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Effect of downstream feedback on the achievable performance of feedback control loops for serial processes

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Abstract— **This paper deals with feedback control of serial processes, that is, processes formed by the series connection of several subsystems. We focus on the determination of conditions under which a full multivariate controller may lead to performance improvements over a triangular controller. In other words, we study conditions under which it is advisable to include a control law with downstream feedback besides local control and feedforward action. The problem is addressed using an $\mathcal{H}_2$ optimal performance criterion in a discrete-time framework. Our main result is a simple condition on the unstable poles, non minimum phase (NMP) zeros and delays of the discrete time plant model that ensures no penalization on the achievable performance under triangular control. As an application of this result, we show that for an application relevant class of serial processes, the condition can be expressed in terms of the time delays in each subsystem only. The derived condition may be useful for both process design and process control.**

Index Terms— **Optimal control, multivariate control, serial processes, process control.**

I. Introduction

The term *serial process* encompasses a wide class of systems which are composed of cascaded subsystems, i.e., subsystems connected in such a way that the output of each one affects the next one either as an input or as a disturbance. This kind of multi-input multi-output (MIMO) plants are often encountered in practical applications related to process industry, such as pH-neutralization tanks and mixing processes [1], [2]. For control design purposes, the common approach is to use linear models obtained around a certain operation point. In this framework, an important feature of serial processes is that their linearized models have lower triangular transfer matrices.

A general diagram of a serial process is shown in Figure 1, where S_i represents the i^{th} subsystem. In that scheme, the output of subsystem S_i acts as a disturbance for subsystem S_{i+1}. In reference to that figure, we refer as *downstream* units (resp. *upstream* units) as those located to the right (left) of a given stage in the diagram.

In this paper we consider the digital control of a serial process. To that end, we study a standard one degree of freedom control architecture, as the one shown in Figure 2. In that figure, $\boldsymbol{C}(z)$ is the controller transfer function, $\boldsymbol{G_o}(s)$ is the continuous-time transfer function of a linearized plant model, T is the sampling interval and $\boldsymbol{G}(z)$ represents an at the samples exact discrete time model for the process. In order to account for the different closed loop performance specifications, the design of the controller $\boldsymbol{C}(z)$ is always an iterative process [3] in which the performance of different control structures must be compared with the given specifications. A key stage in the controller design procedure is controller structure selection, i.e., determining which measurements will be used to build which plant input (see, e.g., [4] and the references therein).

In the particular case of serial processes, it is possible to classify the entries of the MIMO controller $\boldsymbol{C}(z)$ in three categories, as described in [5]:

1) *Local control*: the entries on the diagonal of $\boldsymbol{C}(z)$ controller attempt to control the output of subsystem S_i, y_i, by means of manipulating u_i. Therefore, this control action can be seen as a local control architecture of output y_i.
2) *Feedforward action*: the entries below the main diagonal of $\boldsymbol{C}(z)$ relate the input of each subsystem to the outputs of upstream subsystems. Therefore, they can be interpreted as forming a feedforward based control architecture.
3) *Feedback action*: the entries above the main diagonal of $\boldsymbol{C}(z)$ relate the input of the each subsystem to the outputs of downstream subsystems. Therefore, they can be interpreted as forming a downstream feedback based control architecture.

Different combinations of the control actions described above yield different controller structures. The choice of a particular structure may have a major impact on the performance of the overall control system, which will depend mainly on the interactions between the input/output channels of the plant.

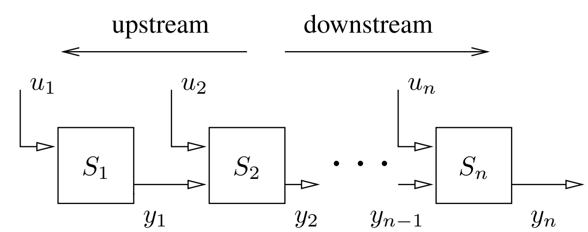

Fig. 1. Standard serial process.

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The extremal cases are *centralized* or full MIMO controllers, which have all the aforementioned control actions and hence use all measurements to build the control input, and *decentralized* controllers, which have diagonal transfer matrices, so they just provide local control for each of the outputs. A family of MIMO controllers that lies between these two extremals are *lower triangular* controllers, which provide both local control and feedforward action.

