cases our results formally prove that, from the loop performance point of view (using J as performance measure), considering the feedback of some *downstream outputs* (i.e., considering a controller with non zero entries above the diagonal) is unnecessary in the case of serial processes. Focusing on MP subsystems only, one can paraphrase the previous discussion stating that, since usually *the delays through the subsystems are large* in serial processes, then there is no need to consider non triangular controller structures. This claim has already been made in the literature [35], with intuitive justifications.

On the other hand, if there is any NMP zero that is not left-canonical, then it is advisable to explore more complex control structures that also include the feedback of some downstream outputs. This adds to the cases, identified in [35], in which feedback of downstream outputs is beneficial from the loop performance point of view. Indeed, our results provide strong analytic conditions that serve as guidelines for controller structure selection in the case of serial processes (and, in general, in the case of triangular models).

Finally, we present a simple numerical example that illustrates the application of the results in this paper to the control of serial processes.

Example 4 *Consider a pH-neutralisation processes as described in [50]. A schematic diagram of such a system with two neutralisation tanks is shown in Fig. 3. Here the inputs $u_1(t)$ and $u_2(t)$ correspond to the reagent flows that neutralises the pH of the inlet flow $d_1(t)$ (disturbance). The outputs $y_1(t)$ and $y_2(t)$ are defined as the deviation of the output flow pH from the neutral condition (pH=7). The control objective in this system is to ensure that the output flow pH is a pre-specified constant. This must be done by means of adjusting the reagent flows to achieve the desired output pH, while compensating*[1]

[1] This is a simplified example, since we are not considering other disturbances in the system that may be important,

the variations in $d_1(t)$. According to the results in [50], a model for the linearised dynamics of each subsystem in Fig. 3 is given by

$$Y_i(s) = H_i(s)U_i(s) + H_{di}(s)D_i(s), \quad i = 1, 2 \tag{63}$$

where the notation is such that $F(s) = \mathcal{L}\{f(t)\}$ and

$$H_1(s) = \frac{k_1}{\tau_1 s + 1} e^{-\theta_1 s}, \tag{64}$$

$$H_2(s) = \frac{k_2}{\tau_2 s + 1} e^{-\theta_2 s}, \tag{65}$$

$$H_{d1}(s) = \frac{k_{d1}}{\tau_1 s + 1} e^{-\theta_{d1} s}, \tag{66}$$

$$H_{d2}(s) = \frac{k_{d2}}{\tau_2 s + 1} e^{-\theta_{d2} s}, \tag{67}$$

with all the parameters depending on the geometry of the system. The MIMO dynamics can then be expressed as

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} H_1(s) & 0 \\ H_{d2}(s)H_1(s) & H_2(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} + \begin{bmatrix} H_{d1}(s) & 0 \\ 0 & H_{d2}(s)H_{d1}(s) \end{bmatrix} D_1(s). \tag{68}$$

Suppose that a digital triangular controller is to be designed for this process and we want to determine when the structural constraint will degrade the achievable tracking performance (in terms of the cost functional considered in this paper). Since we are interested in a one degree freedom control architecture, for the purposes of control design we only need a model relating the reagent flow and output pH. If the system is sampled every T [s], such that $\theta_i = qT$, $q \in \mathbb{N}$, $i = 1, 2$, then this zero-order hold discrete time model is given by

$$\begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} = \begin{bmatrix} G_{11}(z) & 0 \\ G_{12}(z) & G_{22}(z) \end{bmatrix} \begin{bmatrix} U_1(z) \\ U_2(z) \end{bmatrix}, \tag{69}$$

such as variations in the inlet flow pH and reagent pH. For details see [50].

