$\mathcal{H}_2$ optimal ripple-free deadbeat controller design ⋆

Mario E. Salgado [a], Diego A. Oyarzún [a], Eduardo I. Silva [b]

[a] *Department of Electronic Engineering, Universidad Técnica Federico Santa María, Valparaíso, Chile*

[b] *School of Electrical Engineering and Computer Science, University of Newcastle, Australia*

Abstract

This paper deals with ripple-free deadbeat controller design for single-input/single-output sampled-data systems. Given a prescribed settling time, the design problem is tackled by computing a ripple-free deadbeat controller that optimizes an $\mathcal{H}_2$ performance criterion. This criterion accounts for the quality of the tracking response and for the control energy necessary to achieve the deadbeat behavior. Our main result shows that the optimal performance improves as the settling time grows, revealing a fundamental tradeoff inherent to this control technique.

Key words: Deadbeat control; H-2 optimal control; Control design; Quadratic performance; Sampled data control

1 Introduction

Deadbeat control has been a well established discrete-time control technique for several decades [9,10,8]. Its key feature is that it ensures that the tracking error settles to zero in a finite number of samples. Although deadbeat control allows perfect tracking in a finite time horizon and various design methods have been proposed, see for example [20,21,4], it is also known that in the context of sampled-data control systems it may lead to poor loop performance.

Indeed, deadbeat control only ensures that the error sequence vanishes *at the sampling instants* (beyond the settling time) and no considerations about the intersample behavior are done. Therefore, intersample ripple may appear in the continuous-time output, which is certainly an undesirable feature. This issue is dealt with in [17,19,1], where parameterizations of *ripple-free* deadbeat controllers are given. The basic idea behind this approach is that to avoid any intersample ripple after the settling time, the control sequence must also reach its steady state in, at most, the same number of samples [3]. The results in [17,19,1] apply to general plants with arbitrary reference signals, provided that the sampled-data model satisfies the internal model principle for the corresponding reference [7].

If we aim at designing a ripple-free deadbeat controller with certain prescribed settling time, there are, in the general case, an infinite number of solutions. This issue has motivated the search for design methods that allows one to adjust the controller in such a way that different design criteria can be met. This has been tackled in a state-space framework in [2,11], by posing the problem in the context of linear quadratic regulator theory. In [16,15] the problem is faced using a frequency domain approach, leading to optimization problems that deal with robustness and performance objectives that are solved using LMI and linear programming algorithms.

An interesting result in this topic can be found in [1], where the authors propose a systematic procedure to design a ripple-free deadbeat controller that minimizes a combined measure formed by the l_∞ norm of the tracking error and the l_1 norm of the control input. An insightful conclusion of that work is that the optimal value of the mixed l_∞/l_1 index is a decreasing function of the settling time, which reveals an essential tradeoff in the deadbeat design problem. A related result is reported in [22], where a ripple-free deadbeat controller that ensures a minimal control energy is proposed and the authors conclude that the minimal energy can be decreased at the expense of increasing the settling time.

This paper deals with optimal ripple-free deadbeat control for general single-input/single-output linear and time invariant plants in a sampled-data control setup.

⋆ This paper was not presented at any IFAC meeting. Corresponding author D. Oyarzún, Ph. + 353 (0)1 7086100, Fax +353 (0)1 7086269, e-mail: diego.oyarzun@nuim.ie. Postal address: Hamilton Institute, National University of Ireland, Maynooth, Co. Kildare, Ireland

Firstly, we derive a simple characterization of ripple-free deadbeat controllers for constant reference signals. This parameterization is comparable to that in [14] and is useful to derive an analytical solution to an optimal control problem arising from the minimization of a two objective quadratic performance measure. This measure accounts for both the tracking error and control energies. This implies that the derived control law exhibits ripple-free deadbeat behavior while attaining an optimal performance in terms of both tracking quality and control effort. We stress that the latter property is important from an application point of view since excessive control magnitudes have been source of criticism towards this control technique.

The main result of this work is proving that the optimal performance is a non-increasing function of the settling time. This is an interesting feature that emphasizes the design tradeoffs derived in [1] and [22]. In this sense, the contribution of this paper follows the philosophy of that of [1] and stands as a complement to that work. It is revealed that the compromise between settling time/performance appears not only in a l_∞/l_1 framework, but also in a quadratic sense, constituting a fundamental tradeoff inherent to deadbeat control.

The paper is organized as follows: Section 2 defines the main notation and the assumptions made throughout this paper. In Section 3 we derive a characterization of ripple-free deadbeat controllers for step reference signals that is used in Section 4 to propose a design procedure. The main result of this paper is given in Section 5. Finally, the concluding remarks of this work are given in Section 6.

2 Notation and assumptions

In this paper we deal with proper continuous-time models of the form

$$G_c(s) = \frac{B_c(s)}{A_c(s)} e^{-s\tau}, \quad \tau \geq 0, \tag{1}$$

where $B_c(s)$ and $A_c(s)$ are coprime polynomials in s. Since we are interested in a sampled-data control setup, we consider a zero-order hold discrete-time model for $G_c(s)$ given by $G(z) = B(z)/A(z)$, where $B(z)$ and $A(z)$ are coprime, $B(z) = \sum_{i=0}^{m} b_i z^i$, $A(z) = \sum_{i=0}^{n} a_i z^i$ and $m \leq n$. The following assumption is made throughout this paper:

Assumption 1

(a) The sampling is nonpathological, i.e., if p_i, $i = 1, 2, \ldots, n$ is a pole of $G_c(s)$, then

$$\Im\{p_i\} \neq \kappa\omega_s; \quad \forall\kappa \in \mathbb{Z}, \quad i = 1, 2, \ldots, n, \tag{2}$$

where ω_s is the sampling frequency.