Since the transfer matrix of a serial process has a lower triangular structure, it *seems natural* to use the same structure for the controller. Nevertheless, it is also true that a full MIMO controller provides a richer interaction structure among the measurements and the plant inputs. Therefore, the achievable performance of a control system for a serial process should, in principle, be improved when considering a full MIMO controller, instead of a triangular one. This leads us to the main problem addressed in this paper: for the control of a serial process, under which conditions is it advisable to use a full MIMO controller instead of a triangular controller? Or, stated differently, when does the restriction of the controller to be triangular not constraint the achievable performance? Providing an accurate answer to this questions will shed light on when is it advisable to include downstream feedback action in the controller besides local and feedforward control.

The issues described in the previous paragraph have been extensively discussed from an intuitive perspective in [5]. In that work, the authors state that, leaving aside implementation issues, the use of downstream feedback action is recommended just in certain cases, depending mainly on the subsystems' delays.

The main contribution of this paper is to tackle this issue in an analytical fashion. We determine precise conditions on plant dynamic features under which the inclusion of the downstream feedback action is advisable. In particular, considering a standard $\mathcal{H}_2$ measure for loop performance, and assuming that the control loop provides perfect steady state tracking for constant reference signals, we derive simple necessary and sufficient conditions on the unstable poles, non minimum phase (NMP) zeros and delays of the discrete time plant model that ensure no penalization on the best achievable performance when restricting the controller to be triangular. To illustrate the results, we particularize them to a relevant class of serial processes and we provide two illustrative examples. The derived conditions provide a formal complement to the discussion given in [5] and, in our view, may be useful for both process design and control system design.

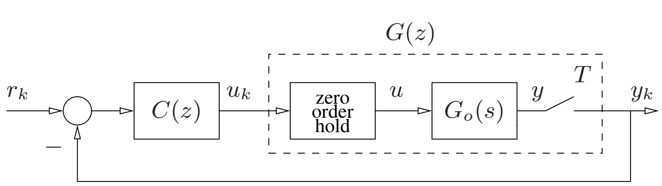

Fig. 2. One degree of freedom sampled data control loop.

The rest of the paper is organized as follows: Section II presents some preliminaries and states the general assumptions considered in this work. Section III defines the problem of our interest. Section IV reviews known results on $\mathcal{H}_2$ optimal performance for general (i.e., not necessarily triangular) plants and, in Section V, those results are used to derive the main result in this paper. Section VI particularizes the results to an application relevant class of serial processes. Finally, Section VII presents the conclusion of this work.

II. Preliminaries and general assumptions

We will denote the set of $n \times n$ discrete-time transfer matrices in the complex variable z as $\mathcal{R}$, the set of stable and proper discrete-time transfer matrices as $\mathcal{RH}_\infty$, the set of stable and strictly proper transfer matrices as $\mathcal{RH}_2$ and its orthogonal complement as $\mathcal{RH}_2^\perp$. The transfer matrix $\boldsymbol{G}(z) \in \mathcal{R}$ has a zero at $z = c \in \mathbb{C}$ if and only if $\boldsymbol{G}(c)$ is singular. This kind of zeros is usually termed in the literature as *transmission zeros*. If $|c| \geq 1$, the zero is called nonminimum phase (NMP), otherwise, it will be referred as minimum phase (MP). Similarly, a transfer matrix having at least one NMP zero is called NMP, otherwise, is termed as MP.

Given any nonsingular almost everywhere (a.e.) transfer matrix $\boldsymbol{A}(z) \in \mathcal{RH}_\infty$, we will refer to $\boldsymbol{\xi_L}(z) \in \mathcal{R}$ as a generalized left unitary interactor (GLUI) for $\boldsymbol{A}(z)$ if and only if $\boldsymbol{\xi_L}(z)$ is non singular a.e., MP, satisfies $\boldsymbol{\xi_L}(z^{-1})^T \boldsymbol{\xi_L}(z) = I$ and $\boldsymbol{\xi_L}(0) = \boldsymbol{I}$, and is such that the transfer matrix

$$\tilde{\boldsymbol{A}}(z) = \boldsymbol{\xi_L}(z)\boldsymbol{A}(z) \tag{1}$$

is biproper, stable and MP (see [6] and the references therein). A GLUI for $\boldsymbol{A}(z)$ exists only if $\boldsymbol{A}(z)$ does not have zeros on the unit circle and can be proven to be unique [7]. An algorithm to build such interactor matrices can be found in [6]. Similarly, we can define a generalized right unitary interactor (GRUI), $\boldsymbol{\xi_R}(z)$. In this case, we require the transfer matrix $\boldsymbol{A}(z)\boldsymbol{\xi_R}(z)$ to be biproper, stable and MP (compare to (1)). Clearly, a GRUI for $\boldsymbol{A}(z)$ is the transpose of a GLUI for $\boldsymbol{A}(z)^T$, and *vice versa*.