where $F(z) = \mathcal{Z}\{f(t)\}$ *and*

$$G_{11}(z) = \frac{K_1}{z^{\theta_1/T}(z+\alpha_1)}, \tag{70}$$

$$G_{12}(z) = \frac{K_{12}(z+\beta)}{z^{(\theta_1+\theta_{d2})/T}(z+\alpha_1)(z+\alpha_2)}, \tag{71}$$

$$G_2(z) = \frac{K_2}{z^{\theta_2/T}(z+\alpha_2)}, \tag{72}$$

and where the constants K_1*,* K_{12}*,* K_2*,* α_1*,* α_2 *and* β *come from the sampling process [51]. From* (69) *it can be noticed that the discrete time model has* $(\theta_1+\theta_2)/T+2$ *zeros at infinity and no finite NMP zeros. Moreover, from Definition 3 and* (68) *it follows that all these zeros at infinity are left-canonical if and only if*

$$\frac{\theta_2}{T} + 1 \leq \frac{\theta_1+\theta_{d2}}{T} + 1, \tag{73}$$

which in turn, using the results of Corollary 2, implies that the performance loss due to the structural constraint on the controller is zero if and only if $\theta_2 \leq \theta_1 + \theta_{d2}$*. This also implies that if this condition is met, then using a triangular controller structure is a natural choice in the sense of minimising the* $\mathcal{L}_2$ *norm of the loop sensitivity. Conversely, as the quantity* $\theta_2 - \theta_{d2} - \theta_1$ *is positive and grows, to achieve an optimal performance a full MIMO controller is required. Since these delays depend on the instrumentation and the geometry of the system, this conclusion reveals an interesting link between the optimal performance and implementation issues. This result provides theoretical support for the intuitive discussion done in [35] regarding the advantages of using a full MIMO controller instead of a triangular one when dealing with general serial processes.*

Figure 4 shows the optimal performance loss when using a triangular controller relative to the performance achieved with an optimal full MIMO controller when $k_1 = k_2 = 3160$*,* $k_{d2} = 1580$*,* $\tau_1 = \tau_2 = 300[s]$*,* $\theta_1 = \theta_{d2} = 1[s]$*,* $T = 1[s]$ *and* $1 \leq \theta_2 \leq 7$*. It can then be observed that, if* $\theta_2 > \theta_1 + \theta_{d2}$*, the optimal performance degrades as the delay* θ_2 *increases, validating our previous discussion. This in turn implies that from an optimal performance point of view, if* $\theta_2 > 2[s]$*, then the usage of a full MIMO controller can have benefits over a triangular one.*

The conclusions drawn in this example regarding a simple serial process are insightful and suggest that the results of this paper may be used in applications in order to derive ideas for both process design *and control design. Additional insight into the meaning of this conditions in terms of pH-neutralisation process design and control should be a topic of future research.*

□

9 Conclusions

In this paper we have focused on sparsity-constrained optimal control problems, for square and stable plant models. We firstly built conditions under which controllers with a prescribed sparse structure can be parameterised with a Youla parameter having the same structure. This has been achieved showing that, for all sparsity structure classes that are closed under inversion, it is necessary and sufficient that the plant model also belongs to that class. This result implies that, if the sparsity constraint is invariant under inversion, then the largest known class of sparse constrained control problems that are amenable to convex synthesis, namely those that consider sparsity constraints that are QI under the plant model, reduce to those problems in which the plant model has the desired controller structure.

We secondly established conditions under which an unrestricted 2-norm optimal controller has the same sparse structure of the plant model and, as a consequence of this, imposing the same structure on the controller has no effect on the optimal loop performance. A condition for this to happen, is that the both the plant and its GLUI belong to the desired controller class. For triangular plant models, we have also shown that this interactor feature is present if and only if the plant NMP zeros are left-canonical.

For the case of stable triangular plant models we have derived explicit expressions for both the Youla parameter that defines the optimal triangular controller, and for the performance loss in which it is incurred because of the controller structural restriction. Our results allow one to see that not only plant NMP zeros and delays, but also certain directionality features that are not present in the unrestricted case, play a significative role in the best achievable performance.