(b) $B(1) \neq 0$.

Assumption 1(a) ensures that no natural modes of the continuous-time model are lost through the sampling process, while Assumption 1(b) is necessary for tracking constant reference signals with an internally stable control loop. Furthermore, to simplify to subsequent derivations, without loss of generality we assume that $B(z)$ is scaled such that $B(1) = 1$. We also introduce the decomposition $A(z) = A_-(z)A_+(z)$, where $A_-(z)$ has all its roots in $|z| < 1$, $A_+(z)$ has all its roots in $|z| \geq 1$ and $\deg\{A_+(z)\} = n_+$.

3 Ripple-free deadbeat control

We are interested in the one degree of freedom sampled-data control architecture represented in Figure 1. In that figure, $G_{h0}(s)$ is the model of the zero order sample and hold device and $C(z)$ is the transfer function of the controller.

If the reference signal is a step function, i.e. $r(k) = v\mu(k)$, $v \in \mathbb{R}$, then achieving a ripple-free deadbeat response means that $y_c(t)$ satisfies $y_c(t) = v\,, \forall t > N\Delta$, where $N \in \mathbb{N}_0$ is called the *deadbeat horizon* of the control system. Otherwise stated, a controller is a ripple-free deadbeat controller if and only if it provides perfect steady state tracking at D.C. and forces the output of the plant to settle in a finite number of samples, while avoiding intersample ripple beyond the deadbeat horizon. Ripple in a deadbeat response arises when the controller cancels the minimum phase zeros of $G(z)$. Those cancelled zeros appear as closed loop poles and generate natural modes in the control input that, in turn, appear in the intersample response of the continuous-time output. In the context of sampled-data control systems, this issue is of major significance, since the discrete-time model $G(z)$ usually contains sampling zeros located in the negative real axis and therefore, its cancellation leads to oscillatory modes in the deadbeat response.

Since the sampling is assumed to be nonpathological, a necessary and sufficient condition to avoid intersample ripple is that the control sequence also settles in, at most, N samples (see [17,3]). This condition is equivalent to $u(k) = u_{ss}\,, \forall k > N$, where u_{ss} is the steady state value of the control sequence. We thus conclude that a stabilizing controller $C(z)$ with integral action provides a ripple-free deadbeat response, with horizon N, if and only if it is such that the complementary sensitivity, $T(z)$, and the control sensitivity, $S_u(z)$, are FIR transfer functions of N^{th} order.

From previous results [14] it is known that the minimum value for N is given by $N_{min} = n + n_+$. Therefore, the deadbeat horizon is, in general, $N = N_{min} + \ell$, with $\ell \in \mathbb{N}$. We next give a parameterization of ripple-free deadbeat controllers for constant reference signals that will prove useful for the purpose of this work.

Lemma 2 *Consider the control system of Figure 1 with $G(z)$ and $G_c(s)$ satisfying Assumption 1. Then, the controller $C(z)$ is stabilizing and achieves ripple-free deadbeat control for step references in $N = N_{min} + \ell$ samples if and only if*

$$C(z) = \frac{P_o(z) + X_\ell(z)A(z)}{L_o(z) - X_\ell(z)B(z)}, \tag{3}$$

where $X_\ell(z)$ is any FIR transfer function of ℓ^{th} order and $C_o(z) = \frac{P_o(z)}{L_o(z)}$ is a biproper controller of n^{th} order achieving minimum horizon ripple free deadbeat control such that

$$A(z)L_o(z) + B(z)P_o(z) = z^{N_{min}} A_-(z), \tag{4}$$

and

$$L_o(1) = X_\ell(1) = 0. \tag{5}$$

Proof

- ***Sufficiency.***
  Here we shall prove that with $C(z)$ defined as in (3), and satisfying (4)-(5), the controller achieves ripple-free deadbeat behavior. We first note that (4) is the Diophantine equation that arises from solving a standard pole assignment problem [7]. Here, N_{min} closed loop poles are assigned to the origin and the rest are equal to the stable poles of $G(z)$. This implies that the controller $C_o(z)$ is stabilizing and must cancel every stable pole of $G(z)$. Hence we may write $P_o(z) = \tilde{P}_o(z)A_-(z)$, so that (4) can be written as

  $$A_+(z)L_o(z) + B(z)\tilde{P}_o(z) = z^{N_{min}}. \tag{6}$$

  Therefore, we have that the complementary and control sensitivities [7] are given by

  $$T(z) = \frac{B(z)(\tilde{P}_o(z) + A_+(z)X_\ell(z))}{z^{N_{min}}}, \tag{7}$$

  $$S_u(z) = \frac{A(z)(\tilde{P}_o(z) + A_+(z)X_\ell(z))}{z^{N_{min}}}. \tag{8}$$

  $C(z)$ is stabilizing since it does not cancel any unstable pole of $G(z)$ and both $T(z)$ and $S_u(z)$ are stable. In addition, we observe that if $X_\ell(z)$ is a FIR transfer function of ℓ^{th} order satisfying (5), then $T(z)$ and $S_u(z)$ are FIR of order $N = N_{min} + \ell$, with $T(1) = 1$, which are necessary and sufficient conditions for ripple-free deadbeat control.
- ***Necessity.***
  From the Youla-Kučera parameterization [7] we have that any stabilizing controller for $G(z)$ can be expressed as