In this paper we assume that the plant model $\boldsymbol{G}(z)$ is such that it satisfies the following:

Assumption 1: $\boldsymbol{G}(z)$ is real rational, proper, square, having non singular DC-gain (i.e., $\boldsymbol{G}(1)$ is non singular) and without poles nor zeros on the unit circle. ■

We note that the condition of $\boldsymbol{G}(1)$ being non singular is necessary for being able to track arbitrary step references or compensate for arbitrary step disturbances [8]. On the other hand, the assumption regarding the absence of poles and zeros in the unit circle is rather technical, and we could remove it at the expense of more involved calculations. Hence, the most restrictive assumption relates to the fact that we are only considering square plants.

Since $\boldsymbol{G}(z)$ is real rational, it admits coprime matrix fraction descriptions (MFD) in $\mathcal{RH}_\infty$ given by [9]

$$\boldsymbol{G}(z) = \boldsymbol{D_I}(z)^{-1}\boldsymbol{N_I}(z) = \boldsymbol{N_D}(z)\boldsymbol{D_D}(z)^{-1}, \tag{2}$$

where $\boldsymbol{D_I}(z), \boldsymbol{N_I}(z), \boldsymbol{N_D}(z)$ and $\boldsymbol{D_D}(z)$ are all in $\mathcal{RH}_\infty$. In this description, every NMP zero of $\boldsymbol{G}(z)$ is also a zero of $\boldsymbol{N_D}(z)$ with the same algebraic and geometric multiplicity [10]. Similarly, every unstable pole of $\boldsymbol{G}(z)$ is a zero of $\boldsymbol{D_I}(z)$. We note that Assumption 1 guarantees the existence of a GLUI for $\boldsymbol{N_D}(z)$ and a GRUI for $\boldsymbol{D_I}(z)$, denoted by $\boldsymbol{\xi_c}(z)$ and $\boldsymbol{\xi_p}(z)$, respectively.

III. Problem Definition

In this paper we consider serial processes with the structure depicted in Figure 1, where every signal u_i and y_i is scalar. The continuous time input-output relation of each subsystem is assumed to be given by

$$\begin{aligned} Y_1(s) &= H_1(s)U_1(s) \\ Y_i(s) &= H_i(s)U_i(s) + H_{di}(s)Y_{i-1}(s),\ i = 2, \dots n, \end{aligned} \tag{3}$$

where upper case denotes Laplace transform in the complex variable s, and $H_i(s)$ and $H_{di}(s)$ are scalar and proper transfer functions. It is straightforward to see that the output of each subsystem, y_i, is related to the subsystems inputs, $\{u_1, u_2, \dots, u_n\}$, by

$$Y_1(s) = H_1(s)U_1(s), \tag{4}$$

$$Y_i(s) = \sum_{j=1}^{i-1} \left(\prod_{\ell=j+1}^{i} H_{d\ell}(s) \right) H_j(s)U_j(s) + H_i(s)U_i(s), \tag{5}$$

for all $i = 2, 3, \dots, n$.

The elements of the discrete time transfer matrix $\boldsymbol{G}(z)$ will be denoted by $G_{ij}(z)$. Without loss of generality, we write

$$G_{ij}(z) = \begin{cases} \dfrac{1}{z^{d_i}} \hat{G}_{ii}(z) & \text{if } j = i \\[2ex] \dfrac{1}{z^{d_{ij}}} \hat{G}_{ij}(z) & \text{if } j < i \\[2ex] 0 & \text{if } j > i, \end{cases} \tag{6}$$

where $0 < |\hat{G}_{ij}(0)| < \infty$, for all i, j, and d_{ij} (resp. d_i) denotes the pure time delay in $G_{ij}(z)$ (resp. $G_{ii}(z)$).

The triangular structure of (6) suggests that it is natural to consider local control enhanced by feedforward of the upstream outputs. The question we seek to answer is when is it advisable to include, in addition, downstream feedback action in the controller. Using intuitive arguments, it *seems reasonable to conjecture* that, if $d_i \le d_{ij}$ for all i, j, then downstream feedback may not be necessary. Indeed, the aim of downstream feedback is to add information to upstream controllers so that they can, indirectly via the off diagonal terms in $\boldsymbol{G}(z)$, correct deviations in the back-fed measurement. If the delays in the off diagonal terms in the same row as the back-fed measurement (i.e., d_{ij}) are greater than the delay in the diagonal term in the same row (i.e., d_i), then the corrective action achieved by downstream feedback will always have more delay that the corrective action of the local controller. Therefore in such cases the inclusion of the downstream feedback might not give any performance improvements. A complementary discussion around these arguments can be found in [5].