The general problem of performance bound evaluation for general plants and an arbitrary controller structure remains open. Interesting topics for future research include the exploration of further connections between invariance under inversion and QI and, in particular, a characterisation of non identically singular sparse restricted controller structures that are QI and not invariant under inversion. Another interesting point is establishing conditions, similar to those considered in this paper (or similar to QI), that allow one to set sparsity constraints on the Youla parameter, that are *not the same* sparsity constraints imposed on the controller. This would lead to a more general characterisation of tractable restricted structure optimal control problems.

Acknowledgements

The authors gratefully acknowledge the support received from UTFSM and from Conicyt through grants Fondecyt 1040313 and 1060437.

A Appendix

Given $\boldsymbol{A}(z) \in \mathcal{RH}_\infty$ non singular a.e., without NMP zeros on the unit circle, and an arbitrary ordering in $\mathcal{C}$, the GLUI for $\boldsymbol{A}(z)$ is given by [17]

$$\boldsymbol{\xi_A}(z) = \prod_{i=1}^{n_c} \boldsymbol{L_{n_c-i+1}}(z), \tag{74}$$

where

$$\boldsymbol{L_i}(z) = f_i(z)\boldsymbol{\eta_i}\boldsymbol{\eta_i}^H + \boldsymbol{U_i}\boldsymbol{U_i}^H, \quad f_i(z) = \begin{cases} \frac{(1-z\bar{c}_i)(1-c_i)}{(z-c_i)(1-\bar{c}_i)} & \text{if } |c_i| < \infty \\ z & \text{if } c_i = \infty \end{cases}, \tag{75}$$

$\boldsymbol{\eta_i}$ is a member of an orthonormal basis of the left null space of $\boldsymbol{\hat{A}_i}(c_i)$, where

$$\boldsymbol{\hat{A}_{i+1}}(z) = \boldsymbol{L_i}(z)\boldsymbol{\hat{A}_i}(z), \qquad i = 1, \cdots, n_c, \tag{76}$$

$\boldsymbol{\hat{A}_1}(z) = \boldsymbol{A}(z)$, and $\boldsymbol{U_i}$ is such that $\begin{bmatrix} \boldsymbol{\eta_i} & \boldsymbol{U_i} \end{bmatrix}$ is unitary. It can be proven that GLUI's are unique [44]. As a consequence, the ordering considered in the construction of a GLUI is completely immaterial.

It is worth noting that, for square and *fat* matrices, the construction of a GLUI is equivalent to an inner-outer factorisation. Since there exist efficient algorithms to build such factorisations [52, 53], it follows that GLUI construction is a numerically viable problem.

Lemma 2 *Consider $\boldsymbol{A}(z) \in \mathcal{RH}_\infty$ non singular a.e., without zeros on the unit circle and $\boldsymbol{\eta_i}$, $\boldsymbol{L_i}(z)$*

defined as in (75)*. Then,*

1. *For a given* i*,* $\boldsymbol{\eta_i}$ *is elementary (i.e., it is zero except for a one in an entry) if and only if* $\boldsymbol{L_i}(z)$ *is diagonal.*

2. *Consider the* j^{th} *row in* $\hat{\boldsymbol{A}}_{\boldsymbol{i}}(z)$ *and define* $m_{ji}^{c_q^\dagger}$ *as* $m_j^{c_q^\dagger}$*, but considering* $\hat{\boldsymbol{A}}_{\boldsymbol{i}}(z)$ *instead of* $\boldsymbol{A}(z)$*. If* $m_{ji}^{c_q^\dagger} > 0$ *for some* q*, then it is always possible to choose an ordering in* $\mathcal{C}$ *such that* $\boldsymbol{\eta_\ell}$ *is elementary for* $\ell = i, i+1, \cdots, i + \sum_{q=1}^{n_c^\dagger} m_{ji}^{c_q^\dagger} - 1$*.*

3. **(Lemma 1)** $\boldsymbol{\xi_A}(z)$ *is diagonal if and only if every NMP zero of* $\boldsymbol{A}(z)$ *is left-canonical.*