  $$C(z) = \frac{P_o(z) + Q(z)A(z)}{L_o(z) - Q(z)B(z)}, \tag{9}$$

  where $Q(z)$ is any stable and proper transfer function and $P_o(z)/L_o(z)$ is any stabilizing controller for $G(z)$. We thus choose that controller to satisfy (4) with $L_o(1) = 0$ and hence

  $$T(z) = \frac{B(z)\left(\tilde{P}_o(z) + A_+(z)Q(z)\right)}{z^{N_{min}}} \tag{10}$$

  $$S_u(z) = \frac{A(z)\left(\tilde{P}_o(z) + A_+(z)Q(z)\right)}{z^{N_{min}}} \tag{11}$$

  However, ripple-free deadbeat requires that $T(z)$ and $S_u(z)$ to be FIR of N^{th} order, i.e., $Q(z) = X_\ell(z)$ must be FIR of ℓ^{th} order, and also $T(1) = 1$, which implies that $X_\ell(1) = 0$. □

Expression (3) shows that the proposed deadbeat controller cancels every stable pole of the plant model (recall that $P_o(z) = A_-(z)\tilde{P}_o(z)$) and that *it does not cancel any zero*, which is necessary to obtain a ripple-free deadbeat response. Also, the cancellation of $A_-(z)$ by $C(z)$ implies that the set of closed loop poles is the union of two subsets: one containing only N poles at the origin, and the other including all stable plant poles. This implies that the response to input disturbances does not exhibit deadbeat behavior. Nonetheless, if the focus is in the response to reference signals or output disturbances, then only the poles at the origin matter and the ripple-free deadbeat response is achieved. The problem of obtaining a deadbeat response even for input disturbances is more general than the one treated in this paper and the reader is referred to [12] for recent results on the topic.

The characterization of Lemma 2 provides the general form for a ripple-free deadbeat controller as a function of the free FIR transfer function $X_\ell(z)$. Since $X_\ell(z)$ must be a FIR transfer function of ℓ^{th} order, we may write it as

$$X_\ell(z) = \frac{D_\ell(z)}{z^\ell}, \tag{12}$$

where $D_\ell(z)$ is a polynomial such that $\deg\{D_\ell(z)\} \leq \ell$ and

$$D_\ell(1) = 0, \tag{13}$$
$$D_\ell(0) \neq 0, \forall\, \ell \neq 0. \tag{14}$$

The parameter $X_\ell(z)$ is completely determined by the polynomial $D_\ell(z)$. As a consequence, instead of $X_\ell(z)$, we will treat $D_\ell(z)$ as the free parameter. In the sequel, we will always consider that $\deg\{D_\ell(z)\} = \ell$. This is advantageous since if we choose a biproper $X_\ell(z)$, then for a fixed ℓ the available number of free parameters in $D_\ell(z)$ is maximized, which is beneficial from an optimization point of view.

If we are interested in achieving a minimum horizon ripple-free deadbeat response, then the associated controller is unique and can be obtained by choosing $\ell = 0$. Therefore $D_0(z) = 0$ is the only feasible choice for the free polynomial in (12). However, if we allow larger deadbeat horizons, additional degrees of freedom appear in the controller. These are given by the polynomial coefficients of $D_\ell(z)$, and may be chosen to satisfy additional design criteria. In this paper we are interested in adjusting the polynomial $D_\ell(z)$ in such a way that a two objective quadratic cost function is optimized.

From (3) it can be seen that $C(z)$ belongs to the class of biproper controllers, which are known to provide implementation benefits such as the inclusion of anti-windup mechanisms [7], making them popular in industrial applications. A parameterization of ripple-free deadbeat controllers such as that of Lemma 2 is not new in the literature and its structure follows the same guidelines as those in [14]. It has been reformulated in this paper using the complex variable z (in [14] the authors work in z^{-1}) in a more compact way, and a simpler proof based on the Youla-Kučera parameterization has been included for completeness.

4 Optimal controller design

In this section we provide a methodology to design ripple-free deadbeat controllers that achieve good transient performance. The main issue to solve this problem is how to choose the polynomial $D_\ell(z)$ in (12), in such a way that an appropriate transient response is achieved. Following a similar idea to that in [1], we tackle this problem by choosing a polynomial $D_\ell(z)$ that minimizes a performance index representative of some standard control objectives. The essential distinction of our approach with respect to [1] is the use of an $\mathcal{H}_2$ measure as the target function to be optimized.

