Two objective optimal multivariable ripple-free deadbeat control

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Abstract

This paper presents novel results on optimal multivariable deadbeat control. Given a discrete-time, stable, linear and time invariant plant model, we give a simple parameterisation of all stabilising ripple-free deadbeat controllers of a given order. The free parameter is then optimised in the sense that a quadratic index is kept minimal. The optimality criterion has the advantage of accounting for both tracking performance and magnitude of the control effort. The proposed design procedure is simple to use and allows the tuning of the controller with a scalar weighting factor. Simulation results are included to illustrate the effectiveness of the proposed design algorithm.

Keywords: Ripple-free deadbeat control, multivariable control, optimal control.

1 Introduction

Deadbeat control is a well established discrete-time control technique since some decades ago. The original formulation in [1] yields a control policy that settles the tracking error sequence to zero in a minimum number of time steps, which is sometimes termed as *minimum prototype* control. Since this formulation is based on pole and zero cancelations between controller and plant model, it has problems when dealing with non minimum phase and/or unstable plants. Later contributions (see [2] and the references therein) solved this problem and allow to extend the settling time of the tracking error while preserving the deadbeat feature.

However, in the context of sampled-data control systems such approach may lead to undesirable loop performance. Indeed, since it only ensures that the error sequence is zero at sampling times (beyond the settling time) and there are no considerations about the intersample behaviour, undesirable ripple may appear in the output. This issue is dealt with in [3, 4], where fair general parameterisations of *ripple-free* deadbeat controllers are given. The basic idea behind this approach is that to avoid any intersample ripple after the settling time, the control sequence must also reach its steady state in, at most, the same number of samples. These results apply to stable and unstable plants with arbitrary reference signals, provided that the sampled-data model satisfies the internal model principle for the corresponding reference [5].

If we look for a ripple-free deadbeat controller with certain pre-specified settling time, there are in the general case, an infinite number of solutions. Ripple-free deadbeat performance is achieved in a number of samples that is equal to the order of the controller. Hence, it turns out that as the settling time increases, there are more degrees of freedom in the controller that may be adjusted in order to obtain a suitable loop response. This issue has motivated researchers to seek for design methods that allow to choose those additional design parameters in such a way that other control objectives may be accounted for. In the single-input/single-output (SISO) case, an interesting result can be found in [6], where the authors optimise the deadbeat control law in order to minimise the $\mathcal{L}_2$ norm of the sensitivity function [5]. An insightful result of that work is that this quantity can always be reduced at the expense of increasing the settling time, revealing a design tradeoff which is inherent of deadbeat control. This issue is dealt with in [7], where the authors use a two degree of freedom controller to achieve minimal time deadbeat response and to minimise the same measure as that of [6]. An extension of this result using an $\mathcal{L}_\infty$ robustness index can be found in [8]. It must be pointed out that the results in [6, 7, 8] rely on deadbeat controller parameterisations that do not ensure a ripple-free response. A state space approach to the optimal design problem of ripple-free deadbeat controllers can be found in [9, 10]. Interesting results in the frequency domain are reported in [11] and [12], where the optimisation problem accounts for both robustness and performance objectives and it is solved by means of standard numerical algorithms. Additionally, in [13] the authors derive a ripple-free deadbeat controller such that the energy of the control signal is kept minimal.

The multivariate (MIMO) results in this topic are less abundant. MIMO extensions of [6] and [7] can be found in [14] and [15], respectively. A rather similar control technique, called *output deadbeat control*, is posed as an optimal control problem in [16, 17]. In those works, the authors solve the problem in a state space framework using standard LQR theory [18] with no cost on the control signal (cheap control). The resulting control law drives the output to zero in minimum number of samples avoiding any intersample ripple after the settling time. A generalisation of these results to descriptor systems can be found in [19].

This paper deals with optimal ripple-free deadbeat control for MIMO, stable, linear and time invariant plants. Firstly, we derive a simple characterisation of ripple-free deadbeat controllers for stable plants and constant reference signals. Despite our characterisation is less general than that of [4], it proves to be useful in solving an optimal control problem that minimises a two objective performance measure. This measure accounts for the energy of the tracking error, which has been widely used as a performance measure [20], and also for the energy of the control signal. This implies that the derived control law exhibits ripple-free deadbeat behaviour while attaining an optimal performance in terms of both tracking behaviour and control effort. We stress that the latter property is important from an application point of view and that the excessive control magnitudes, typical from deadbeat control, have been one of the main criticisms to this control technique. The results of this paper are the multivariate extensions of those of [21]. We also note that this research work follows the philosophy of [13] and [14], but we introduce the novelty of dealing with a combined optimality criterion and ensuring a MIMO *ripple-free* deadbeat response.

The paper is organised as follows: Section 2 defines the main notation and the assumptions made throughout this paper. Section 3 summarises the fundamentals ideas on deadbeat control and in Section 4 we derive a parameterization of a general MIMO ripple-free deadbeat controller for stable plants and constant reference signals. The main result of this article is given in Section 5. Finally, an illustrative numerical example and the concluding remarks of this work are given in Section 6 and 7, respectively.

2 Assumptions and definitions

Given a polynomial matrix $\boldsymbol{K}(z) = \sum_{i=0}^{n} \boldsymbol{K_i} z^i$ with $\boldsymbol{K_i} \in \mathbb{R}^{p\times p}$ such that $\boldsymbol{K_n}$ is not identically the zero matrix, we will refer to n as the degree of $\boldsymbol{K}(z)$. Consequently, $\mathbb{P}_n$ is defined as the set of polynomial matrices of degree n. The $(i,j)^{th}$ entry of $\boldsymbol{K}(z)$ is denoted as $K_{ij}(z)$ and its degree as $\deg\{K_{ij}(z)\}$. This implies that the degree of $\boldsymbol{K}(z)$ is equal to $n = \max_{i,j}\{\deg\{K_{ij}(z)\}\}$.

For notational purposes, the $p \times p$ null matrix will be denoted as $\boldsymbol{0}_{p\times p}$ (when the dimension is clear from the context, the subscript will be omitted). Similar convention holds for the identity matrix $\boldsymbol{I_{p\times p}}$.

In this paper we consider the plant model $\boldsymbol{G}(z)$ to be a stable $p\times p$ discrete-time transfer matrix. We will assume that $\boldsymbol{G}(z)$ is represented in right coprime polynomial matrix fraction description [22, 23] as

$$\boldsymbol{G}(z) = \boldsymbol{B}(z)\boldsymbol{A}(z)^{-1}, \tag{1}$$

where $\boldsymbol{A}(z)$ and $\boldsymbol{B}(z)$ are right coprime polynomial matrices of dimension $p\times p$. Algorithms to build coprime polynomial factorisations of transfer matrices can be found in [24, 25, 26]. Throughout this paper, we assume that $\boldsymbol{G}(z)$ has no zeros on $z = 1$, i.e. $\boldsymbol{B}(1)$ is nonsingular and, without loss of generality, we further assume that $\boldsymbol{B}(1) = \boldsymbol{I}$. Given that $\boldsymbol{B}(1)$ is nonsingular, this last assumption can always be satisfied by introducing an appropriate scaling matrix. The nonsingularity of $\boldsymbol{B}(1)$ is a standard condition necessary for being able to track constant reference signals.

Given a proper transfer matrix $\boldsymbol{M}(z)$, we define the *right degree interactor* (RDI) of $\boldsymbol{M}(z)$ as a polynomial matrix $\boldsymbol{E}(z)$ such that the product $\boldsymbol{M}(z)\boldsymbol{E}(z)$ is biproper, i.e.

$$\lim_{z\to\infty} \boldsymbol{M}(z)\boldsymbol{E}(z) = \boldsymbol{D}, \tag{2}$$

where $0 < \det\{\boldsymbol{D}\} < \infty$. Algorithms to build different types of RDI matrices can be found in [27, 28].