4. $\boldsymbol{\xi_A}(z)$ *is diagonal if and only if* $\boldsymbol{\eta_i}$ *is elementary for all* i*.*

Proof:

1. Given the definition of $\boldsymbol{L_i}(z)$, it follows that

$$\boldsymbol{L_i}(z) = \begin{bmatrix} \boldsymbol{\eta_i} & \boldsymbol{U_i} \end{bmatrix} \begin{bmatrix} f_i(z) & 0 \\ 0 & \boldsymbol{I_{n-1}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta_i} & \boldsymbol{U_i} \end{bmatrix}^H . \tag{77}$$

   Since $\begin{bmatrix} \boldsymbol{\eta_i} & \boldsymbol{U_i} \end{bmatrix}$ is unitary, it follows that $\boldsymbol{\eta_i}$ is elementary if and only if $\begin{bmatrix} \boldsymbol{\eta_i} & \boldsymbol{U_i} \end{bmatrix}$ is a permutation matrix, which is equivalent to having $\boldsymbol{L_i}(z)$ diagonal.

2. If $m_{ji}^{c_q^\dagger} > 0$ for at least one $q = q_o$, then it is possible choose an ordering in $\mathcal{C}$ such that $c_i = c_{q_o}$. Therefore, $\boldsymbol{\eta_i}$ can be chosen elementary, $\boldsymbol{L_i}(z)$ diagonal and

$$\hat{\boldsymbol{A}}_{\boldsymbol{i+1}}(z) = \begin{bmatrix} [\hat{\boldsymbol{A}}_{\boldsymbol{i}}(z)]_{1*} \\ \vdots \\ f_i(z)[\hat{\boldsymbol{A}}_{\boldsymbol{i}}(z)]_{j*} \\ \vdots \\ [\hat{\boldsymbol{A}}_{\boldsymbol{i}}(z)]_{n*} \end{bmatrix} . \tag{78}$$

   If $m_{ji}^{c_{q_o}^\dagger} = 1$ and $m_{ji}^{c_q^\dagger} = 0$ for all $q \neq q_o$, then the procedure stops and the proof is ready. If this is not the case, then $\hat{\boldsymbol{A}}_{\boldsymbol{i+1}}(z)$ is such that $m_{ji+1}^{c_q^\dagger} > 0$ for $q = q_1$ (q_1 may be equal to q_o).

This implies that if an ordering is enforced on $\mathcal{C}$ such that $c_{i+1} = c_{q_1}$, then $\boldsymbol{\eta_{i+1}}$ is elementary, $\boldsymbol{L_{i+1}}(z)$ diagonal and

$$\boldsymbol{\hat{A}_{i+2}}(z) = \begin{bmatrix} [\boldsymbol{\hat{A}_i}(z)]_{1*} \\ \vdots \\ f_{i+1}(z)f_i(z)[\boldsymbol{\hat{A}_i}(z)]_{j*} \\ \vdots \\ [\boldsymbol{\hat{A}_i}(z)]_{n*} \end{bmatrix}. \tag{79}$$

Repeating last procedure until all j^{th} row concentrated zeros of $\boldsymbol{\hat{A}_i}(z)$ are removed, the result follows. Note that $\boldsymbol{\eta}_{\boldsymbol{i}+\sum_{q=1}^{n_c^\dagger} \boldsymbol{m}_{\boldsymbol{ji}}^{c_q^\dagger}}$ may be non elementary.