We can measure the quality of the transient performance through the cost function

$$J_e = \sum_{k=0}^{N} e(k)^2, \tag{15}$$

where $e(k)$ is the tracking error sequence. The quantity J_e is the energy of the tracking error sequence and has been widely used as a tracking performance measure [13,18] in the context of performance bounds computation. On the other hand, a common criticism made to deadbeat control is that it usually demands a large control energy. This observation suggests the introduction of an optimality criterion that weights the control effort necessary to achieve the deadbeat response. To that end, in an analogous fashion as in [22], we define the cost functional

$$J_u = \sum_{k=0}^{N} (u(k) - u_{ss})^2. \tag{16}$$

This measure accounts for the control effort by means of the energy of its deviation from the steady state value. It is worth noting that in the definitions of both J_e and J_u, the deadbeat feature of the transient response has been implicitly taken into account, since both sums are made only up to the N^{th} sample. It is simple to prove that these measures can be conveniently expressed in the frequency domain as

$$J_e\left(D_\ell(z)\right) = \left|\left| \frac{S(z)}{z-1} v \right|\right|_2^2, \tag{17}$$

$$J_u\left(D_\ell(z)\right) = \left|\left| \frac{S_u(z) - A(1)}{z-1} v \right|\right|_2^2, \tag{18}$$

where $S(z)$ is the sensitivity function and the argument of both cost functions has been made explicit to emphasize their dependence on the free polynomial $D_\ell(z)$. In the sequel, and without loss of generality, we assume that $v = 1$. We are interested in including both partial costs, tracking error and control energies, in our optimal control problem. To that end, we define the two objective cost functional

$$\mathcal{J}\left(D_\ell(z)\right) = \lambda J_e\left(D_\ell(z)\right) + (1-\lambda) J_u\left(D_\ell(z)\right), \tag{19}$$

where $0 \le \lambda \le 1$. We are now able to state the optimization problem of our interest.

Problem 3 *Given a discrete-time plant model $G(z)$ satisfying Assumption 1 and a prescribed deadbeat horizon $N = N_{min} + \ell$, $\ell \in \mathbb{N}_0$, find a polynomial $D_\ell^o(z)$ such that*

$$D_\ell^o(z) = \arg \min_{D_\ell(z) \in \mathbb{P}^\ell} \mathcal{J}\left(D_\ell(z)\right), \tag{20}$$

where $\mathbb{P}_\ell$ is the set of polynomials of ℓ^{th} order and $D_\ell^o(z)$ satisfies (13) *and* (14).

As stated in the previous section, if we choose $\ell = 0$ there is a unique feasible polynomial that solves Problem 3. Hence, in such a case, the controller is uniquely determined and no optimization is possible; if we choose larger deadbeat horizons, then the coefficients of $D_\ell(z)$ may be computed as a solution of Problem 3. For that purpose we define

$$H^e(z) = B(z)A_+(z) = \sum_{i=0}^{m+n_+} h_i^e z^i, \tag{21}$$

$$H^u(z) = A(z)A_+(z) = \sum_{i=0}^{n+n_+} h_i^u z^i, \tag{22}$$

$$F^e(z) = \frac{z^N - z^\ell B(z)\tilde{P}_o(z)}{z-1}, \tag{23}$$

$$F^u(z) = \frac{z^N A(1) - z^\ell A(z)\tilde{P}_o(z)}{z-1}, \tag{24}$$

where $\tilde{P}_o(z) = P_o(z)/A_-(z)$ and $P_o(z)$ is defined in Lemma 2. From the proof of Lemma 2 we have that $\tilde{P}_o(z)$ satisfies (6), so that $\tilde{P}_o(1) = 1$ and therefore both $F^e(z)$ and $F^u(z)$ are always polynomials (recall that $B(1) = 1$). Thus, we may let $F^e(z) = \sum_{i=0}^{N-1} f_i^e z^i$ and $F^u(z) = \sum_{i=0}^{N-1} f_i^u z^i$. With these definitions, an analytic solution to Problem 3 can be found, as shown next.

Theorem 4 *Consider the parameterization of ripple-free deadbeat controllers given in Lemma 2. Then, the optimal polynomial that solves Problem 3 is given by*

$$D_\ell^o(z) = (z-1)\begin{bmatrix}1 & z & \cdots & z^{\ell-1}\end{bmatrix}\begin{bmatrix}\sqrt{\lambda}\boldsymbol{\Gamma_\ell} \\ \sqrt{1-\lambda}\boldsymbol{\Psi_\ell}\end{bmatrix}^\dagger \begin{bmatrix}\sqrt{\lambda}\boldsymbol{\gamma_\ell} \\ \sqrt{1-\lambda}\boldsymbol{\psi_\ell}\end{bmatrix}, \tag{25}$$

where $(\cdot)^\dagger$ denotes the Moore-Penrose pseudoinverse [5], and $\boldsymbol{\Gamma_\ell} \in \mathbb{R}^{N\times\ell}$, $\boldsymbol{\Psi_\ell} \in \mathbb{R}^{N\times\ell}$, $\boldsymbol{\gamma_\ell} \in \mathbb{R}^{N\times 1}$ and $\boldsymbol{\psi_\ell} \in \mathbb{R}^{N\times 1}$ are defined as

$$\boldsymbol{\Gamma_\ell} = \left[\begin{array}{cccc} h_0^e & 0 & \cdots & 0 \\ h_1^e & h_0^e & \cdots & 0 \\ \vdots & h_1^e & \ddots & \vdots \\ h_{m+n_+}^e & \vdots & \ddots & 0 \\ 0 & h_{m+n_+}^e & \ddots & h_0^e \\ 0 & 0 & \ddots & h_1^e \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & h_{m+n_+}^e \\ \hdashline \multicolumn{4}{c}{\mathbf{0}_{(n-m)\times\ell}} \end{array}\right], \tag{26}$$