3 Deadbeat control essentials

We briefly recall the fundamental ideas underlying deadbeat control theory. Consider a continuous-time plant with transfer function $\boldsymbol{G_c}(s)$ which is digitally controlled through a zero order sample and hold device with transfer function $\boldsymbol{G_{h0}}(s)$, by a linear discrete-time feedback controller with transfer function $\boldsymbol{C}(z)$. A block diagram of the sampled data control loop is shown in Figure 1. Also assume that the sampling frequency $\omega_s = 2\pi/\Delta$ is such that the sampling is nonpathological [29], i.e. the imaginary part of every plant poles p_i, $i = 1, 2, \ldots, n_p$ satisfies

$$\Im\{p_i\} \neq \kappa\omega_s; \quad \forall\kappa \in \mathbb{Z}, \quad i = 1, 2, \ldots, n_p. \tag{3}$$

We then have that the pulse transfer function $\boldsymbol{G}(z)$ of the sampled data open loop system is

$$\boldsymbol{G}(z) = \mathcal{Z}\left\{\boldsymbol{G_{h0}}(s)\boldsymbol{G_c}(s)\right\} = \boldsymbol{B}(z)\boldsymbol{A}(z)^{-1}, \tag{4}$$

where $\boldsymbol{B}(z)$ and $\boldsymbol{A}(z)$ are right coprime polynomial matrices. Given that the reference vector signal is a assumed to be step function, i.e. $\boldsymbol{r}(k) = \boldsymbol{v}\mu(k)$, $\boldsymbol{v} \in \mathbb{R}^p$, then having a ripple-free deadbeat control loop means that $\boldsymbol{y_c}(t)$ satisfies

$$\boldsymbol{y_c}(t) = \boldsymbol{v} \quad , \forall t > N\Delta, \tag{5}$$

where $N \in \mathbb{N}$ is called the *deadbeat horizon* of the control system. Hence, a controller is of ripple-free deadbeat class if it provides perfect steady state tracking at D.C. and makes the output of the plant to settle in a finite number of samples, while avoiding any intersample ripple beyond the deadbeat horizon. Ripple in a deadbeat response arises when the controller cancels the minimum phase zeros of $\boldsymbol{G}(z)$. Those cancelled zeros appear as closed loop poles and generate natural modes in the control input that in turn, appear in the intersample response of the continuous-time output. Such behaviour can be seen in Figure 2, where it is noticeable that, although the sampled output settles in one sample, the continuous-time output exhibits considerable ripple. In the context of sampled-data control systems, this issue is of considerable importance, since the discrete-time model $\boldsymbol{G}(z)$ usually contains sampling zeros located in the negative real axis and therefore, its cancellation leads to oscillatory modes in the deadbeat response.

Since the sampling is assumed to be nonpathological, a necessary and sufficient condition to avoid the intersample ripple is that the control sequence also settles in, at most, N samples. This condition is equivalent to

$$\boldsymbol{u}(k) = \boldsymbol{u}_{ss} \quad , \forall k > N, \tag{6}$$

where the constant vector $\boldsymbol{u_{ss}}$ is the steady state value of the control sequence. In the SISO case, standard results [30] report that the ripple-free deadbeat behaviour can be achieved only if N is at least equal to the order of the discrete-time plant model. Next section addresses the issue of finding a general form of a MIMO ripple-free deadbeat controller, which is the key to solve the optimal control problem set in this paper.

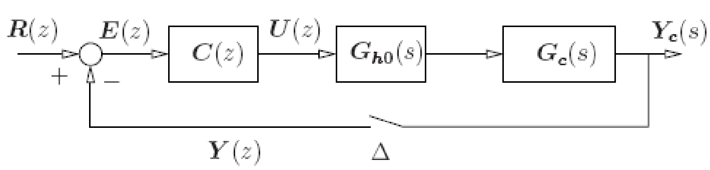

Figure 1: Multivariable sampled data control loop.

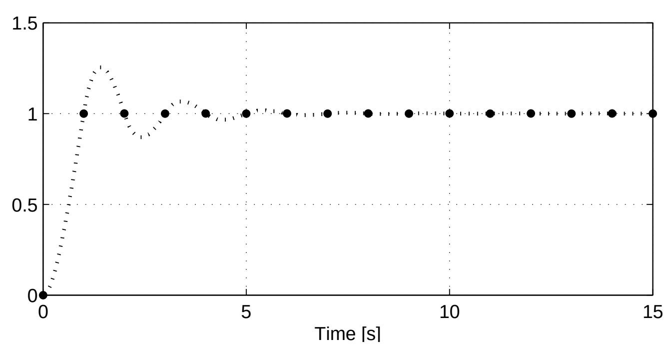

Figure 2: Continuous-time deadbeat response with ripple.

4 Characterisation of MIMO ripple-free deadbeat controllers

The ripple-free condition (6) implies that the control sensitivity [5] must have the form

$$\boldsymbol{S_u}(z) = \frac{\boldsymbol{K}(z)}{z^N}, \tag{7}$$

where the deadbeat horizon $N \in \mathbb{N}$ and $\boldsymbol{K}(z)$ is a polynomial matrix with degree such that $\boldsymbol{S_u}(z)$ is proper. On the other hand, the tracking error signal must also settle in a finite horizon, so that the complementary sensitivity function

$$\boldsymbol{T}(z) = \boldsymbol{G}(z)\boldsymbol{S_u}(z) = \boldsymbol{B}(z)\boldsymbol{A}(z)^{-1}\frac{\boldsymbol{K}(z)}{z^N}, \tag{8}$$

must have all its poles at $z = 0$. This means that $\boldsymbol{K}(z)$ must be factored as $\boldsymbol{K}(z) = \boldsymbol{A}(z)\boldsymbol{V}(z)$, with $\boldsymbol{V}(z)$ being a polynomial matrix. Thus

$$\boldsymbol{S_u}(z) = \frac{\boldsymbol{A}(z)\boldsymbol{V}(z)}{z^N}, \tag{9}$$

For convenience, we set $N = n + \ell$ where n is the degree of $\boldsymbol{A}(z)$ and $\ell \in \mathbb{N}_0$. Let also $\boldsymbol{V}(z)$ be written as $\boldsymbol{V}(z) = \boldsymbol{E}(z)\boldsymbol{W}(z)$, with $\boldsymbol{E}(z)$ being a RDI of $\boldsymbol{A}(z)/z^n$, hence making $\boldsymbol{A}(z)\boldsymbol{E}(z)/z^n$ biproper. These definitions lead to

$$\boldsymbol{S_u}(z) = \frac{\boldsymbol{A}(z)}{z^n}\boldsymbol{E}(z)\frac{\boldsymbol{W}(z)}{z^\ell}, \tag{10}$$

from where we have that the properness of $\boldsymbol{S_u}(z)$ (and hence, the properness of the controller) depends on the properness of $\boldsymbol{W}(z)/z^\ell$. It is worth noting that the degree of $\boldsymbol{A}(z)$ is always equal to that of $\boldsymbol{A}(z)\boldsymbol{E}(z)$. The form of $\boldsymbol{T}(z)$, $\boldsymbol{S_u}(z)$ and the multivariable deadbeat controller $\boldsymbol{C}(z)$ can then be obtained simply as

$$\boldsymbol{T}(z) = \frac{\boldsymbol{B}(z)\boldsymbol{E}(z)\boldsymbol{W}(z)}{z^N}, \tag{11}$$

$$\boldsymbol{S_u}(z) = \frac{\boldsymbol{A}(z)\boldsymbol{E}(z)\boldsymbol{W}(z)}{z^N}, \tag{12}$$

$$\boldsymbol{C}(z) = \boldsymbol{S_u}(z)\left(\boldsymbol{I} - \boldsymbol{T}(z)\right)^{-1} \tag{13}$$

$$= \boldsymbol{A}(z)\boldsymbol{E}(z)\boldsymbol{W}(z)\left(z^N\boldsymbol{I} - \boldsymbol{B}(z)\boldsymbol{E}(z)\boldsymbol{W}(z)\right)^{-1}. \tag{14}$$

Since we also need perfect steady state tracking of constant references, we must force $\boldsymbol{T}(1) = \boldsymbol{I}$, which, using (11) and the fact that $\boldsymbol{B}(1) = \boldsymbol{I}$, implies that $\boldsymbol{W}(1) = \boldsymbol{E}(1)^{-1}$. The controller given in (14) is then a general form of a MIMO deadbeat controller for stable plants and constant reference signals. It can be noticed that for design purposes, $\boldsymbol{C}(z)$ has two free parameters, namely the polynomial matrix $\boldsymbol{W}(z)$ and the constant $\ell \in \mathbb{N}_0$. Also, as stated before, to guarantee a proper controller, we need $\boldsymbol{W}(z)/z^\ell$ to be a proper transfer matrix. Since biproper controllers provide several implementation benefits [5], in the remaining of this paper we will consider $\boldsymbol{W}(z)/z^\ell$ biproper, i.e. the free matrix $\boldsymbol{W}(z)$ is a polynomial matrix of degree ℓ.