3. We proceed by parts:

- ($\Rightarrow$) Given the definition of $\boldsymbol{\xi_A}(z)$,

$$\boldsymbol{A}(z) = \boldsymbol{\xi_A}(z)^{-1}\boldsymbol{\tilde{A}}(z) \tag{80}$$

  where $\boldsymbol{\tilde{A}}(z)$ is stable, biproper and MP. Therefore, the only values of $|z| > 1$ that make $\boldsymbol{A}(z)$ singular are the zeros of $\boldsymbol{\xi_A}(z)^{-1}$. Since $\boldsymbol{\xi_A}(z)^{-1}$ is diagonal, this means that for every NMP zero of $\boldsymbol{A}(z)$ at $z = c_i^\dagger$,

$$\alpha_{c_i^\dagger} = \sum_{j=1}^{n} \bar{m}_j^{c_i^\dagger} \tag{81}$$

  where $\bar{m}_j^{c_i^\dagger}$ is defined as $m_j^{c_i^\dagger}$, but considering $\boldsymbol{\xi_A}(z)^{-1}$ instead of $\boldsymbol{A}(z)$. Since $\boldsymbol{\xi_A}(z)^{-1}$ is diagonal and $\boldsymbol{\tilde{A}}(z)$ stable, it follows that $\bar{m}_j^{c_i^\dagger} = m_j^{c_i^\dagger}$ $\forall i$ and the result follows.

- ($\Leftarrow$) It is always possible consider an ordering in $\mathcal{C}^\dagger$ such that $c_{n_c^\dagger} = \infty$ and therefore, for every $j = 1, \cdots, n$

$$[\boldsymbol{A}(z)]_{j*} = \left(\frac{1}{z^{m_j^\infty}} \prod_{i=1}^{n_c^\dagger - 1} (z - c_i^\dagger)^{m_j^{c_i^\dagger}} \right) \boldsymbol{E_j}(z), \tag{82}$$

where $0 < \left\|\boldsymbol{E_j}(c_i^\dagger)^T\right\| < \infty$ for all $i = 1, \cdots, n_c^\dagger$ and all $j = 1, \cdots, n$. Define

$$\boldsymbol{\xi}(z) = \operatorname{diag}\left\{ z^{m_j^\infty} \prod_{i=1}^{n_c^\dagger - 1} \left(\frac{(1 - z\bar{c}_i^\dagger)(1 - c_i^\dagger)}{(z - c_i^\dagger)(1 - \bar{c}_i^\dagger)} \right)^{m_j^{c_i^\dagger}} \right\}_{j=1,\cdots,n} . \tag{83}$$

It is clear that $\boldsymbol{\xi}(z)$ is MP, unitary and such that $[\boldsymbol{\xi}(z)\boldsymbol{A}(z)]_{j*}$ is not identically zero and is finite at $z = c_i^\dagger$, $\forall i$. This means that $\boldsymbol{\xi}(z)\boldsymbol{A}(z)$ does not have any zeros at $z = c_i^\dagger$ concentrated by rows, for every i. Moreover, $\boldsymbol{\xi}(z)\boldsymbol{A}(z)$ has $\sum_{i=1}^{n_c^\dagger}\sum_{j=1}^{n} m_j^{c_i^\dagger}$ NMP zeros less that $\boldsymbol{A}(z)$. But, since $\boldsymbol{A}(z)$ has only left-canonical NMP zeros, it follows that

$$\sum_{j=1}^{n} m_j^{c_i^\dagger} = \alpha_{c_i^\dagger} \tag{84}$$

and therefore, the number of extracted NMP zeros equals n_c, which implies that $\boldsymbol{\xi}(z)\boldsymbol{A}(z)$ is MP and therefore, $\boldsymbol{\xi}(z)$ is a diagonal GLUI for $\boldsymbol{A}(z)$ and it is possible to write $\boldsymbol{\xi}(z) = \boldsymbol{\xi}_{\boldsymbol{A}}(z)$.

4. If $\boldsymbol{\xi}_{\boldsymbol{A}}(z)$ is diagonal, then it may written as in (83). Therefore it follows that all zeros of $\boldsymbol{A}(z)$ are concentrated by rows and, using part 2 for every row, this implies that $\boldsymbol{\eta_i}$ is elementary for all i. The converse is immediate from part 1.

□□□

Further properties of GLUI's can be found in [44].