The rest of this paper is devoted to the analytical justification of the conclusions drawn in the previous paragraph. Using a standard quadratic performance measure, we explore whether it is possible to improve the best performance achievable by a lower triangular controller by enhancing it with downstream feedback (i.e., considering a full MIMO controller). In this way, our results provide a formal complement to the intuitive arguments given in [5] regarding the inclusion of downstream feedback action in the design of controllers for serial processes.

IV. $\mathcal{H}_2$ optimal performance for general plants

This section reviews known facts about $\mathcal{H}_2$ optimal performance of feedback control systems. We measure the performance of the loop in Figure 2 by means of the weighted $\mathcal{H}_2$ norm of the loop sensitivity function [11]

$$\boldsymbol{S_o}(z) = (\boldsymbol{I} + \boldsymbol{G}(z)\boldsymbol{C}(z))^{-1}. \tag{7}$$

In particular, we define the functional

$$J = \left\| \frac{\boldsymbol{S_o}(z)}{z-1} \right\|_2^2. \tag{8}$$

Given the fact that $\boldsymbol{G}(1)$ is assumed to be non singular, it follows that J is well defined. We note that J can be interpreted as the reference-direction averaged energy of the tracking error for step references [12].

A key step in the minimization of J is the appropriate parameterization of $\boldsymbol{S_o}(z)$. Usually this has been made using the Youla parameterization of all stabilizing controllers [8], [9], [11]. Here we consider an alternative formulation, originally described in [12]. If we consider $\boldsymbol{G}(z)$ satisfying Assumption 1, and denote by $\boldsymbol{S_{oo}}(z)$ *any* admissible[1] sensitivity function for $\boldsymbol{G}(z)$, then *every* admissible sensitivity function for $\boldsymbol{G}(z)$ can be written as

$$\boldsymbol{S_o}(z) = \boldsymbol{S_{oo}}(z) - \boldsymbol{\xi_c}(z)^{-1}\boldsymbol{X}(z)\boldsymbol{\xi_p}(z)^{-1}, \tag{9}$$

where $\boldsymbol{X}(z) \in \mathcal{RH}_\infty$ is a free parameter.

Equation (9) allows one to write the cost functional as

$$J = \left\| \frac{\boldsymbol{S_{oo}}(z) - \boldsymbol{\xi_c}(z)^{-1}\boldsymbol{X}(z)\boldsymbol{\xi_p}(z)^{-1}}{z-1} \right\|_2^2. \tag{10}$$

Defining the optimal parameter $\boldsymbol{X^{opt}}(z)$ as

$$\boldsymbol{X^{opt}}(z) = \arg \min_{\boldsymbol{X}(z) \in \mathcal{RH}_\infty} J, \tag{11}$$

it is possible to prove the following lemma [12].

Lemma 1: Consider $\boldsymbol{G}(z)$ satisfying Assumption 1. Then,

$$\boldsymbol{X^{opt}}(z) = \boldsymbol{F}_\perp(1) + \boldsymbol{F_\infty}(z), \tag{12}$$

[1] A sensitivity function is called admissible if and only if it is generated by a proper and stabilizing controller.

where $\boldsymbol{F}_\perp(z) \in \mathcal{RH}_2^\perp$ and $\boldsymbol{F_\infty}(z) \in \mathcal{RH}_\infty$ are such that

$$\boldsymbol{F}_\perp(z) + \boldsymbol{F_\infty}(z) = (\boldsymbol{I} - \boldsymbol{\xi_c}(z)\boldsymbol{T_{oo}}(z))\,\boldsymbol{\xi_p}(z) \triangleq \boldsymbol{F}(z), \tag{13}$$

with $\boldsymbol{T_{oo}}(z) = \boldsymbol{I} - \boldsymbol{S_{oo}}(z)$. ■

Given (7), (9) and (11) we have that the controller that minimizes J is given by

$$\boldsymbol{C^{opt}}(z) \triangleq \min_{\boldsymbol{C}(z)\in\mathcal{S}} J \tag{14}$$

$$= \boldsymbol{G}(z)^{-1}\left[(\boldsymbol{S_{oo}}(z) - \boldsymbol{\xi_c}(z)^{-1}\boldsymbol{X^{opt}}(z)\boldsymbol{\xi_p}(z)^{-1})^{-1} - \boldsymbol{I}\right], \tag{15}$$

where $\mathcal{S}$ is the set of stabilizing and proper controllers for $\boldsymbol{G}(z)$. It is worth noting that, since we have not imposed any specific structure on $\boldsymbol{X^{opt}}(z)$, $\boldsymbol{C^{opt}}(z)$ is, in general, a full MIMO controller. For further reference we define the performance achieved by the optimal controller as

$$J^{opt} \triangleq \min_{X(z)\in\mathcal{RH}_\infty} J = \min_{\boldsymbol{C}(z)\in\mathcal{S}} J. \tag{16}$$