Remark 1 *There are several ways [27, 28] to build the RDI $\boldsymbol{E}(z)$ needed in* (14)*. A class of RDI that has proven to be useful in solving certain quadratic optimisation problems is given in [28]. This formulation has the advantage to provide a unitary $\boldsymbol{E}(z)$ that also satisfies $\boldsymbol{E}(1) = \boldsymbol{I}$, which certainly simplifies the condition imposed on $\boldsymbol{W}(z)$ to $\boldsymbol{W}(1) = \boldsymbol{I}$. In the sequel, we will always assume that $\boldsymbol{E}(z)$ is a unitary RDI, as those described in [28].*

Expression (14) shows that the proposed deadbeat controller cancels every pole of the plant model and that *it does not cancel any zero*. This feature allows a ripple-free deadbeat response and sets a key difference of our work with respect to previous related research works [14, 15]. Also, the cancelation of $\boldsymbol{A}(z)$ between $\boldsymbol{G}(z)$ and $\boldsymbol{C}(z)$ implies that all closed loop poles are located at the origin and at the poles of the plant (which means that the response to input disturbances does not exhibit deadbeat behaviour). However, if one is only interested in the response to reference signals or output disturbances, then only the poles at the origin matter and the deadbeat response is achieved. The problem of obtaining a deadbeat response even for input disturbances is more general than the one treated in this paper and the reader is referred to [31] for interesting results on the topic.

Additionally, from (12) it is clear that since $N = n + \ell$, then the *minimum deadbeat horizon* is $N_{min} = n$, that is, the degree of $\boldsymbol{A}(z)$. This is the multivariate version of the fact that for SISO systems, the minimum deadbeat horizon is given by the plant order [30]. Next lemma gives a characterisation of all polynomial matrices $\boldsymbol{W}(z)$ that yield the minimum horizon ripple-free deadbeat controller.

Lemma 1 *Consider a stable transfer matrix $\boldsymbol{G}(z)$ and the MIMO deadbeat controller of* (14)*. Then the minimum horizon deadbeat controller is given by*

$$\boldsymbol{C_{min}}(z) = \boldsymbol{A}(z)\boldsymbol{E}(z)\left(z^n\boldsymbol{I} - \boldsymbol{B}(z)\boldsymbol{E}(z)\right)^{-1}, \tag{15}$$

and it is achieved by choosing $\boldsymbol{W}(z) = z^\ell\boldsymbol{I}$, $\forall\, \ell \in \mathbb{N}_0$ in (14)*.*

Proof:

Suppose $\boldsymbol{W}(z) = z^\ell\boldsymbol{I}$ with $\ell \in \mathbb{N}_0$, then $\boldsymbol{W}(1) = \boldsymbol{I}$ and substituting in (12) gives

$$\boldsymbol{S_u}(z) = \frac{\boldsymbol{A}(z)\boldsymbol{E}(z)}{z^n}, \tag{16}$$

which is equivalent to choose $N = N_{min} = n$. Substituting $\boldsymbol{W}(z)$ in (14) gives $\boldsymbol{C_{min}}(z)$ as in (15).

□□□

From Lemma 1 it holds that we can achieve a minimum horizon deadbeat response if we choose $\boldsymbol{W}(z) = z^\ell\boldsymbol{I}$. Nevertheless, larger deadbeat horizons can be attained with different choices for $\boldsymbol{W}(z) \in \mathbb{P}_\ell$. Hence, if we write

$$\boldsymbol{W}(z) = \sum_{i=0}^{\ell} \boldsymbol{W_i} z^i, \tag{17}$$

and consider the constraint $\boldsymbol{W}(1) = \boldsymbol{E}(1)^{-1}$, we conclude that there is a set of ℓ free design parameters yielding a ripple-free response of $N = n + \ell$ samples. The polynomial matrix $\boldsymbol{W}(z)$ can be used for many purposes, such as to avoid the cancelation of certain plant poles or to choose the zeros of the complementary sensitivity function (recall from (11) that every zero of $\boldsymbol{W}(z)$ is also a zero of $\boldsymbol{T}(z)$). For the purposes of this paper, we are interested in using $\boldsymbol{W}(z)$ to build a ripple-free deadbeat controller that minimises a two objective quadratic cost function.

5 Proposed optimal ripple-free deadbeat controller

We next aim to design a ripple-free deadbeat controller, which at the same time achieves a good transient performance. For this purpose, it is required to define an optimality criterion meaningful as a measure of the loop's performance. A quantity widely used [20, 32] as a tracking performance measure, is the energy of the tracking error, i.e. we define the cost functional

$$J_1 = \sum_{k=0}^{N} \boldsymbol{e}(k)^T \boldsymbol{e}(k), \tag{18}$$

where $\boldsymbol{e}(k) \in \mathbb{R}^p$ is the (vector) error sequence. On the other hand, a common criticism made to deadbeat control is that it usually demands a large control energy. This observation motivates using an optimality criterion that considers the control effort necessary to achieve the deadbeat behaviour. In an analogous fashion as in [13], we will measure the control effort by means of the energy of its deviation from the steady state, i.e. we define the cost functional

$$J_2 = \sum_{k=0}^{N} \left(\boldsymbol{u}(k) - \boldsymbol{u_{ss}}\right)^T \left(\boldsymbol{u}(k) - \boldsymbol{u_{ss}}\right). \tag{19}$$

If the reference signal is $\boldsymbol{r}(k) = \boldsymbol{v}\mu(k)$, $\boldsymbol{v} \in \mathbb{R}^p$, then it is clear that J_1 and J_2 must both depend on $\boldsymbol{v}$. Hence, any controller that is designed for minimising J_1, J_2 or a mixture of both, will necessarily depend on the reference direction. From a practical point of view, this situation is undesirable since it forces to compute a new controller for any change in the command signal. A useful way to overcome this problem is to average J_1 and J_2 over all possible reference directions. To that end, it is plausible to assume that $\boldsymbol{v}$ is a random vector such that

$$\mathcal{E}_{\boldsymbol{v}}\left\{\boldsymbol{v}\right\} = \boldsymbol{0}, \tag{20}$$

$$\mathcal{E}_{\boldsymbol{v}}\left\{\boldsymbol{v}\boldsymbol{v}^T\right\} = \boldsymbol{I}, \tag{21}$$

where $\mathcal{E}_{\boldsymbol{v}}\{\cdot\}$ denotes the expectation operator with respect to $\boldsymbol{v}$. Then, we define the reference averaged cost functionals

$$J_e(\boldsymbol{W}(z)) = \mathcal{E}_{\boldsymbol{v}}\{J_1\}, \tag{22}$$

$$J_u(\boldsymbol{W}(z)) = \mathcal{E}_{\boldsymbol{v}}\{J_2\}, \tag{23}$$

where the notation is such that the dependence of both indexes on the free polynomial matrix $\boldsymbol{W}(z)$ is emphasized. A closed form of $J_e(\boldsymbol{W}(z))$ and $J_u(\boldsymbol{W}(z))$ in terms of the plant model and $\boldsymbol{W}(z)$ is given in the next lemma.

Lemma 2 *Consider a stable discrete-time transfer matrix $\boldsymbol{G}(z)$ and the ripple-free deadbeat controller in* (14)*, then it holds*

$$J_e(\boldsymbol{W}(z)) = \left\|\frac{z^N - \hat{\boldsymbol{B}}(z)\boldsymbol{W}(z)}{z-1}\right\|_2^2, \tag{24}$$

$$J_u(\boldsymbol{W}(z)) = \left\|\frac{\hat{\boldsymbol{A}}(z)\boldsymbol{W}(z) - z^N\boldsymbol{A}(1)}{(z-1)}\right\|_2^2, \tag{25}$$

where $\hat{\boldsymbol{B}}(z) = \boldsymbol{B}(z)\boldsymbol{E}(z)$, $\hat{\boldsymbol{A}}(z) = \boldsymbol{A}(z)\boldsymbol{E}(z)$ and $\boldsymbol{E}(z)$ is a RDI of $\boldsymbol{A}(z)/z^\ell$.