$$\boldsymbol{\Psi_\ell} = \begin{bmatrix} h_0^u & 0 & \cdots & 0 \\ h_1^u & h_0^u & \cdots & 0 \\ \vdots & h_1^u & \ddots & \vdots \\ h_{n+n_+}^u & \vdots & \ddots & 0 \\ 0 & h_{n+n_+}^u & \ddots & h_0^u \\ 0 & 0 & \ddots & h_1^u \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & h_{n+n_+}^u \end{bmatrix}, \tag{27}$$

$$\boldsymbol{\gamma_\ell} = \begin{bmatrix} f_0^e \\ f_1^e \\ \vdots \\ f_{N-1}^e \end{bmatrix}, \quad \boldsymbol{\psi_\ell} = \begin{bmatrix} f_0^u \\ f_1^u \\ \vdots \\ f_{N-1}^u \end{bmatrix}, \tag{28}$$

with $\mathbf{0}_{(n-m)\times\ell} \in \mathbb{R}^{(n-m)\times\ell}$ denoting the zero matrix of the specified dimensions.

Proof:

From Lemma 2 and (12) we have that

$$S(z) = \frac{z^N - B(z)P(z)}{z^N}, \tag{29}$$

$$S_u(z) = \frac{A(z)P(z)}{z^N}, \tag{30}$$

where $P(z) = z^\ell \tilde{P}_o(z) + A_+(z)D_\ell(z)$. The constraint (13) can be satisfied if we write $D_\ell(z)$ as

$$D_\ell(z) = (z-1)\tilde{D}_\ell(z), \tag{31}$$

where $\tilde{D}_\ell(z)$ is an arbitrary polynomial such that $\deg\left\{\tilde{D}_\ell(s)\right\} = \ell - 1$. Substituting (29)-(31) in (17) and (18) yields

$$J_e\left(D_\ell(z)\right) = \left|\left|\frac{z^N - z^\ell B(z)\tilde{P}_o(z)}{z-1} - B(z)A_+(z)\tilde{D}_\ell(z)\right|\right|_2^2, \tag{32}$$

$$= \left|\left|F^e(z) - H^e(z)\tilde{D}_\ell(z)\right|\right|_2^2, \tag{33}$$

$$J_u\left(D_\ell(z)\right) = \left|\left|\frac{z^N A(1) - z^\ell A(z)Q^o(z)}{z-1} - A(z)A_+(z)\tilde{D}_\ell(z)\right|\right|_2^2, \tag{34}$$

$$= \left|\left|F^u(z) - H^u(z)\tilde{D}_\ell(z)\right|\right|_2^2, \tag{35}$$

where $H^e(z)$, $H^u(z)$, $F^e(z)$ and $F^u(z)$ are defined in (21)-(24), and we have used the fact that z^{-N} preserves the $\mathcal{H}_2$ norm. Expressions (33) and (35) can be vectorized as follows

$$J_e(D_\ell(z)) = \left|\left|\begin{bmatrix}1 & z & \dots & z^{N-1}\end{bmatrix}\left(\boldsymbol{\gamma_\ell} - \boldsymbol{\Gamma_\ell}\tilde{\boldsymbol{d}}_{\boldsymbol{\ell}}\right)\right|\right|_2^2, \quad (36)$$

$$J_u(D_\ell(z)) = \left|\left|\begin{bmatrix}1 & z & \dots & z^{N-1}\end{bmatrix}\left(\boldsymbol{\psi_\ell} - \boldsymbol{\Psi_\ell}\tilde{\boldsymbol{d}}_{\boldsymbol{\ell}}\right)\right|\right|_2^2, \quad (37)$$

where $\boldsymbol{\gamma_\ell}$, $\boldsymbol{\Gamma_\ell}$, $\boldsymbol{\psi_\ell}$ and $\boldsymbol{\Psi_\ell}$ are defined in (26)-(28), and $\tilde{\boldsymbol{d}}_{\boldsymbol{\ell}} \in \mathbb{R}^{\ell\times 1}$ is such that

$$\tilde{\boldsymbol{d}}_{\boldsymbol{\ell}} = \begin{bmatrix}\tilde{d}_0 & \tilde{d}_1 & \dots & \tilde{d}_{\ell-1}\end{bmatrix}^T, \quad (38)$$

and $\tilde{D}_\ell(z) = \sum_{i=0}^{\ell-1}\tilde{d}_i z^i$. Substituting (36) and (37) in (19) and using elementary properties of the $\mathcal{H}_2$ norm yields

$$\mathcal{J}(D_\ell(z)) = \left|\left|\begin{bmatrix}\sqrt{\lambda}\boldsymbol{\gamma_\ell} \\ \sqrt{1-\lambda}\boldsymbol{\psi_\ell}\end{bmatrix} - \begin{bmatrix}\sqrt{\lambda}\boldsymbol{\Gamma_\ell} \\ \sqrt{1-\lambda}\boldsymbol{\Psi_\ell}\end{bmatrix}\tilde{\boldsymbol{d}}_{\boldsymbol{\ell}}\right|\right|_e^2, \quad (39)$$

where $||\cdot||_e$ denotes the Euclidean norm. Thus, the original optimization problem can be posed in a standard least squares framework as

$$\tilde{\boldsymbol{d}}^o_{\boldsymbol{\ell}} = \arg\min_{\tilde{\boldsymbol{d}}_{\boldsymbol{\ell}}\in\mathbb{R}^{\ell\times 1}} \mathcal{J}(D_\ell(z)). \quad (40)$$