V. $\mathcal{H}_2$ OPTIMAL PERFORMANCE FOR TRIANGULAR PLANT MODELS AND TRIANGULAR CONTROLLERS

So far we have not included any structural constraints on the optimal controller $\boldsymbol{C^{opt}}(z)$. We are now interested in an optimal controller with triangular structure, i.e., a controller defined by

$$\boldsymbol{C_t^{opt}(z)} = \arg\min_{\boldsymbol{C}(z)\in\mathcal{S}_t} J, \tag{17}$$

where $\mathcal{S}_t$ is the set of triangular, stabilizing and proper controllers for $\boldsymbol{G}(z)$.

We define the performance achieved by the optimal triangular controller as

$$J_t^{opt} = \min_{\boldsymbol{C}(z)\in\mathcal{S}_t} J. \tag{18}$$

Results in [2], [13] show that, since $\boldsymbol{G}(z)$ is triangular, the optimization problem whose solution leads to $\boldsymbol{C_t^{opt}}(z)$ can be recast as a convex problem and, hence, it can be solved with standard techniques.

We are interested in determining when the full MIMO controller $\boldsymbol{C^{opt}}(z)$ outperforms the triangular controller $\boldsymbol{C_t^{opt}}(z)$. It should be clear that, since $\mathcal{S}_t \subset \mathcal{S}$, then in general $J^{opt} \leq J_t^{opt}$. This is intuitively reasonable because a full MIMO controller has a richer interaction structure among measurements and control signals as compared to a triangular one. The previous fact might lead one to think that optimal performance can always be improved using a full MIMO controller instead of a triangular one. However, results in [13] show that this is not always the case. In particular, if $\boldsymbol{G}(z)$ is triangular and stable, then $J^{opt} = J_t^{opt}$ if and only if the plant NMP zeros satisfy certain algebraic condition that is equivalent to having a diagonal GLUI for $\boldsymbol{G}(z)$. In those cases, using a full MIMO controller does not provide any improvement in the best achievable performance, as measured by J.

We next present a definition taken from [13], which is key to the derivation of the main result of this paper.

***Definition** 1:* Consider any $n \times n$ transfer matrix $\boldsymbol{A}(z) \in \mathcal{R}$. A zero at $z = c$ of $\boldsymbol{A}(z)$, with algebraic multiplicity α_c, is said left canonical if and only if

$$\alpha_c = \sum_{i=1}^{n} m_i^c, \tag{19}$$

where m_i^c is such that the i^{th} row of $\boldsymbol{A}(z)$ satisfies

$$[\boldsymbol{A}(z)]_{i*} = \begin{cases} (z-c)^{m_i^c}\boldsymbol{F_i^c}(z) & \text{if } |c| < \infty \\ \\ \frac{1}{z^{m_i^c}}\boldsymbol{F_i^c}(z) & \text{if } c = \infty \end{cases}, \tag{20}$$

and $0 < \left|\left|\boldsymbol{F_i^c}(c)^T\right|\right| < \infty$. ■

Last definition encompasses those NMP zeros of a transfer matrix that are *concentrated* by rows. In an analogous fashion, we can define the notion of *right canonical zeros* in terms of the columns of the transfer matrix. For the purposes of this paper it suffices to note that a zero of $\boldsymbol{A}(z)$ is left (resp. right) canonical if and only if it is a right (resp. left) canonical zero of $\boldsymbol{A}(z)^T$. This consideration, together with the MFD in (2) allow one to state the following definition:

***Definition** 2:* Consider a real rational plant model $\boldsymbol{G}(z) \in \mathcal{R}$ with coprime MFD as in (2). Then:

1) A NMP zero of $\boldsymbol{G}(z)$ is said left canonical if and only if it is a left canonical zero of $\boldsymbol{N_D}(z)$.
2) An unstable pole of $\boldsymbol{G}(z)$ is said right canonical if and only if it is a left canonical zero of $\boldsymbol{D_I}(z)^T$. ■

Next theorem extends some results in [13] to unstable plant models, and constitutes the main result in this paper.