Proof:

To prove (24) and (25), we firstly note that since $\boldsymbol{e}(k)$ and $\boldsymbol{u}(k) - \boldsymbol{u}_{\boldsymbol{ss}}$ vanish to zero in N samples, then

$$J_1 = \sum_{k=0}^{\infty} \boldsymbol{e}(k)^T\boldsymbol{e}(k), \tag{26}$$

$$J_2 = \sum_{k=0}^{\infty} (\boldsymbol{u}(k) - \boldsymbol{u}_{\boldsymbol{ss}})^T (\boldsymbol{u}(k) - \boldsymbol{u}_{\boldsymbol{ss}}). \tag{27}$$

Using the fact that the reference is a step function and Parseval's relation it holds that

$$J_1 = \left\|\boldsymbol{S}(z)\frac{z\boldsymbol{v}}{z-1}\right\|_2^2, \tag{28}$$

$$J_2 = \left\|(\boldsymbol{S}_{\boldsymbol{u}}(z) - \boldsymbol{A}(1))\frac{z\boldsymbol{v}}{z-1}\right\|_2^2,. \tag{29}$$

where $\boldsymbol{S}(z)$ is the loop sensitivity function, $\boldsymbol{S_u}(z)$ is the control sensitivity function and we have used the fact that

$$\boldsymbol{u_{ss}} = \boldsymbol{G}(1)^{-1}\boldsymbol{v} = \boldsymbol{A}(1)\boldsymbol{v}, \tag{30}$$

Using $\boldsymbol{S}(z) = \boldsymbol{I} - \boldsymbol{T}(z)$, (11) and (12) in (28) and (29) yields

$$J_1 = \left\| \frac{z^N - \hat{\boldsymbol{B}}(z)\boldsymbol{W}(z)}{z^{N-1}(z-1)}\boldsymbol{v} \right\|_2^2, \tag{31}$$

$$J_2 = \left\| \frac{\hat{\boldsymbol{A}}(z)\boldsymbol{W}(z) - z^N\boldsymbol{A}(1)}{z^{N-1}(z-1)}\boldsymbol{v} \right\|_2^2, \tag{32}$$

Relations (24) and (25) are obtained by taking expectation operator and using (20) and (21) in the definition of the $\mathcal{L}_2$ norm. Note that the factor z^{N-1} can be removed from (31) and (32) since it is unitary and hence, it preserves the norm.

□□□

We are interested in including both partial costs, tracking error and control effort, in our optimal control problem. Hence we define the two objective cost functional

$$\mathcal{J}(\lambda, \boldsymbol{W}(z)) = \lambda J_e(\boldsymbol{W}(z)) + (1-\lambda)J_u(\boldsymbol{W}(z)), \tag{33}$$

where $0 \leq \lambda \leq 1$. The inclusion of the scalar weight λ has been made in such a way that the control effort has a greater impact on the cost functional as λ approaches to zero. We are now able to formulate the optimisation problem of our interest.

Problem 1 *Given a stable transfer matrix $\boldsymbol{G}(z)$ and $\ell \in \mathbb{N}_0$, find $\boldsymbol{W}^*(z)$ such that*

$$\boldsymbol{W}^*(z) = \arg \min_{\substack{\boldsymbol{W}(z)\in\mathbb{P}_\ell \\ \boldsymbol{W}(1)=\boldsymbol{E}(1)^{-1}}} \mathcal{J}(\lambda, \boldsymbol{W}(z)), \tag{34}$$

where $\mathcal{J}(\lambda, \boldsymbol{W}(z))$ is defined in (33) and $\boldsymbol{E}(z)$ is a RDI of $\boldsymbol{A}(z)/z^\ell$.

Solving Problem 1 allows us to build an optimal ripple-free deadbeat controller using (14) and taking $\boldsymbol{W}(z) = \boldsymbol{W}^*(z)$. It can be observed that if we choose $\ell = 0$, i.e. the minimum deadbeat horizon, then according to Lemma 1, the minimum horizon deadbeat controller is parameterised by the family of polynomials $\boldsymbol{W}(z) = z^\ell \boldsymbol{I}$, $\ell \in \mathbb{N}_0$ and they are the only feasible polynomials in Problem 1. In such case, the deadbeat controller in (15) is uniquely determined and hence no optimisation is possible. However, if we choose larger deadbeat horizons, then the matrix coefficients of $\boldsymbol{W}(z)$ may be adjusted and computed as a solution of Problem 1. Next theorem provides an analytic solution for this problem and is the main result of this paper.

Theorem 1 *Let $\boldsymbol{G}(z)$ be a stable transfer matrix and consider the statement of Problem 1. Define*

$$\hat{\boldsymbol{B}}(z) = \sum_{i=0}^{m} \hat{\boldsymbol{B}}_{\boldsymbol{i}} z^i, \quad \hat{\boldsymbol{A}}(z) = \sum_{i=0}^{n} \hat{\boldsymbol{A}}_{\boldsymbol{i}} z^i, \tag{35}$$

where $\hat{\boldsymbol{B}}(z) = \boldsymbol{B}(z)\boldsymbol{E}(z)$, $\hat{\boldsymbol{A}}(z) = \boldsymbol{A}(z)\boldsymbol{E}(z)$ and $\boldsymbol{E}(z)$ is a unitary RDI of $\boldsymbol{A}(z)/z^\ell$ satisfying $\boldsymbol{E}(1) = \boldsymbol{I}$. Then, the solution to Problem 1 is given by

$$\boldsymbol{W}^*(z) = \boldsymbol{I} + (z-1)\bar{\boldsymbol{W}}^*(z), \tag{36}$$

where

$$\bar{\boldsymbol{W}}^*(z) = \begin{bmatrix} 1 & z & \cdots z^{\ell-1} \end{bmatrix} \underbrace{\begin{bmatrix} \bar{\boldsymbol{W}}_{\boldsymbol{0}}{}^* \\ \bar{\boldsymbol{W}}_{\boldsymbol{1}}{}^* \\ \vdots \\ \bar{\boldsymbol{W}}_{\boldsymbol{\ell-1}}{}^* \end{bmatrix}}_{\boldsymbol{\Omega}^*}, \tag{37}$$

with $\bar{\boldsymbol{W}}_{\boldsymbol{i}} \in \mathbb{R}^{p\times p}$, $\forall i = 0, 1, \ldots, \ell-1$, and

$$\boldsymbol{\Omega}^* = \begin{bmatrix} \sqrt{\lambda}\,\boldsymbol{\Gamma} \\ \sqrt{1-\lambda}\,\boldsymbol{\Psi} \end{bmatrix}^\dagger \begin{bmatrix} \sqrt{\lambda}\,\boldsymbol{\gamma} \\ \sqrt{1-\lambda}\,\boldsymbol{\psi} \end{bmatrix}, \tag{38}$$

where $(\cdot)^\dagger$ denotes the Moore-Penrose pseudoinverse [33]. The matrices $\boldsymbol{\Gamma} \in \mathbb{R}^{p(m+\ell)\times p\ell}$, $\boldsymbol{\gamma} \in$