The optimal vector $\tilde{\boldsymbol{d}}^o_{\boldsymbol{\ell}}$ is then given by [6]

$$\tilde{\boldsymbol{d}}^o_{\boldsymbol{\ell}} = \begin{bmatrix}\sqrt{\lambda}\boldsymbol{\Gamma_\ell} \\ \sqrt{1-\lambda}\boldsymbol{\Psi_\ell}\end{bmatrix}^\dagger \begin{bmatrix}\sqrt{\lambda}\boldsymbol{\gamma_\ell} \\ \sqrt{1-\lambda}\boldsymbol{\psi_\ell}\end{bmatrix}. \quad (41)$$

The result (25) follows by substituting (41) in (31). □

Last theorem is comparable to the results in [1,16,15], in which design methodologies based on optimization routines are proposed. Nevertheless, the framework used throughout this paper allows one to obtain an *analytical* solution to the two objective optimal control problem of interest. This draws a difference with respect to previous work on this topic. This issue, together with the fact we use an $\mathcal{H}_2$ performance index, constitute the main distinctions of this work with respect to [1,16,15]. Theorem 4 provides a ripple-free deadbeat controller design procedure that can be summarized as follows:

i. Choose a deadbeat horizon $N = N_{min} + \ell$ and a scalar weight $0 \leq \lambda \leq 1$.
ii. Compute the unique solution $(P_o(z), L_o(z))$ of the diophantine equation (4) such that $P_o(z) = \tilde{P}_o(z)A_-(z)$, $L_o(1) = 0$ and $\deg\{P_o(z)\} = \deg\{L_o(z)\} = n$.
iii. Compute the polynomials $F^e(z)$, $H^e(z)$, $F^u(z)$ and $H^u(z)$ using (21)-(24).
iv. Compute $D^o_\ell(z)$ from (25) and substitute it in (3) using (12) to obtain the optimal controller.

5 Optimal performance

In the previous section, an explicit formula for the optimal ripple-free deadbeat controller has been derived. We now turn our attention to the performance properties of the optimal design. In particular, this section focuses on a key property of the optimal controller that sheds light on the design tradeoffs that appear in ripple-free deadbeat control. This constitutes the main result of this paper and is stated in next theorem.

Theorem 5 *Let $G(z)$ be a discrete-time plant model and suppose that the ripple-free deadbeat controller defined by Lemma 2 is computed using the optimal polynomial $D^o_\ell(z)$ that solves Problem 3. Then, the optimal cost $\mathcal{J}(D^o_\ell(z))$ is a non-increasing function of ℓ.*

Proof:

To keep a compact notation let us define

$$\mathbf{M}_\ell = \begin{bmatrix}\sqrt{\lambda}\boldsymbol{\Gamma_\ell} \\ \sqrt{1-\lambda}\boldsymbol{\Psi_\ell}\end{bmatrix}, \quad (42)$$

$$\mathbf{m}_\ell = \begin{bmatrix}\sqrt{\lambda}\boldsymbol{\gamma_\ell} \\ \sqrt{1-\lambda}\boldsymbol{\psi_\ell}\end{bmatrix}. \quad (43)$$

It should be observed that $\mathbf{M}_\ell \in \mathbb{R}^{2N\times\ell}$ and has full column rank (which can be verified from (26) and (27)). Hence

$$\mathbf{M}_\ell{}^\dagger = \left(\mathbf{M}_\ell{}^T\mathbf{M}_\ell\right)^{-1}\mathbf{M}_\ell{}^T. \quad (44)$$

Therefore, from (39) and (41), it is simple to prove [6] that the optimal cost for a horizon N is given by

$$\mathcal{J}(D^o_\ell(z)) = \mathbf{m}_\ell{}^T\left(\mathbf{I_{2N}} - \mathbf{M}_\ell\left(\mathbf{M}_\ell{}^T\mathbf{M}_\ell\right)^{-1}\mathbf{M}_\ell{}^T\right)\mathbf{m}_\ell. \quad (45)$$

Analogously, if we increase the deadbeat horizon in one sample, the optimal cost can be computed from

$$\mathcal{J}\left(D^o_{\ell+1}(z)\right) = \mathbf{m}_{\ell+1}{}^T\Big(\mathbf{I_{2N+2}} - \mathbf{M}_{\ell+1}\left(\mathbf{M}_{\ell+1}{}^T\mathbf{M}_{\ell+1}\right)^{-1}\mathbf{M}_{\ell+1}{}^T\Big)\mathbf{m}_{\ell+1}. \quad (46)$$

From the definition of $\boldsymbol{\Gamma_\ell}$ and $\boldsymbol{\Psi_\ell}$ in (26) and (27) we observe that

$$\mathbf{M}_{\ell+\mathbf{1}} = \left[\begin{array}{c:c} \sqrt{\lambda}\, h_o^e & \mathbf{0}_{\mathbf{1}\times\boldsymbol{\ell}} \\ \hdashline \mathbf{q} & \mathbf{T}_\ell \mathbf{M}_\ell \end{array}\right], \tag{47}$$

where $\boldsymbol{q} \in \mathbb{R}^{(2N+1)\times 1}$ is given by

$$\begin{aligned} \boldsymbol{q} = \Big[&\sqrt{\lambda}\, h_1^e \;\cdots\; \sqrt{\lambda}\, h_{m+n_+}^e \;\; \mathbf{0}_{\mathbf{1}\times(\boldsymbol{n}-\boldsymbol{m}+\boldsymbol{\ell})} \\ &\sqrt{1-\lambda}\, h_0^u \;\cdots\; \sqrt{1-\lambda}\, h_{n+n_+}^u \;\; \mathbf{0}_{\mathbf{1}\times\boldsymbol{\ell}} \Big]^T, \end{aligned} \tag{48}$$