***Theorem** 1:* Consider a lower triangular plant model $\boldsymbol{G}(z)$ satisfying Assumption 1.

1) If, in addition, $\boldsymbol{G}(z)$ admits a stabilizing and proper triangular controller $\boldsymbol{C_{oo}}(z)$, then $J_t^{opt} = J^{opt}$ *if* every NMP zero of $\boldsymbol{G}(z)$ is left canonical and every unstable pole of $\boldsymbol{G}(z)$ is right canonical.
2) If, in addition, $\boldsymbol{G}(z)$ is stable, then $J_t^{opt} = J^{opt}$ *if and only if* every NMP zero of $\boldsymbol{G}(z)$ is left canonical.

Proof:

1) Since $\boldsymbol{C_{oo}}(z)$ and $\boldsymbol{G}(z)$ are triangular, it follows that both $\boldsymbol{S_{oo}}(z) = (\boldsymbol{I} + \boldsymbol{G}(z)\boldsymbol{C_{oo}}(z))^{-1}$ and $\boldsymbol{T_{oo}}(z) = \boldsymbol{I} - \boldsymbol{S_{oo}}(z)$ are triangular. On the other hand, since every NMP zero of $\boldsymbol{G}(z)$ is left canonical and every unstable pole of $\boldsymbol{G}(z)$ is right canonical, it follows from Lemma 1 in [13] that both $\boldsymbol{\xi_c}(z)$ and $\boldsymbol{\xi_p}(z)$ are diagonal. Therefore, $\boldsymbol{F}(z)$ defined in (13) is triangular and so are $\boldsymbol{F}_\perp(z)$ and $\boldsymbol{F_\infty}(z)$. As a consequence, $\boldsymbol{X^{opt}}(z)$ in (12) is triangular. Upon using (15) and the fact that triangular matrices are closed under inversion and standard matrix product, we have that $\boldsymbol{C^{opt}}(z)$ is triangular.
From the previous paragraph, it turns out that $\boldsymbol{C^{opt}}(z)$ is inherently triangular when all the NMP zeros of $\boldsymbol{G}(z)$ are left canonical and all the unstable poles of $\boldsymbol{G}(z)$ are right canonical. Therefore, it follows that in those cases constraining the structure of the controller to be triangular does not have any deleterious effect on the best achievable performance and, as a consequence, $J^{opt} = J_t^{opt}$.

2) Due to length constraints, we refer the reader to [13]. ■

Last theorem states that, if the unstable poles and NMP zeros of a given triangular plant model are sufficiently simple (i.e., all NMP zeros and unstable poles are left and right canonical, respectively), then the restriction of the controllers to be lower triangular *poses no limit on the achievable performance*, as measured by J. It should be emphasized that, as shown in [13], the aforementioned condition on the plant NMP zeros and unstable poles is equivalent to requiring that the right numerators ($\boldsymbol{N_D}(z)$) and left denominators ($\boldsymbol{D_I}(z)$) of $\boldsymbol{G}(z)$ admit diagonal GLUI and GRUI, respectively. In those cases we have that the full MIMO optimal controller $\boldsymbol{C^{opt}}(z)$ will not outperform the triangular optimal controller $\boldsymbol{C_t^{opt}}(z)$. In other words, if the conditions of Theorem 1 are satisfied, the inclusion of downstream feedback is, in principle, not required.

Theorem 1 provides an interesting link between the loss in the optimal performance due to the triangular controller constraint and dynamic plant features such as unstable poles and NMP zeros (including time delays as a special case of these). We stress that Theorem 1 is a formal complement to the arguments given in [5] regarding conditions under which it is advisable to include downstream feedback action in a controller for serial processes.

A direct consequence of Theorem 1 is that, in the cases in which the conditions of the result are not met, the performance loss in which it is incurred by constraining the controller to be lower triangular, i.e., the quantity $J_t^{opt} - J^{opt}$, should only depend on the existence of non canonical NMP zeros and non canonical unstable poles. As a matter of fact, this feature is revealed by the results in [13], where an explicit form for $J_t^{opt} - J^{opt}$ is derived for the stable case.

VI. Application to a class of serial processes

The results presented in the previous section considered general triangular plant models, without any specific dynamical structure. In this section we turn our attention to a specific class of serial processes.