$\mathbb{R}^{p(m+\ell)\times p}$, $\boldsymbol{\Psi} \in \mathbb{R}^{p(n+\ell)\times p\ell}$ *and* $\boldsymbol{\psi} \in \mathbb{R}^{p(n+\ell)\times p}$ *are defined as*

$$\boldsymbol{\Gamma} = \begin{bmatrix} \hat{\boldsymbol{B}}_{\boldsymbol{0}} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \hat{\boldsymbol{B}}_{\boldsymbol{1}} & \hat{\boldsymbol{B}}_{\boldsymbol{0}} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \vdots & \hat{\boldsymbol{B}}_{\boldsymbol{1}} & \ddots & \ddots & \vdots \\ \hat{\boldsymbol{B}}_{\boldsymbol{m}} & \vdots & \ddots & \ddots & \hat{\boldsymbol{B}}_{\boldsymbol{0}} \\ \boldsymbol{0} & \hat{\boldsymbol{B}}_{\boldsymbol{m}} & \ddots & \ddots & \hat{\boldsymbol{B}}_{\boldsymbol{1}} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \cdots & \hat{\boldsymbol{B}}_{\boldsymbol{m}} \end{bmatrix}, \quad \boldsymbol{\gamma} = \begin{bmatrix} \hat{\boldsymbol{B}}_{\boldsymbol{0}} \\ \hat{\boldsymbol{B}}_{\boldsymbol{0}} + \hat{\boldsymbol{B}}_{\boldsymbol{1}} \\ \vdots \\ \hat{\boldsymbol{B}}_{\boldsymbol{0}} + \hat{\boldsymbol{B}}_{\boldsymbol{1}} + \cdots + \hat{\boldsymbol{B}}_{\boldsymbol{m-1}} \\ \boldsymbol{I} \\ \vdots \\ \boldsymbol{I} \end{bmatrix}, \tag{39}$$

$$\boldsymbol{\Psi} = \begin{bmatrix} \hat{\boldsymbol{A}}_{\boldsymbol{0}} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \hat{\boldsymbol{A}}_{\boldsymbol{1}} & \hat{\boldsymbol{A}}_{\boldsymbol{0}} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \vdots & \hat{\boldsymbol{A}}_{\boldsymbol{1}} & \ddots & \ddots & \vdots \\ \hat{\boldsymbol{A}}_{\boldsymbol{n}} & \vdots & \ddots & \ddots & \hat{\boldsymbol{A}}_{\boldsymbol{0}} \\ \boldsymbol{0} & \hat{\boldsymbol{A}}_{\boldsymbol{n}} & \ddots & \ddots & \hat{\boldsymbol{A}}_{\boldsymbol{1}} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \cdots & \hat{\boldsymbol{A}}_{\boldsymbol{n}} \end{bmatrix}, \quad \boldsymbol{\psi} = \begin{bmatrix} \hat{\boldsymbol{A}}_{\boldsymbol{0}} \\ \hat{\boldsymbol{A}}_{\boldsymbol{0}} + \hat{\boldsymbol{A}}_{\boldsymbol{1}} \\ \vdots \\ \hat{\boldsymbol{A}}_{\boldsymbol{0}} + \hat{\boldsymbol{A}}_{\boldsymbol{1}} + \cdots + \hat{\boldsymbol{A}}_{\boldsymbol{n-1}} \\ \boldsymbol{A}(1) \\ \vdots \\ \boldsymbol{A}(1) \end{bmatrix}. \tag{40}$$

Proof:

The proof follows upon expressing the cost functionals $J_e\left(\boldsymbol{W}(z)\right)$ and $J_u\left(\boldsymbol{W}(z)\right)$ in a vectorised fashion, so that the optimisation problem can be recast into a standard unconstrained least squares problem. To that end, since $\boldsymbol{E}(1) = \boldsymbol{I}$, we require that $\boldsymbol{W}(1) = \boldsymbol{I}$. This implies that $\boldsymbol{W}(z)$ may be expressed as

$$\boldsymbol{W}(z) = \boldsymbol{I} + (z-1)\bar{\boldsymbol{W}}\boldsymbol{(z)}, \tag{41}$$

where $\bar{\boldsymbol{W}}\boldsymbol{(z)}$ is a polynomial matrix of degree $(\ell-1)$. Substituting (41) in (24) and (25), and after some algebra we have that

$$J_e\left(\boldsymbol{W}(z)\right) = \left\|\frac{(z^N-1)\boldsymbol{I}}{z-1} - \frac{\hat{\boldsymbol{B}}(z)-\boldsymbol{I}}{z-1} - \hat{\boldsymbol{B}}(z)\bar{\boldsymbol{W}}(z)\right\|_2^2 \tag{42}$$

$$= \left\|\sum_{i=0}^{N-1} z^i\boldsymbol{I} - \sum_{i=0}^{m-1}\left(\sum_{j=i+1}^{m}\hat{\boldsymbol{B}}_{\boldsymbol{j}}\right)z^i - \hat{\boldsymbol{B}}(z)\bar{\boldsymbol{W}}(z)\right\|_2^2, \tag{43}$$

$$J_u\left(\boldsymbol{W}(z)\right) = \left\|\frac{\hat{\boldsymbol{A}}(z)-z^N\boldsymbol{A}(1)}{z-1} + \hat{\boldsymbol{A}}(z)\bar{\boldsymbol{W}}(z)\right\|_2^2 \tag{44}$$

$$= \left\|-\sum_{i=0}^{\ell-1}\left(\sum_{j=0}^{i}\hat{\boldsymbol{A}}_{\boldsymbol{j}}\right)z^i - \sum_{i=\ell}^{n+\ell-1}\boldsymbol{A}(1)z^i + \hat{\boldsymbol{A}}(z)\bar{\boldsymbol{W}}(z)\right\|_2^2. \tag{45}$$

If we define $\bar{\boldsymbol{W}}(z) = \sum_{i=0}^{\ell-1}\bar{\boldsymbol{W}}_{\boldsymbol{i}}z^i$, then the matrix products $\hat{\boldsymbol{B}}(z)\bar{\boldsymbol{W}}(z)$ and $\hat{\boldsymbol{A}}(z)\bar{\boldsymbol{W}}(z)$ can be written as

$$\hat{\boldsymbol{B}}(z)\bar{\boldsymbol{W}}(z) = \boldsymbol{z}_{\boldsymbol{m+\ell-1}}\boldsymbol{\Gamma\Omega}, \tag{46}$$

$$\hat{\boldsymbol{A}}(z)\bar{\boldsymbol{W}}(z) = \boldsymbol{z}_{n+\ell-1}\boldsymbol{\Psi\Omega}, \tag{47}$$

where $\boldsymbol{z}_{\boldsymbol{q}} = \begin{bmatrix}1 & z & \cdots & z^q\end{bmatrix}$, $\boldsymbol{\Omega} = \begin{bmatrix}\bar{\boldsymbol{W}}_{\boldsymbol{0}}{}^T & \bar{\boldsymbol{W}}_{\boldsymbol{1}}{}^T & \cdots & \bar{\boldsymbol{W}}_{\boldsymbol{\ell-1}}{}^T\end{bmatrix}^T$ and the matrices $\boldsymbol{\Gamma}$ and $\boldsymbol{\Psi}$ are defined in (39) and (40), respectively. Similarly, we may write the vectorised version of the first two terms in (43) and (45) as

$$\frac{(z^N-1)\boldsymbol{I}}{z-1} - \frac{\hat{\boldsymbol{B}}(z)-\boldsymbol{I}}{z-1} = \sum_{i=m+\ell}^{N-1} z^i\boldsymbol{I} - \boldsymbol{z}_{\boldsymbol{m+\ell-1}}\boldsymbol{\gamma}, \tag{48}$$

$$-\sum_{i=0}^{\ell-1}\left(\sum_{j=0}^{i}\hat{\boldsymbol{A}}_{\boldsymbol{j}}\right)z^i - \sum_{i=\ell}^{n+\ell-1}\boldsymbol{A}(1)z^i = -\boldsymbol{z}_{\boldsymbol{n+\ell-1}}\boldsymbol{\psi}, \tag{49}$$

where the matrices $\boldsymbol{\gamma}$ and $\boldsymbol{\psi}$ are defined in (39) and (40), respectively. Substituting (46)-(49) in (43) and (45) yields

$$J_e\left(\boldsymbol{W}(z)\right) = \left|\left|\sum_{i=m+\ell}^{N-1} z^i \boldsymbol{I} - \boldsymbol{z_{m+\ell-1}}\left(\boldsymbol{\gamma} - \boldsymbol{\Gamma\Omega}\right)\right|\right|_2^2 \tag{50}$$

$$= p(N-m-\ell) + \left|\left|\boldsymbol{z_{m+\ell-1}}\left(\boldsymbol{\gamma} - \boldsymbol{\Gamma\Omega}\right)\right|\right|_2^2, \tag{51}$$

$$J_u\left(\boldsymbol{W}(z)\right) = \left|\left|\boldsymbol{z_{n+\ell-1}}\left(\boldsymbol{\psi} - \boldsymbol{\Psi\Omega}\right)\right|\right|_2^2, \tag{52}$$

Using standard properties of the $\mathcal{L}_2$ norm of polynomial matrices in (51) and (52), it follows that the two objective cost functional in (33) can be written as

$$\mathcal{J}\left(\boldsymbol{W}(z)\right) = \lambda p(N-m-\ell) + \left|\left|\begin{bmatrix}\sqrt{\lambda}\,\boldsymbol{\gamma} \\ \sqrt{1-\lambda}\,\boldsymbol{\psi}\end{bmatrix} - \begin{bmatrix}\sqrt{\lambda}\,\boldsymbol{\Gamma} \\ \sqrt{1-\lambda}\,\boldsymbol{\Psi}\end{bmatrix}\boldsymbol{\Omega}\right|\right|_F^2, \tag{53}$$

where $||\cdot||_F$ denotes the Frobenius matrix norm. Therefore, solving Problem 1 is equivalent to find the optimal matrix $\boldsymbol{\Omega}^* \in \mathbb{R}^{p\ell\times p}$ that minimises (53). This optimisation is a standard least squares problem, so that its solution [33] is given by (38) and hence, the result follows.