and

$$\mathbf{T}_\ell = \Big[\mathbf{e_1} \;\cdots\; \mathbf{e_N} \;\; \mathbf{e_{N+2}} \;\cdots\; \mathbf{e_{2N+1}}\Big]^T \in \mathbb{R}^{(2N+1)\times 2N}, \tag{49}$$

with $\mathbf{e_i}$ denoting the i^{th} element of the canonical basis of $\mathbb{R}^{2N+1}$. Similarly, from (23), (24), (28) and (43), it follows that the vector $\mathbf{m}_{\ell+\mathbf{1}}$ can be conveniently expressed as

$$\mathbf{m}_{\ell+\mathbf{1}} = \begin{bmatrix} 0 \\ \mathbf{T}_\ell \mathbf{m}_\ell \end{bmatrix}. \tag{50}$$

Equation (47) implies that

$$\mathbf{M}_{\ell+\mathbf{1}}{}^T \mathbf{M}_{\ell+\mathbf{1}} = \begin{bmatrix} \lambda {h_0^e}^2 + \mathbf{q}^T\mathbf{q} & \mathbf{q}^T \mathbf{T}_\ell \mathbf{M}_\ell \\ \mathbf{M}_\ell{}^T \mathbf{T}^T \mathbf{q} & \mathbf{M}_\ell{}^T \mathbf{M}_\ell \end{bmatrix}. \tag{51}$$

With some algebra after substituting (47), (50)-(51) in (46) and applying the matrix inversion lemma [5] it follows that

$$\begin{aligned} \mathcal{J}\left(D_{\ell+1}^o(z)\right) &= \left(\mathbf{T}_\ell \mathbf{m}_\ell\right)^T \boldsymbol{\Lambda} \left(\mathbf{T}_\ell \mathbf{m}_\ell\right) - \\ &\frac{1}{\kappa}\left(\mathbf{T}_\ell \mathbf{m}_\ell\right)^T \boldsymbol{\Lambda} \mathbf{q}\mathbf{q}^T \boldsymbol{\Lambda} \left(\mathbf{T}_\ell \mathbf{m}_\ell\right), \end{aligned} \tag{52}$$

where

$$\boldsymbol{\Lambda} = \boldsymbol{I}_{\mathbf{2N+1}} - \left(\mathbf{T}_\ell \mathbf{M}_\ell\right)\left(\left(\mathbf{T}_\ell \mathbf{M}_\ell\right)^T \left(\mathbf{T}_\ell \mathbf{M}_\ell\right)\right)^{-1} \left(\mathbf{T}_\ell \mathbf{M}_\ell\right)^T, \tag{53}$$

$$\kappa = \lambda {h_0^e}^2 + \mathbf{q}^T \boldsymbol{\Lambda} \mathbf{q}. \tag{54}$$

Using the fact that $\mathbf{T}_\ell{}^T \mathbf{T}_\ell = \mathbf{I_{2N}}$ in (52) and comparing with (45), we recognize the optimal cost for horizon N in the first term of (52), so that $\mathcal{J}\left(D_{\ell+1}^o(z)\right)$ can be expressed as

$$\mathcal{J}\left(D_{\ell+1}^o(z)\right) = \mathcal{J}\left(D_\ell^o(z)\right) - \left(\frac{1}{\sqrt{\kappa}} \mathbf{m}_\ell{}^T \left(\mathbf{I_{2N}} - \mathbf{M}_\ell \left(\mathbf{M}_\ell{}^T \mathbf{M}_\ell\right)^{-1} \mathbf{M}_\ell{}^T\right) \mathbf{T}^T \mathbf{q}\right)^2. \tag{55}$$

Therefore it can be concluded that $\mathcal{J}\left(D_{\ell+1}^o(z)\right) \leq \mathcal{J}\left(D_\ell^o(z)\right),\ \forall \ell \geq 0$ and the result follows.

□

The result in Theorem 5 reveals an essential tradeoff in the design of ripple-free deadbeat control systems. If the focus of the designer is to have a deadbeat response with short settling time, then last theorem implies that this can only be done at the expense of sacrificing performance. Similarly, one can only improve the transient performance by extending the settling time (and hence the complexity of the controller).

This kind of design tradeoff has already been reported in [1] for a mixed l_∞/l_1 performance measure that accounts for the same elements as those in this work, i.e., tracking error and control input measures. Therefore, since in this paper an $\mathcal{H}_2$ performance index is used, our result stands as a complement to those previous findings and adds an insightful conclusion to the topic: performance vs. settling time tradeoffs are inherent to ripple-free deadbeat controller design. It should be emphasized that our conclusion cannot be intuitively derived from the results in [1], since l_∞/l_1 and $\mathcal{H}_2$ measures are not related to the same aspects of the transient response.

6 Conclusions

In this paper a ripple-free deadbeat controller design methodology for step references has been proposed and analyzed. The ripple-free property of the controller means that after the settling time, no ripple appears in the output of the underlying continuous time plant. The design algorithm is applicable to general SISO plants under sampled data control and is such that, for a fixed settling time, the controller is adjusted to minimize a quadratic cost function. This cost function penalizes a combination of the energy of both the tracking error and the control signal, and has been minimized with aid of the standard properties of the $\mathcal{H}_2$ norm. This also allowed us to derive an explicit formula for the optimal controller.