Consider a serial process as in (3) such that

$$H_i(s) = \frac{k_i e^{-\theta_i s}}{\tau_i s + 1}, \tag{21}$$

$$H_{di}(s) = \frac{k_{di} e^{-\theta_{di} s}}{\tau_{di} s + 1}, \tag{22}$$

with $k_i \neq 0$, $\tau_i \geq 0$, $\tau_{di} \geq 0$, $\theta_i \geq 0$ and $\theta_{di} \geq 0$. These models, although very simple, are commonly used for pH-neutralization processes [1] and other application relevant cases such as heat exchangers [14]. Assuming that the process is sampled every T time units, and that

$$T = r_i \theta_i = r_i' \theta_{di}, \ \forall i = 1, 2, \ldots, n, \tag{23}$$

for some $r_i, r_i' \in \mathbb{N}$, then the $(i,j)^{th}$ entry of the discrete time transfer matrix $\boldsymbol{G}(z)$ can be computed as

$$G_{ij}(z) = \begin{cases} \dfrac{K_i}{z^{\theta_i/T}(z + \alpha_i)}, & \forall j = i \\[2ex] \dfrac{B_{ij}(z)}{z^{\theta_{ij}/T} A_{ij}(z)}, & \forall j < i \\[2ex] 0, & \forall j > i \end{cases} \tag{24}$$

where the constants K_i and α_i, as well as the polynomials $B_{ij}(z)$ and $A_{ij}(z)$, depend on the original subsystem parameters and

$$\theta_{ij} = \theta_j + \sum_{\ell=j+1}^{i} \theta_{d\ell}. \tag{25}$$

We note that $A(0) \neq 0$, $K_i \neq 0$ for all i, and that $G_{ij}(z)$ is stable for all (i,j).

We now draw conclusions regarding controller structure selection for the class of serial processes defined above:

***Theorem** 2:* Consider a serial process as in Figure 1, satisfying (3), (21), (22) and (24), with $k_i \neq 0$, $\tau_i \geq 0$, $\tau_{di} \geq 0$, $\theta_i \geq 0$ and $\theta_{di} \geq 0$. Assume a sampling period T satisfying (23). Then, $J_t^{opt} = J^{opt}$ if and only the time delays of each subsystem satisfy

$$\theta_i \leq \theta_{ij}, \tag{26}$$

where $i = 2, 3, \ldots, n$ and $j = 1, 2, \ldots, i-1$.

Proof: The proof follows from a straightforward application of Theorem 1, and the structure of the plant model $\boldsymbol{G}(z)$ for the class of systems under consideration. From (24) it can be noticed that all the NMP zeros of $G_{ii}(z)$ are located at infinity and that there are no unstable poles. Hence, using the definition of left canonical NMP zeros and Part 2) of Lemma 1, it follows that $J_t^{opt} = J^{opt}$ if and only if

$$reld\{G_{ii}(z)\} \leq reld\{G_{ij}\}, \tag{27}$$

for $i = 2, 3, \ldots, n$ and $j = 1, 2, \ldots, i-1$. In (27), $reld\{G_{ij}(z)\}$ denotes the relative degree of $G_{ij}(z)$ [11]. The result follows using the definitions of $G_{ij}(z)$ in (24). ■

Theorem 2 relates the loss in the optimal performance due to the triangular controller constraint and the time delays in each subsystem of Figure 1, for the class of systems defined in (21) and (22). The derived condition is simple and allows one to determine whether the inclusion of a downstream feedback action in the control law may or may not improve the overall achievable performance, as measured by J. Moreover, in the context of application relevant processes such as pH-neutralization tanks [1], condition (26) may reveal useful ideas for both process design and process control. We note that Theorem 2 formally justifies the argumentation made in Section III regarding conditions on the delays of the subsystems in a serial process, under which it is advisable to include downstream feedback action in the controller. We note, however, that the discussion in Section III is far from being complete: it only considers delays, whilst, as Theorem 1 shows, NMP zeros and unstable poles may also play a role.

We next present two simple examples to illustrate our results.

Example 1: Consider a serial process satisfying (21) and (22) with 3 inputs and 3 outputs. A straightforward use of Theorem 2 implies that better loop performance *may* be achieved with a full MIMO controller instead of a triangular one if and only if any of the following inequalities is satisfied

$$\begin{aligned}\theta_2 &> \theta_1 + \theta_{d2},\\ \theta_3 &> \theta_1 + \theta_{d2} + \theta_{d3},\\ \theta_3 &> \theta_2 + \theta_{d3}.\end{aligned}$$

Conversely, if no one of these conditions is met, then the best achievable performance can be attained with a triangular controller and hence, the inclusion of a downstream feedback does not lead to any improvement in the best achievable performance, as measured by J. ■

Example 2: A relevant case in which the results of this paper apply relates to pH-neutralization processes. For simplicity we consider a 2×2 case, which arises when considering two cascaded mixing tanks, as shown in Figure 3 [1].