□□□

Theorem 1 provides the solution for the optimisation problem associated to finding the ripple-free deadbeat controller that minimizes a combined measure of tracking performance and control effort. Hence, given a deadbeat horizon $N = n + \ell$ and a tuning weight λ, we can compute the optimal polynomial $\boldsymbol{W}^*(z)$ using Theorem 1 and then from (14), the optimal ripple-free deadbeat controller is obtained. This methodology is the multivariate generalisation of the results derived in [21].

It is interesting to note that the cost functional $J_e\left(\boldsymbol{W}(z)\right)$ has been commonly used as a discrete-time tracking performance index [20, 28, 32, 34]. Therefore, by using $\lambda = 1$ in Problem 1, our solution yields a ripple-free deadbeat controller that achieves optimal tracking performance under cheap control.

The result provided by Theorem 1 is novel in the literature in the sense that it explicitly solves a MIMO two objective optimal control problem constraining the solution to achieve a ripple-free deadbeat behaviour. Previous results [14, 15] have dealt with the MIMO optimal deadbeat control problem, but no considerations regarding the intersample ripple have been done. In the SISO case, the ripple-free approach has been tackled in [11, 12, 13], but the MIMO extensions of those results are still absent. We also note that, although the proposed solution is constrained to the stable case, it stands as a first step in dealing with the general problem and provides a simple to use controller design tool that might be used in an important class of applications.

6 Illustrative example

To illustrate the controller design procedure proposed in this paper, consider the continuous time plant model

$$\boldsymbol{G_c}(s) = \begin{bmatrix} \dfrac{24}{(s+1)(s+5)} & \dfrac{108}{(s+5)(s+6)} \\ \\ \dfrac{-162}{(s+5)(s+6)} & \dfrac{30}{(s+1)(s+6)} \end{bmatrix}. \tag{54}$$

The zero-order hold discrete-time version of $\boldsymbol{G_c}(s)$ using a sampling time of 0.1[s] is given by

$$\boldsymbol{G}(z) = \begin{bmatrix} \dfrac{0.098812(z+0.8189)}{(z-0.9048)(z-0.6065)} & \dfrac{0.37755(z+0.6928)}{(z-0.6065)(z-0.5488)} \\ \\ \dfrac{-0.56632(z+0.6928)}{(z-0.6065)(z-0.5488)} & \dfrac{0.11979(z+0.7922)}{(z-0.9048)(z-0.5488)} \end{bmatrix}. \tag{55}$$

In this case, the right coprime polynomial factors of $\boldsymbol{G}(z)$ are given by

$$\boldsymbol{A}(z) = \begin{bmatrix} -2.05+9.39z-14.04z^2+6.81z^3 & 1.48-6.76z+10.11z^2-4.91z^3 \\ \\ -2.22+10.14z-15.16z^2+7.36z^3 & -1.97+9.01z-13.47z^2+6.54z^3 \end{bmatrix}, \tag{56}$$

$$\boldsymbol{B}(z) = \begin{bmatrix} -2.04 - 0.41z + 3.45z^2 & -1.33 - 0.65z + 1.98z^2 \\ \\ 1.99 + 0.98z - 2.98z^2 & -2.12 - 0.44z + 3.56z^2 \end{bmatrix}. \tag{57}$$

Note that $\boldsymbol{B}(z)$ satisfies $\boldsymbol{B}(1) = \boldsymbol{I}$ and in this case $n = 3$. Hence, the minimum achievable deadbeat horizon is $N_{min} = 3$ and the result of Theorem 1 can be used in order to design ripple-free deadbeat controllers with larger horizons. Figures 2 and 3 show the simulation results with $\boldsymbol{r}(k) = [\mu(k) \quad \mu(k-20)]^T$ for the minimum horizon deadbeat controller $\boldsymbol{C_{min}}(z)$ in Lemma 1. It can be seen that the ripple-free response is achieved in the minimum number of samples. The transient performance of this deadbeat controller may be improved upon enlarging the deadbeat horizon. Using Theorem 1 optimal controllers are computed for $N = \{6,\ 10,\ 15\}$ and with three representative choices of the weight λ, namely $\lambda = \{1, 0.9, 0.1\}$. Simulation results for all designs are presented in Figures 5-10. These results illustrate the effectiveness of the proposed design procedure since: (a) all the time responses exhibit a ripple-free deadbeat behaviour with the chosen horizon and, (b) by inspecting Figures 6, 8 and 10 it can be seen that decreasing the weight λ generates less energetic control sequences. This latter feature implies, as expected, that the transient responses of the plant outputs are smoother as λ decreases, as it can be readily noticed in Figures 5, 7 and 9. Note also that reductions on λ generate designs with larger rise times, but no necessarily with a smaller overshoot. This is connected to an interesting feature that can be noticed from the simulation results, namely, the effect of reducing λ is less important for small values of N (this is particularly evident in Figure 5). This phenomenon is due to the fact that for larger horizons, the optimisation procedure has more degrees of freedom to reduce the control energy while preserving the ripple-free deadbeat response. Figures 11 and 12 show a surface plot of $\log_{10}(J_u(\boldsymbol{W}^*(z)))$ and $J_e(\boldsymbol{W}^*(z))$ for different optimal designs as a function of N and λ. The plots reveal that as N diminishes, the energy of the optimal control signal becomes less sensitive to reductions on the weight λ and that there is an inescapable trade off between the importance of $J_u(\boldsymbol{W}(z))$ and $J_e(\boldsymbol{W}(z))$ in the optimised cost functional.

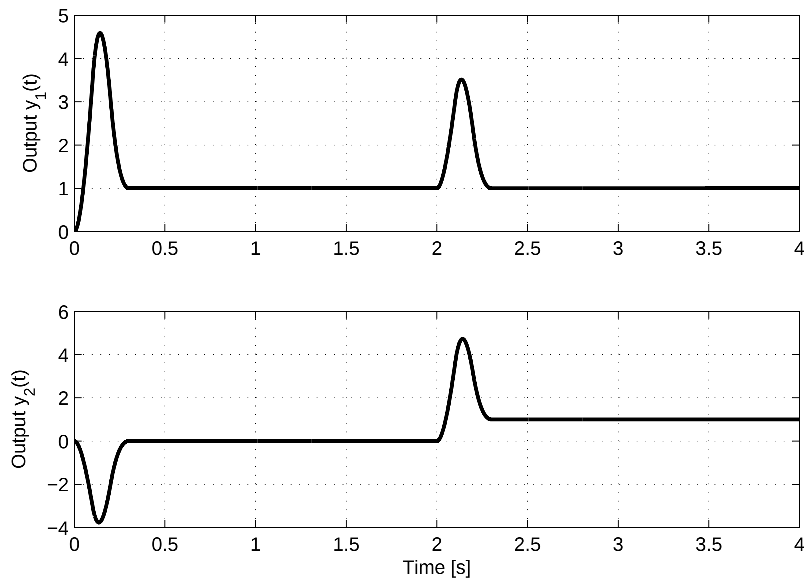

Figure 3: Continuous time output of optimal deadbeat control loop with $N = 3$.