Since the controller is obtained from an optimization procedure, the only design parameters in this strategy are the settling time and a scalar weighting factor. The most significative result of this paper is showing that the optimal cost function is a non-increasing function of the settling time for any value of the weighting factor. This conclusion is a complement to existent results in the literature [1] that use different norms and therefore, reveals that the settling time/performance tradeoff is essential in ripple-free deadbeat control.

Acknowledgements

The authors gratefully acknowledge the support received from Universidad Técnica Federico Santa María (PAC program and grant UTFSM 230623) and from CONICYT through grant Fondecyt 1060437.

References

[1] A. Casavola, E. Mosca, and P. Zecca. Robust ripple-free deadbeat control design. *International Journal of Control*, 72(6):564–573, 1999.

[2] A. Emami-Naeini and G. Franklin. Deadbeat control and tracking of discrete-time systems. *IEEE Transactions on Automatic Control*, 27(1):176–181, 1982.

[3] G. F. Franklin and A. Emami-Naeini. Design of ripple free multivariable robust servomechanisms. *IEEE Transactions on Automatic Control*, 31(7):661–664, 1986.

[4] Y. Funahashi and H. Katoh. Robust-tracking deadbeat control. *International Journal of Control*, 56(1):213–225, 1992.

[5] G.H. Golub and C.F. Van Loan. *Matrix computations*. The John Hopkins University Press, Baltimore, Maryland, 3rd edition, 1996.

[6] G.C. Goodwin and R.L. Payne. *Dynamic System Identification: Experiment Design and Data Analysis*. Academic Press, New York, 1977.

[7] Graham C. Goodwin, Stefan Graebe, and Mario E. Salgado. *Control System Design*. Prentice Hall, New Jersey, 2001.

[8] T. Hartley, R. Veillette, and G. Cook. Techniques in deadbeat and one-step ahead control. In *Digital Control Systems Implementation and Computational Techniques: Control and Dynamic Systems & Advances in Theory and Applications*, volume 79, chapter 3. Academic Press NY, 1996.

[9] V. Kučera. *Discrete Linear Control: The Polynomial Equation Approach*. John Wiley & Sons, Inc. New York, NY, USA, 1979.

[10] V. Kučera. Diophantine equations in control — a survey. *Automatica*, 29:1361–1375, 1993.

[11] J. Lam, H. K. Tso, and N. K. Tsing. Robust deadbeat regulation. *International Journal of Control*, 67(4):587–602, 1997.

[12] E. Milonidis and E. Karcanias. Finite settling time stabilisation for multivariable discrete-time systems: a polynomial equation approach. *International Journal of Control*, 79(12):1565–1580, 2006.

[13] M. Morari and E. Zafiriou. *Robust process control*. Prentice Hall Inc., Englewood Cliffs, New Jersey, 1989.

[14] E. Nobuyama. Parametrization of all ripple-free deadbeat controllers. *Systems and Control Letters*, 21(3):217–244, 1993.

[15] R. Paz. Ripple-free tracking with robustness. *International Journal of Control*, 79(6):543–568, 2006.

[16] R. Paz and H. Elaydi. Optimal ripple-free deadbeat controllers. *International Journal of Control*, 71(6):1087–1104, 1998.

[17] H. R. Sirisena. Ripple-free deadbeat control of SISO discrete systems. *IEEE Transactions on Automatic Control*, 30(2):168–170, 1985.

[18] O. Toker, L. Chen, and L. Qiu. Tracking performance limitations in LTI multivariable discrete-time systems. *IEEE Transactions on Circuits and Systems—Part I: Fundamental Theory and Applications*, 49(5):657–670, 2002.

[19] S. Urikura and A. Nagata. Ripple-free deadbeat control for sampled-data systems. *IEEE Transactions on Automatic Control*, 32(6):474–482, 1987.

[20] Y. Zhao and H. Kimura. Dead-beat control with robustness. *International Journal of Control*, 43(5):1427–1440, 1986.

[21] Y. Zhao and H. Kimura. Two-degrees-of-freedom dead-beat control system with robustness. *International Journal of Control*, 48(1):303–315, 1988.

[22] Y. Zhao and H. Kimura. Optimization of control input for deadbeat control systems. *International Journal of Control*, 59(4):1107–1117, 1994.

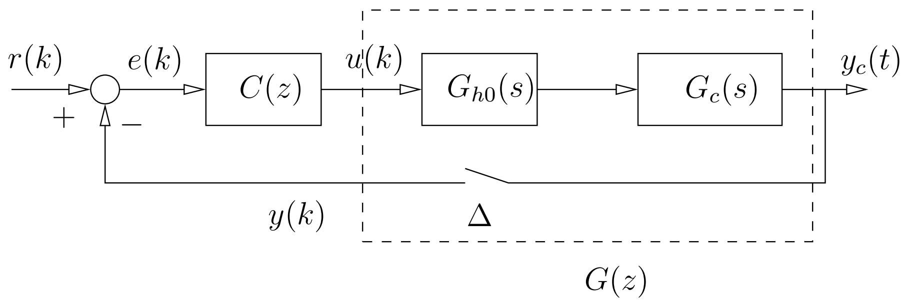

Fig. 1. Sampled data control loop.