In Figure 3, u_1 and u_2 correspond to the reagent flows that neutralize the pH of the inlet flow d_1, which acts as an input disturbance to the first tank. The outputs y_1 and y_2 are defined as the deviation of the output flow pH from the neutral condition ($pH = 7$). The control objective in this system is to ensure that the output flow pH is a pre-specified constant. This must be done by means of adjusting the reagent flows to achieve the desired output pH, while compensating[2] the variations in the pH of the inlet flow. According to the results in [1], a suitable model for the linearized dynamics of each tank is given by (3), (21) and (22). We can then use Theorem 2, so that from (26) it follows that downstream feedback leads to an improvement in the best achievable performance if and only if

$$\theta_2 > \theta_1 + \theta_{d2}. \tag{28}$$

Consider the numerical parameters $k_1 = k_2 = 3160$, $k_{d2} = 1580$, $\tau_1 = \tau_2 = 300[s]$, $\theta_1 = \theta_{d2} = 1[s]$ and $T = 1[s]$. For this set of parameters, Figure 4 shows the relative optimal performance degradation, i.e., $100 \cdot (J_t^{opt} - J^{opt})/J^{opt}$, when using a triangular controller instead of a full MIMO one, as a function of θ_2. It can be appreciated that when $\theta_2 > \theta_1 + \theta_{d2} = 2$, the performance achieved by a triangular controller is significatively worse than the full MIMO performance. This result validates our theoretical conclusion in (28). Moreover, as it could be expected, when θ_2 grows, the performance loss increases monotonically. Consequently, as θ_2 becomes larger, the inclusion of downstream feedback in the controller becomes more and more advisable. ■

[2]This is a simplified example, since we are not considering other disturbances in the system that may be important, such as variations in the inlet flow pH and reagent pH. For details see [1].

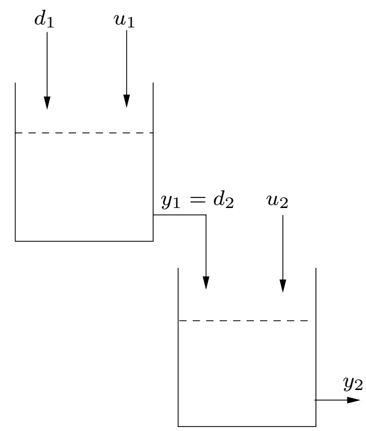

Fig. 3. Schematic diagram of a two-stage pH-neutralization process.

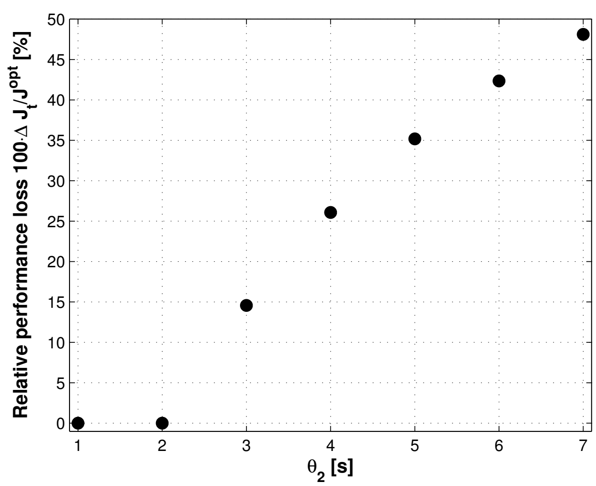

Fig. 4. Relative performance degradation when using a triangular optimal controller ($\Delta_J \triangleq J_t^{opt} - J^{opt}$).

VII. Conclusions

This paper studied the control of serial processes and, in particular, addressed the determination of conditions under which the best performance achievable by a full MIMO controller is better than the best performance achievable by a triangular controller, i.e., conditions under which the inclusion of downstream feedback is advisable. We derive simple sufficient (and in the case of stable plant models also necessary) conditions on the unstable poles, non minimum phase (NMP) zeros and delays of the discrete time plant model that ensures no penalization on the achievable performance, when restricting the controller to be triangular. We believe that the derived conditions could be useful in both process design and control system design, serving as a guideline for controller structure selection.

Several questions remain open. An interesting one relates to whether the conclusions of this paper also hold in the face of model uncertainty. In addition, it would be useful to develop a more comprehensive framework that encompasses plants and controllers with several structures, not only triangular ones.

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