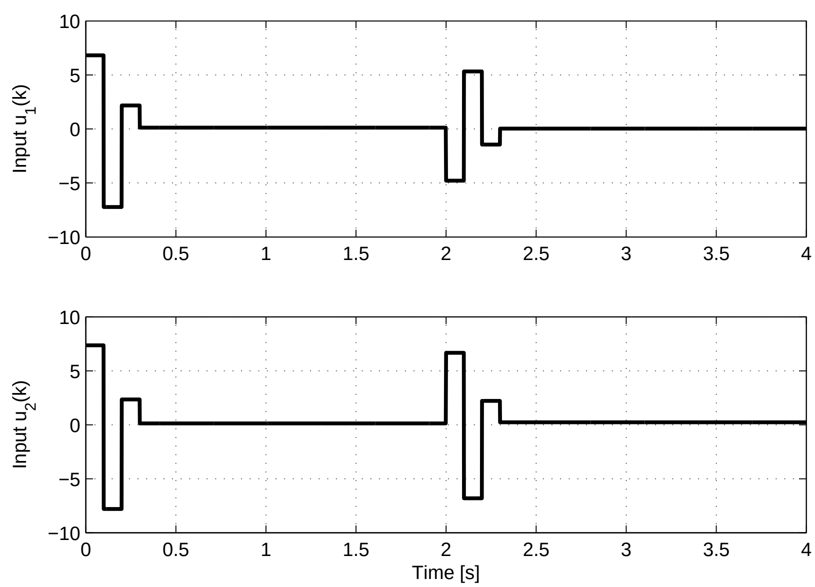

Figure 4: Control sequences of optimal deadbeat control loop with $N = 3$.

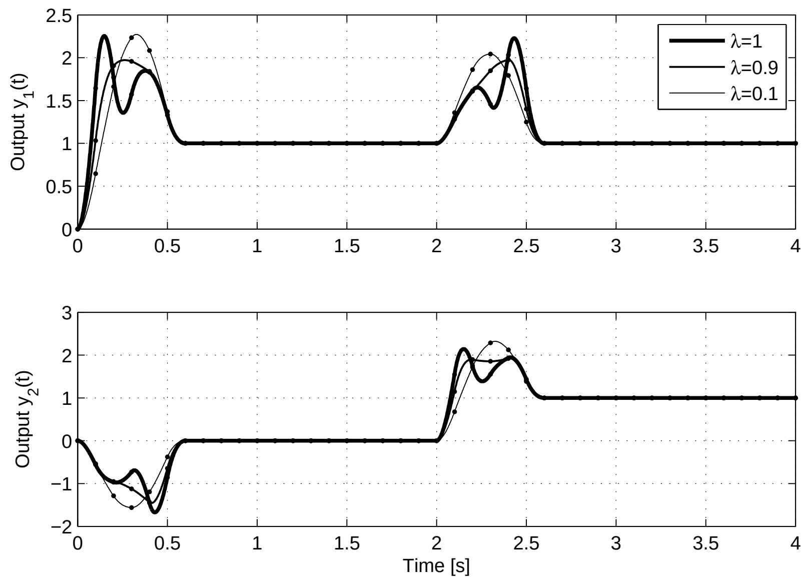

Figure 5: Continuous time output of optimal deadbeat control loop with $N = 6$.

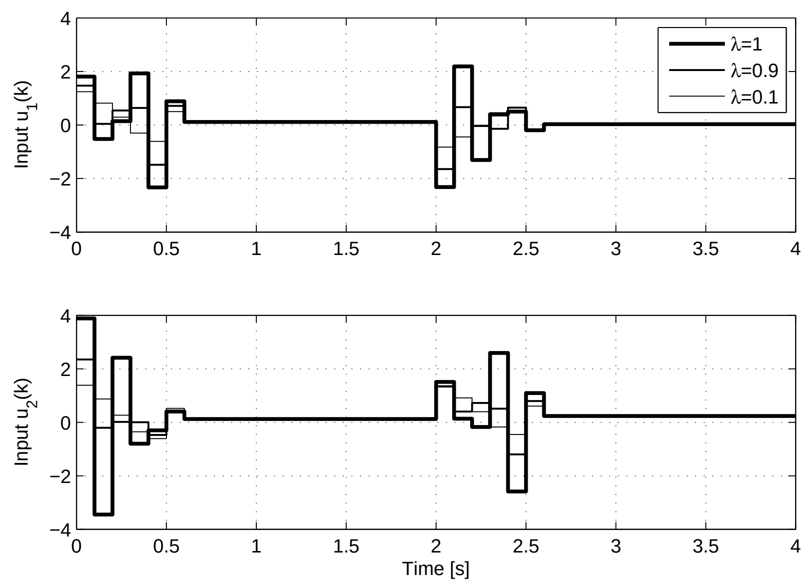

Figure 6: Control sequences of optimal deadbeat control loop with $N = 6$.

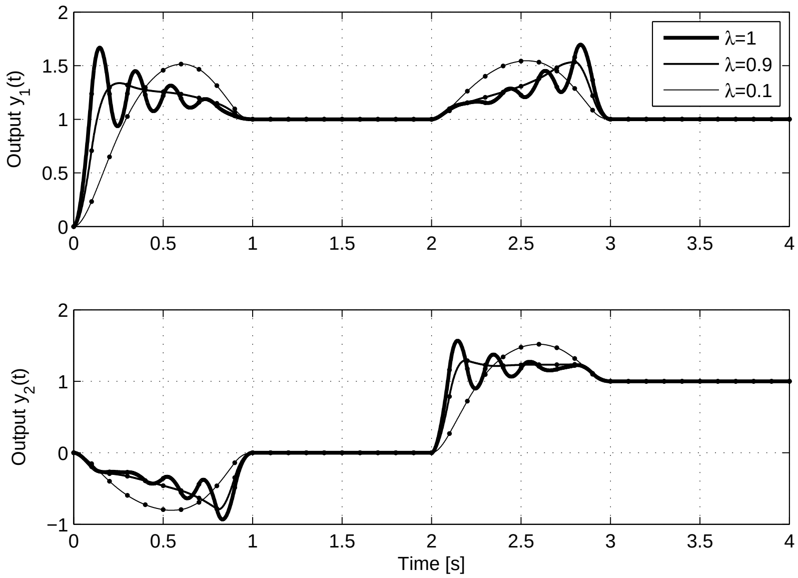

Figure 7: Continuous time output of optimal deadbeat control loop with $N = 10$.

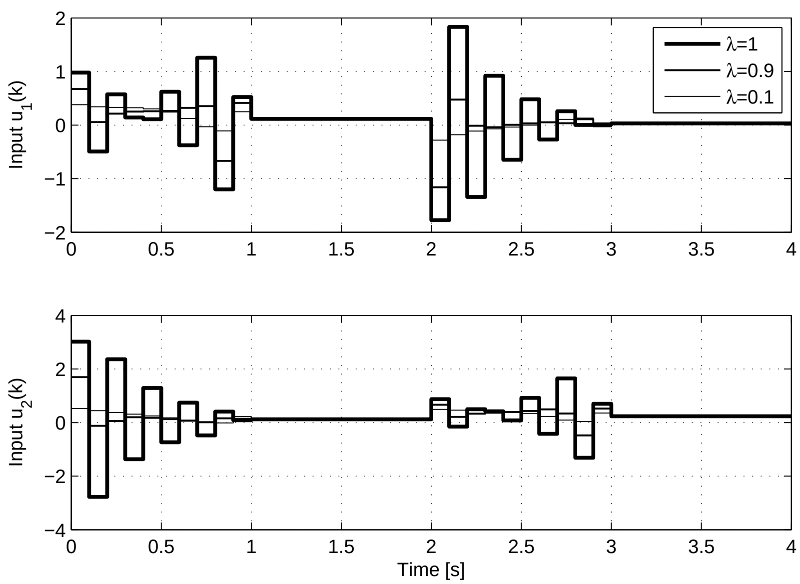

Figure 8: Control sequences of optimal deadbeat control loop with $N = 10$.

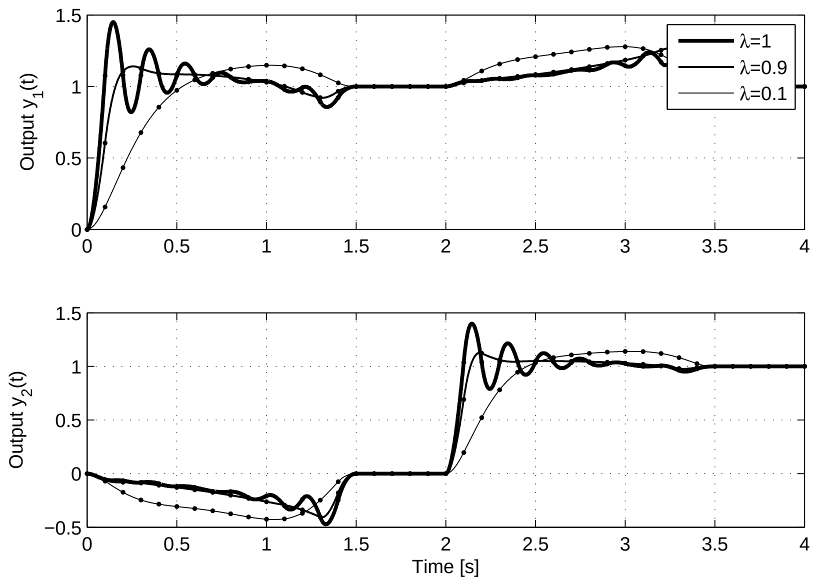

Figure 9: Continuous time output of optimal deadbeat control loop with $N = 15$.

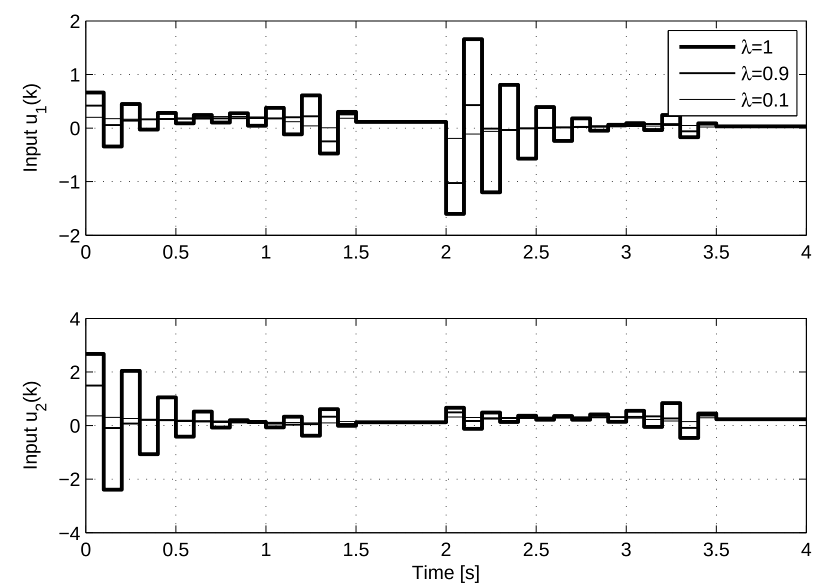

Figure 10: Control sequences of optimal deadbeat control loop with $N = 15$.

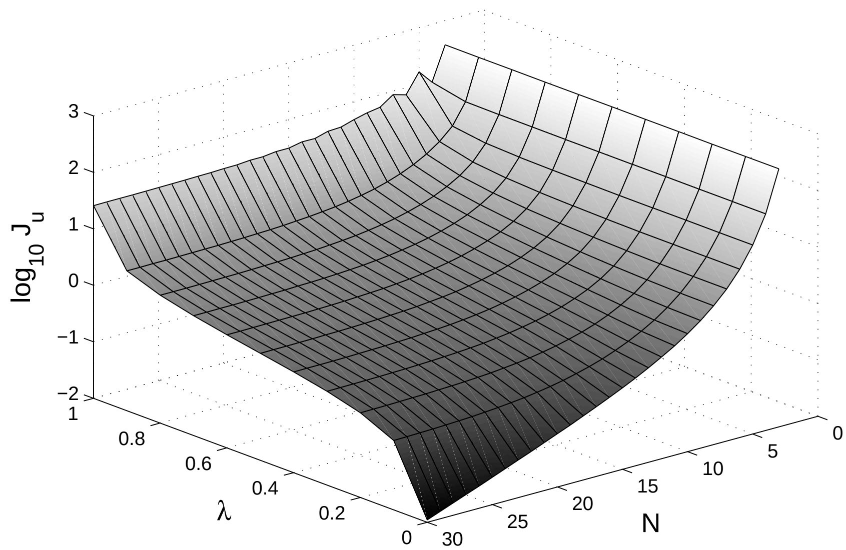

Figure 11: Reference averaged energy of the sequence $\boldsymbol{u}(k) - \boldsymbol{u_{ss}}$ as a function of N and λ.

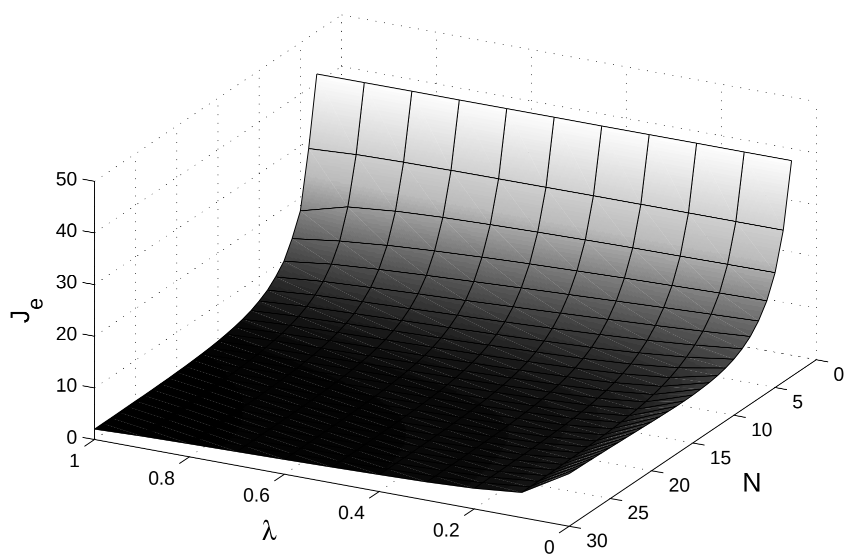

Figure 12: Reference averaged energy of the optimal tracking error sequence as a function of N and λ.

7 Conclusions

In this paper we have derived a methodology to design multivariable ripple-free deadbeat controllers for stable, linear and time-invariant plant models. The methodology relies upon a simple parameterisation of all ripple-free deadbeat controllers for stable plant models. The resulting controller provides integral action and avoids any intersample ripple after the settling time in the response to constant reference signals and output disturbances. The ripple-free feature is one the main distinguishing features of this work and it is tackled by ensuring that not only tracking error settles in a finite time horizon, but also the control input. This avoids that the control input has any natural modes distinct from those arising from the poles at $z = 0$ and hence, our procedure generates a controller that does not cancel any zero of the plant model. We must stress that, although the ripple-free behavior is of no interest when dealing with purely discrete-time control systems, in the case of sampled-data systems it may be key to attain a good transient performance. This is emphasized by the fact that the sampled-data models usually contain sampling zeros located in the negative real axis, thus if a deadbeat controller without the ripple-free feature is used, it will surely generate damped oscillatory modes in both the control input sequence and continuous-time output.

In addition, the derived control law is optimal in the sense that it accounts for the minimisation of the energy of both the tracking error and the control signal. This strategy yields transient responses that satisfy specific design requirements. The solution of the optimal control problem is solved analytically by taking advantage of $\mathcal{L}_2$ norm properties and conveniently posing the problem in a standard unconstrained least squares framework. The usage of the proposed design methodology is simple since it only requires the deadbeat horizon and a scalar tuning factor as input parameters.

In this line of research, future studies should consider the unstable case and how the optimal cost depends on dynamical features of the plant such as poles, zeros and time delays.

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Figure captions

- Figure 1. Multivariable sampled data control loop.
- Figure 2. Continuous-time deadbeat response with ripple.
- Figure 3. Continuous-time output of minimum horizon ripple-free deadbeat control loop.
- Figure 4. Control sequences of minimum horizon ripple-free deadbeat control loop.
- Figure 5. Continuous-time output of optimal ripple-free deadbeat control loop with $N = 6$.
- Figure 6. Control sequences of optimal ripple-free deadbeat control loop with $N = 6$.
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- Figure 11. Reference averaged energy of the optimal sequence $\boldsymbol{u}(k) - \boldsymbol{u_{ss}}$ as a function of N and λ.
- Figure 12. Reference averaged energy of the optimal tracking error sequence as a function of N and λ.