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Figure captions

- Figure 1: Standard one degree of freedom control loop.
- Figure 2: Relative extra cost for the plant of Example 3 for different choices of α and β.
- Figure 3: Schematic diagram of a two-stage pH-neutralisation process.
- Figure 4: Relative performance loss for Example 4 for different values of θ_2.

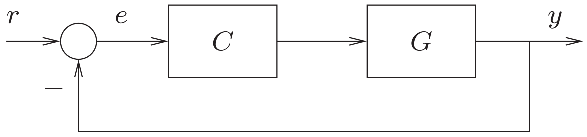
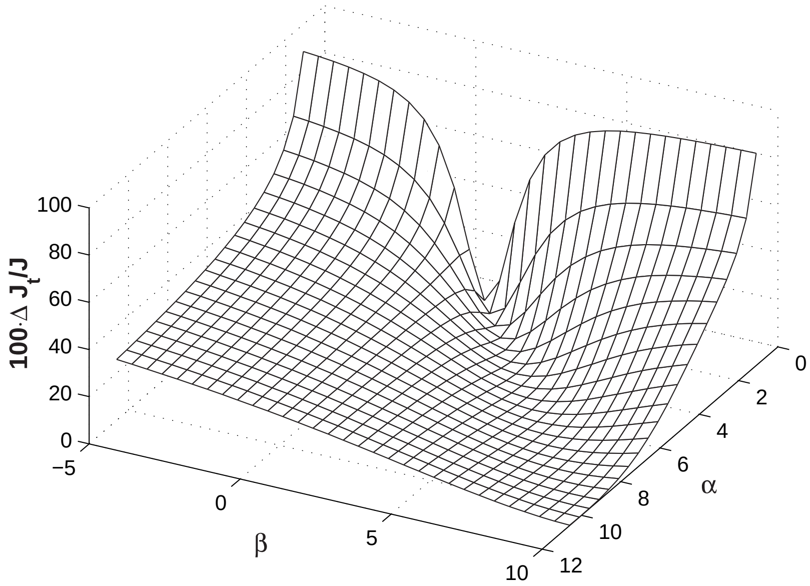


Figure 1:



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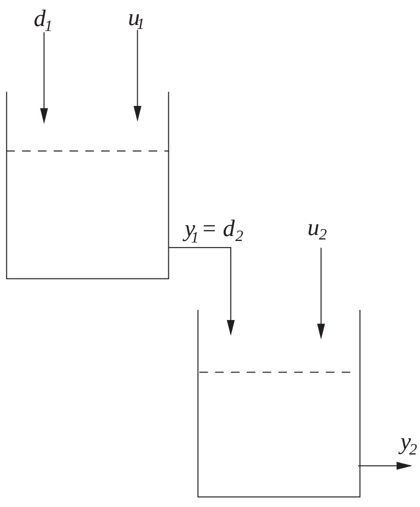

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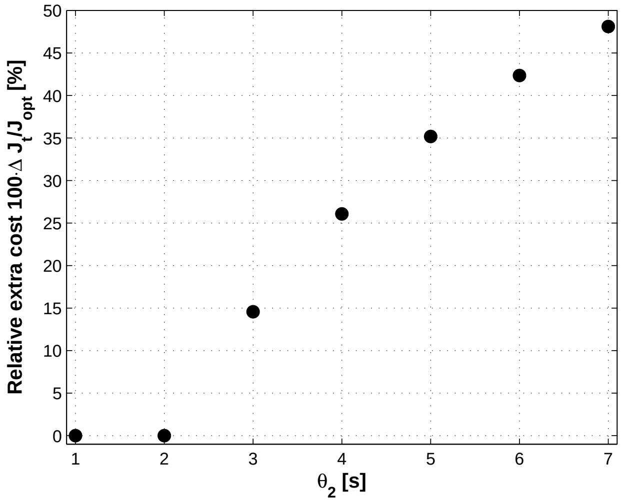

Relative extra cost 100·Δ J_t/J_opt [%]
θ_2 [s]
0
5
10
15
20
25
30
35
40
45
50
1
2
3
4
5
6
7

Figure 4: